



Abstract. Recently, in September 2025, Roland van der Veen and myself released a paper titled “A Fast ☺, Strong ☺, Topologically Meaningful ☺, and Fun ☺ Knot Invariant” [BV3]. More recently, in January 2025 in Les Diablerets [BN3], I gave a talk in which I suggested 27 homework tasks related to that same invariant Θ (plus a bonus task). Today, after a brief review, I’ll talk a bit more about just 4 of those tasks:



van der Veen

- ☺ Figure out the relationship of Θ with Chern Simons theory.
- ☺ Complete the discussion of the relationship of Θ with sl_3 .
- ☺ Find formulas for Θ corresponding to other presentations of the Alexander module.
- ☺ Fully understand Θ-like formulas and their non-uniqueness (i.e., clean the mess).

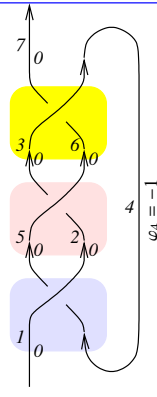
Acknowledgement. This work was supported by NSERC grant RGPIN-2025-06718 and by the Chu Family Foundation (NYC).

Preparation. Draw an n -crossing knot K as a diagram D as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n+1\}$ and with rotation numbers φ_k .

A. With T an indeterminate, start from a presentation matrix A for the Alexander module of K , coming from the Wirtinger presentation of $\pi_1(K)$: $A := I_{2n+1} + \sum_c A_c$, where

$$i^+ \begin{array}{c} \nearrow \\ \searrow \end{array} j^+ \quad i^+ \begin{array}{c} \nwarrow \\ \searrow \end{array} j^+ \quad \rightarrow \quad \begin{array}{c|c|c} A_c & i+1 & j+1 \\ \hline i & -T^s & T^s-1 \\ j & 0 & -1 \end{array}$$

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Note. Alexander’s Δ is $\Delta = T^{(-\varphi-w)/2} \det(A)$, with $\varphi = \sum_k \varphi_k$, $w = \sum_c s_c$.

G. Let $G = (g_{\alpha\beta}) := A^{-1}$, the “two point function”:

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



the “traffic function”

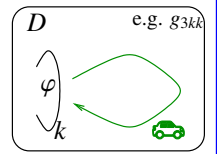
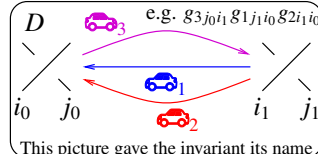
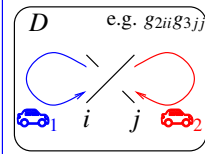
Let T_1 and T_2 be new indeterminates, let $T_3 = T_1 T_2$, and let $G_\nu = (g_{\nu\alpha\beta})$ be G with $T \rightarrow T_\nu$, for $\nu = 1, 2, 3$. Likewise for Δ_ν .

$$F_1(c) = s \left[1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \right. \\ \left. + (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj} \right] \\ + \frac{s}{T_2^s - 1} \left[(T_1^s - 1) T_2^s (g_{3ji} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}) \right. \\ \left. + (T_3^s - 1) (g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2ji} g_{3ji}) \right. \\ \left. - (T_1^s - 1) (T_2^s + 1) (T_3^s - 1) g_{1ji} g_{3ji} \right] \\ F_2(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1) (T_3^{s_1} - 1) g_{1ji} g_{3ji}}{T_2^{s_1} - 1} (T_2^{s_0} g_{2i i_0} + g_{2j_1 j_0} - T_2^{s_0} g_{2j_1 i_0} - g_{2i_1 j_0}) \\ F_3(\varphi, k) = \varphi(-1/2 + g_{3kk})$$

Ouch.

Theorem [BV3]. With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic $F_1(c) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$, a cubic $F_2(c_0, c_1) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $F_3(\varphi, k)$ such that θ is a knot invariant:

$$\theta(D) := \Delta_1 \Delta_2 \Delta_3 \left(\sum_c F_1(c) + \sum_{c_0, c_1} F_2(c_0, c_1) + \sum_k F_3(\varphi_k, k) \right),$$



This picture gave the invariant its name

If these pictures remind you of Feynman diagrams, it’s because they are Feynman diagrams [BN2].

☺ Fast.

Data (ouch)

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T3 := T1 T2;
CF[ $\varphi$ ] := ExpandCollect[ $\varphi$ ,  $\varphi$ , F] /. F -> Factor;
 $\varphi$  := (1/2 -  $\varphi_{3ii}$  +  $T_2^s$   $\varphi_{1ii}$   $\varphi_{2ji}$  -  $T_2^s$   $\varphi_{3jj}$   $\varphi_{2ji}$  - ( $T_2^s$  - 1)  $\varphi_{3ii}$   $\varphi_{2ji}$  + ( $T_3^s$  - 1)  $\varphi_{2ji}$   $\varphi_{3ji}$  -  $\varphi_{1ii}$   $\varphi_{2jj}$  + 2  $\varphi_{3ii}$   $\varphi_{2jj}$  +  $\varphi_{1ii}$   $\varphi_{3jj}$  -  $\varphi_{2ii}$   $\varphi_{3jj}$ ) + (( $T_1^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ji}$  -  $T_2^s$   $\varphi_{2jj}$  +  $T_2^s$   $\varphi_{3jj}$ ) + ( $T_2^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ii}$   $\varphi_{3ji}$  + ( $T_2^s$  - 2)  $\varphi_{2ij}$   $\varphi_{3ji}$  - ( $T_1^s$  - 1) ( $T_2^s$  + 1)  $\varphi_{1ji}$   $\varphi_{3ji}$ )) / ( $T_2^s$  - 1);
F2[ $\varphi$ ] :=  $\varphi$  { $\varphi_{00}$ ,  $\varphi_{01}$ ,  $\varphi_{02}$ ,  $\varphi_{11}$ ,  $\varphi_{12}$ ,  $\varphi_{21}$ ,  $\varphi_{22}$ };
CF[ $\varphi$ ] :=  $\varphi$  { $\varphi_{00}$ ,  $\varphi_{01}$ ,  $\varphi_{02}$ ,  $\varphi_{11}$ ,  $\varphi_{12}$ ,  $\varphi_{21}$ ,  $\varphi_{22}$ };
 $\varphi$  := (1/2 -  $\varphi_{3ii}$  +  $T_2^s$   $\varphi_{1ii}$   $\varphi_{2ji}$  -  $T_2^s$   $\varphi_{3jj}$   $\varphi_{2ji}$  - ( $T_2^s$  - 1)  $\varphi_{3ii}$   $\varphi_{2ji}$  + ( $T_3^s$  - 1)  $\varphi_{2ji}$   $\varphi_{3ji}$  -  $\varphi_{1ii}$   $\varphi_{2jj}$  + 2  $\varphi_{3ii}$   $\varphi_{2jj}$  +  $\varphi_{1ii}$   $\varphi_{3jj}$  -  $\varphi_{2ii}$   $\varphi_{3jj}$ ) + (( $T_1^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ji}$  -  $T_2^s$   $\varphi_{2jj}$  +  $T_2^s$   $\varphi_{3jj}$ ) + ( $T_2^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ii}$   $\varphi_{3ji}$  + ( $T_2^s$  - 2)  $\varphi_{2ij}$   $\varphi_{3ji}$  - ( $T_1^s$  - 1) ( $T_2^s$  + 1)  $\varphi_{1ji}$   $\varphi_{3ji}$ )) / ( $T_2^s$  - 1);
F3[ $\varphi$ ,  $k$ ] :=  $\varphi$  { $\varphi_{3ii}$ ,  $\varphi_{2ji}$ };

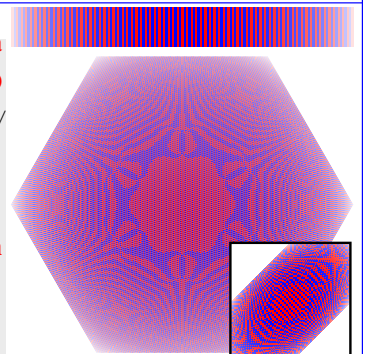
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Program

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CF[ $\varphi$ ] := ExpandCollect[ $\varphi$ ,  $\varphi$ , F] /. F -> Factor;
 $\varphi$  := (1/2 -  $\varphi_{3ii}$  +  $T_2^s$   $\varphi_{1ii}$   $\varphi_{2ji}$  -  $T_2^s$   $\varphi_{3jj}$   $\varphi_{2ji}$  - ( $T_2^s$  - 1)  $\varphi_{3ii}$   $\varphi_{2ji}$  + ( $T_3^s$  - 1)  $\varphi_{2ji}$   $\varphi_{3ji}$  -  $\varphi_{1ii}$   $\varphi_{2jj}$  + 2  $\varphi_{3ii}$   $\varphi_{2jj}$  +  $\varphi_{1ii}$   $\varphi_{3jj}$  -  $\varphi_{2ii}$   $\varphi_{3jj}$ ) + (( $T_1^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ji}$  -  $T_2^s$   $\varphi_{2jj}$  +  $T_2^s$   $\varphi_{3jj}$ ) + ( $T_2^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ii}$   $\varphi_{3ji}$  + ( $T_2^s$  - 2)  $\varphi_{2ij}$   $\varphi_{3ji}$  - ( $T_1^s$  - 1) ( $T_2^s$  + 1)  $\varphi_{1ji}$   $\varphi_{3ji}$ )) / ( $T_2^s$  - 1);
F2[ $\varphi$ ] :=  $\varphi$  { $\varphi_{00}$ ,  $\varphi_{01}$ ,  $\varphi_{02}$ ,  $\varphi_{11}$ ,  $\varphi_{12}$ ,  $\varphi_{21}$ ,  $\varphi_{22}$ };
CF[ $\varphi$ ] :=  $\varphi$  { $\varphi_{00}$ ,  $\varphi_{01}$ ,  $\varphi_{02}$ ,  $\varphi_{11}$ ,  $\varphi_{12}$ ,  $\varphi_{21}$ ,  $\varphi_{22}$ };
 $\varphi$  := (1/2 -  $\varphi_{3ii}$  +  $T_2^s$   $\varphi_{1ii}$   $\varphi_{2ji}$  -  $T_2^s$   $\varphi_{3jj}$   $\varphi_{2ji}$  - ( $T_2^s$  - 1)  $\varphi_{3ii}$   $\varphi_{2ji}$  + ( $T_3^s$  - 1)  $\varphi_{2ji}$   $\varphi_{3ji}$  -  $\varphi_{1ii}$   $\varphi_{2jj}$  + 2  $\varphi_{3ii}$   $\varphi_{2jj}$  +  $\varphi_{1ii}$   $\varphi_{3jj}$  -  $\varphi_{2ii}$   $\varphi_{3jj}$ ) + (( $T_1^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ji}$  -  $T_2^s$   $\varphi_{2jj}$  +  $T_2^s$   $\varphi_{3jj}$ ) + ( $T_2^s$  - 1)  $\varphi_{3ji}$  ( $T_2^s$   $\varphi_{1ii}$   $\varphi_{3ji}$  + ( $T_2^s$  - 2)  $\varphi_{2ij}$   $\varphi_{3ji}$  - ( $T_1^s$  - 1) ( $T_2^s$  + 1)  $\varphi_{1ji}$   $\varphi_{3ji}$ )) / ( $T_2^s$  - 1);
F3[ $\varphi$ ,  $k$ ] :=  $\varphi$  { $\varphi_{3ii}$ ,  $\varphi_{2ji}$ };

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A random 300 xing knot from [DHOEBL]. For most invariants, 300 is science fiction.

☺ Strong. Θ vs. a slew of other reasonably-computable invariants (deficits shown):

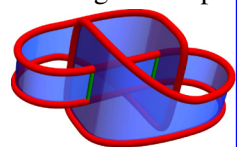
n	≤ 10	≤ 11	≤ 12	≤ 13	≤ 14	≤ 15
knots	249	801	2,977	12,965	59,937	313,230
Δ	(38)	(250)	(1,204)	(7,326)	(39,741)	(236,326)
σ_{LT}	(108)	(356)	(1,525)	(7,736)	(40,101)	(230,592)
J	(7)	(70)	(482)	(3,434)	(21,250)	(138,591)
Kh	(6)	(65)	(452)	(3,226)	(19,754)	(127,261)
H	(2)	(31)	(222)	(1,839)	(11,251)	(73,892)
Vol	(~6)	(~25)	(~113)	(~1,012)	(~6,353)	(~43,607)
(Kh, H, Vol)	(~0)	(~14)	(~84)	(~911)	(~5,917)	(~41,434)
(Δ, ρ_1)	(0)	(14)	(95)	(959)	(6,253)	(42,914)
(Δ, ρ_1, ρ_2)	(0)	(14)	(84)	(911)	(5,926)	(41,469)
$(\rho_1, \rho_2, Kh, H, Vol)$	(0)	(~14)	(~84)	(~911)	(~5,916)	(~41,432)
Θ	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, ρ_2)	(0)	(3)	(10)	(169)	(982)	(6,341)
(Θ, σ_{LT})	(0)	(3)	(19)	(194)	(1,118)	(6,758)
(Θ, Kh)	(0)	(3)	(18)	(185)	(1,062)	(6,555)
(Θ, H)	(0)	(3)	(18)	(185)	(1,064)	(6,563)
(Θ, Vol)	(0)	(~3)	(~10)	(~169)	(~973)	(~6,308)
$(\Theta, \rho_2, Kh, H, Vol)$	(0)	(~3)	(~10)	(~169)	(~972)	(~6,304)

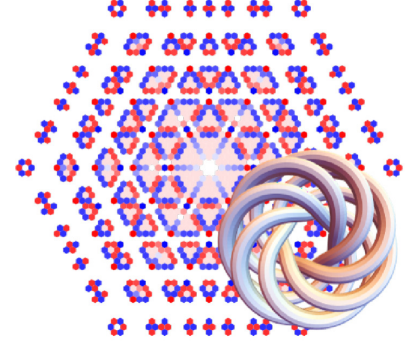
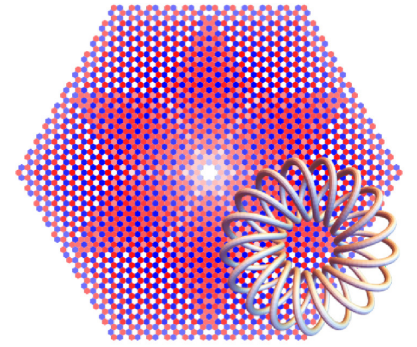
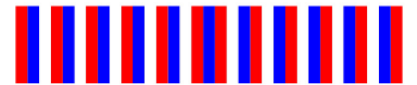
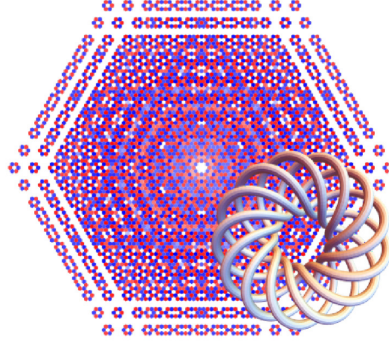
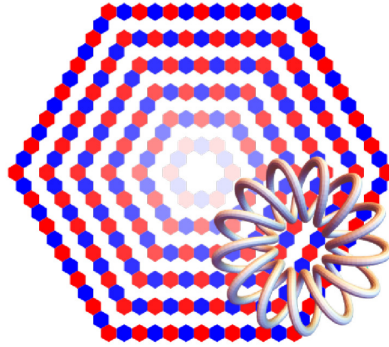
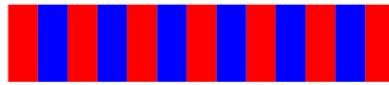
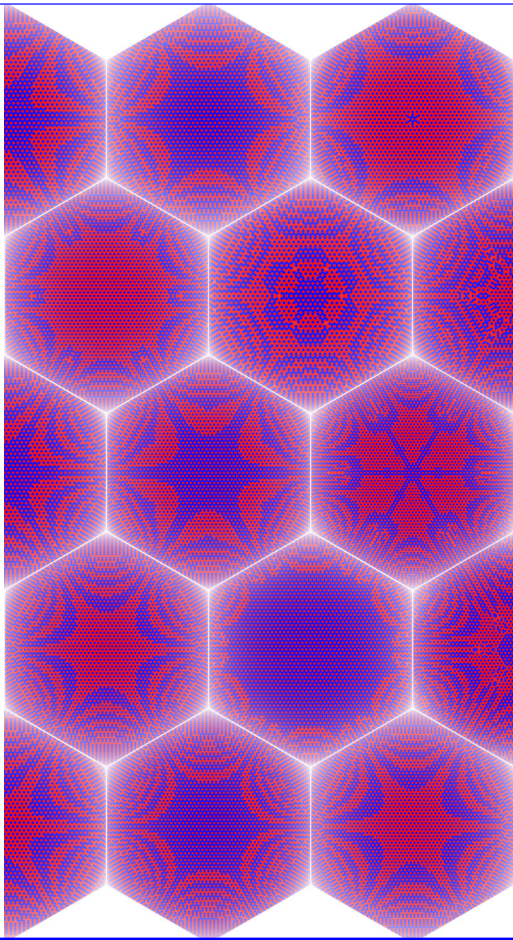
☺ Topologically Meaningful. θ is near Δ and we dream that anything Δ can do, θ does too (sometimes better).

Theorem. $\deg_{T_1} \theta(K) \leq 2g(K)$.

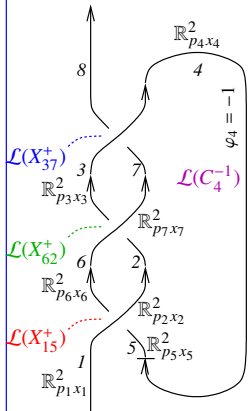
Conjecture (verified to 13 xings). If K is a fibered knot and d is the degree of $\Delta(K)$ (the highest power of T), then the coefficient of T_2^{2d} in $\theta(K)$, which is a polynomial in T_1 , is an integer multiple of $T_1^d \Delta(K)|_{T \rightarrow T_1}$.

Dream. θ has something to say about ribbon knots.





The $sl_2^{\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2 = 0$:



$$Z = \oint_{\mathbb{R}^{14}_{p_i x_i}} \mathcal{L}(X_{15}^+) \mathcal{L}(X_{62}^+) \mathcal{L}(X_{37}^+) \mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}$ standard

where $\mathcal{L}(X_{ij}^s) = T^{s/2} e^{L(X_{ij}^s)}$ and $\mathcal{L}(C_i^\varphi) = T\varphi/2 e^{L(C_i^\varphi)}$, and

$$L(X_{ij}^s) = x_i(p_{i+1} - p_i) + x_j(p_{j+1} - p_j) + (T^s - 1)x_i(p_{i+1} - p_{j+1}) + \frac{\epsilon s}{2} \left(x_i(p_i - p_j) \left(\frac{(T^s - 1)x_i p_j}{+2(1 - x_j p_j)} \right) - 1 \right)$$

$$L(C_i^\varphi) = x_i(p_{i+1} - p_i) + \epsilon\varphi(1/2 - x_i p_i)$$

So $Z = T \oint_{\mathbb{R}^{14}} d p_1 \dots d p_7 d x_1 \dots d x_7$, where $L(\odot) = \sum_{i=1}^7 x_i(p_{i+1} - p_i) + (T-1)(x_1(p_2 - p_6) + x_6(p_7 - p_3) + x_3(p_4 - p_8))$

$$+ \frac{\epsilon}{2} \left(\begin{aligned} & x_1(p_1 - p_5)((T-1)x_1 p_5 + 2(1 - x_5 p_5)) - 1 \\ & + x_6(p_6 - p_2)((T-1)x_6 p_2 + 2(1 - x_2 p_2)) - 1 \\ & + x_3(p_3 - p_7)((T-1)x_3 p_7 + 2(1 - x_7 p_7)) - 1 \\ & + 2x_4 p_4 - 1 \end{aligned} \right),$$

and so $Z = (T - 1 + T^{-1})^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right) = \Delta^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})p_1}{\Delta^2}\right)$. Here Δ is the Alexander polynomial and ρ_1 is the Rozansky-Overbay polynomial [Ro1, Ro2, Ro3, Ov, BV1, BV2]. It is a reduction of θ .

☺ **Combinatorics:** We still don't really understand the θ formulas $F_{1,2,3}$. How unique are they? What others like them there are? What special games they play when the knot is non-virtual? Perhaps with Alexander numbering? Is there an indigenous theory here, or only exogenous results?

The $sl_3^{\epsilon^2}$ Example [BV3]. Here we have two formal variables T_1 and T_2 , we set $T_3 := T_1 T_2$, we integrate over 6 variables for each edge: $p_{1i}, p_{2i}, p_{3i}, x_{1i}, x_{2i}$, and x_{3i} , with the Lagrangian given by:

$$\begin{aligned} \mathcal{L}[\mathbf{X}_{i-}, j_- [\mathbf{s}_-]] &:= T_3^{\mathbf{s}} \mathbb{E} [\mathbf{CF@Plus} [\\ & \sum_{v=1}^3 (x_{vi} (p_{vi}^+ - p_{vi}) + x_{vj} (p_{vj}^+ - p_{vj}) + \\ & (T_v^s - 1) x_{vi} (p_{vi}^+ - p_{vj}^+)), \\ & (T_1^s - 1) p_{3j} x_{1i} (T_2^s x_{2i} - x_{2j}), \\ & \in \mathbf{s} (T_3^s - 1) p_{1j} (p_{2i} - p_{2j}) x_{3i} / (T_2^s - 1), \\ & \in \mathbf{s} (1/2 + T_2^s p_{1i} p_{2j} x_{1i} x_{2i} - p_{1i} p_{2j} x_{1i} x_{2j} - \\ & p_{3i} x_{3i} - (T_2^s - 1) p_{2j} p_{3i} x_{2i} x_{3i} + \\ & (T_3^s - 1) p_{2j} p_{3j} x_{2i} x_{3i} + 2 p_{2j} p_{3i} x_{2j} x_{3i} + \\ & p_{1i} p_{3j} x_{1i} x_{3j} - p_{2i} p_{3j} x_{2i} x_{3j} - T_2^s p_{2j} p_{3j} x_{2i} x_{3j} + \\ & ((T_1^s - 1) p_{1j} x_{1i} \\ & (T_2^s p_{2j} x_{2i} - T_2^s p_{2j} x_{2j} - (T_2^s + 1) (T_3^s - 1) p_{3j} x_{3i} + \\ & T_2^s p_{3j} x_{3j}) + (T_3^s - 1) p_{3j} x_{3i} \\ & (1 - T_2^s p_{1i} x_{1i} + p_{2i} x_{2j} + (T_2^s - 2) p_{2j} x_{2j})) / \\ & (T_2^s - 1))]] \end{aligned}$$

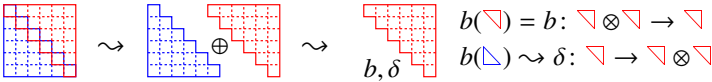
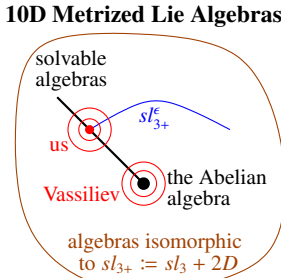
$$\mathcal{L}[\mathbf{C}_i [\varphi_-]] := T_3^{\varphi} \mathbb{E} [\sum_{v=1}^3 x_{vi} (p_{vi}^+ - p_{vi}) + \epsilon \varphi (p_{3i} x_{3i} - 1/2)]$$

Theorem. Here, $Z = \frac{1}{\Delta_1 \Delta_2 \Delta_3} \exp\left(\epsilon \frac{\theta}{\Delta_1 \Delta_2 \Delta_3}\right)$.

☺ **Topology:** We don't have formulas coming from other presentations of the Alexander module. Such may give the definitive proofs of the genus theorem and fibered conjecture, explain the hexagonal symmetry, and clarify the relation with the two-loop invariant [GR, Kr, Oh]. Why should Wirtinger be special? Are there efficient 3D formulas for θ [BBHS]?

⊙ Quantum Algebra: We still haven't implemented the relationship with sl_3 and with other quantum groups.

Co-Commutative Limit / Solvable Approximation. In sl_n , half is enough! Indeed $sl_n \oplus a_{n-1} = \mathcal{D}(\nabla, b, \delta)$. Now define $sl_{n+}^\epsilon := \mathcal{D}(\nabla, b, \epsilon\delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\Delta, \Delta] = \epsilon\Delta$, and $[\nabla, \Delta] = \Delta + \epsilon\nabla$. The same process works for all semi-simple Lie algebras $\mathfrak{g}$, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.



Note. $\epsilon = 0$ is w-knots [BD, BN1].
By PBW, $\mathcal{U}(\mathfrak{g}_+^\epsilon)$ and $\mathcal{U}_q(\mathfrak{g}_+^\epsilon)$ are isomorphic as vector spaces to polynomial rings. So we care for maps between polynomial rings.
Docile Perturbed Gaussians. For a finite set A , let $z_A := \{z_i\}_{i \in A}$ and let $\zeta_A := \{z_i^* = \zeta_i\}_{i \in A}$ with $\langle z_i, \zeta_j \rangle = \delta_{ij}$. We have

$$\begin{aligned} \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) &\simeq \mathbb{Q}[z_A]^* \otimes \mathbb{Q}[z_B] \\ &\simeq \mathbb{Q}[\zeta_A] \otimes \mathbb{Q}[z_B] \simeq \mathbb{Q}[z_B][\zeta_A] \subset \mathbb{Q}[\zeta_A, z_B], \end{aligned}$$

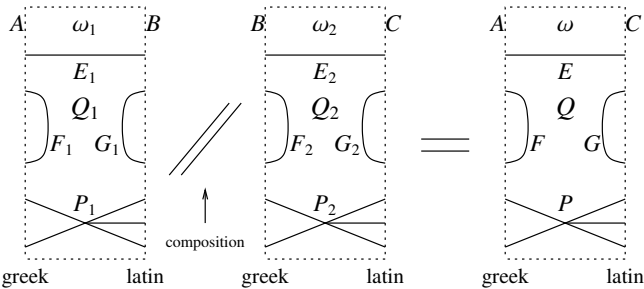
so every morphism has a *generating function*.
Future Theorem. The generating functions of all the relevant morphisms for $\mathcal{U}(\mathfrak{g}_+^\epsilon)$ and $\mathcal{U}_q(\mathfrak{g}_+^\epsilon)$ (meaning, m, Δ, S, R , etc.) are docile perturbed $\left(\frac{\text{two}}{\text{step}}\right)$ Gaussians.

Theorem. The composition of two docile perturbed Gaussians is again a docile perturbed Gaussian, computable efficiently.

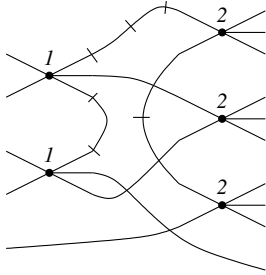
Proof Sketch. In $\text{mor}(A \rightarrow B)$,

$$Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i z_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} z_i z_j,$$

and so



- where $\bullet E = E_1(I - F_2G_1)^{-1}E_2$.
- $\bullet F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
- $\bullet G = G_2 + E_2^TG_1(I - F_2G_1)^{-1}E_2$.
- $\bullet \omega = \omega_1\omega_2 \det(I - F_2G_1)^{-1}$.
- $\bullet P$ is computed as the solution of a messy PDE or using “connected Feynman diagrams” (yet we’re still in pure algebra!). Docility is preserved.



All this is fully implemented for sl_2 and in the works for sl_n . Hopefully Roland and I will have news by my (generalized) 61th birthday.

⊙ Quantum Field Theory: Chern-Simons-Witten theory is screaming at us and we aren't listening. There is an alternative formula for θ , and it just can't be that it isn't directly related to Chern-Simons-Witten with gauge group $\mathfrak{g}_\epsilon$,

$$\int_{A \in \Omega^1(\mathbb{R}^3; \mathfrak{g}_\epsilon)} \mathcal{D}A \otimes \frac{i}{4\pi} \int_{\mathbb{R}^3} \text{tr}(A \wedge dA + \frac{2\sqrt{\hbar}}{3} A \wedge A \wedge A) \mathcal{P} \exp_\gamma(\sqrt{\hbar} A) \in \mathcal{U}(\mathfrak{g}_\epsilon)[[\hbar]].$$

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, Acta Mathematica Vietnamica **40-2** (2015) 271–329, [arXiv:1308.1721](#).
[BN2] D. Bar-Natan, *Knot Invariants from Zero-Dimensional QFT*, talk in Bonn, May 2025. Handout and video at [oeqf/bo25](#).
[BN3] —, *Homework*, talk in Les Diablerets, January 2026. Handout and video at [oeqf/ld26](#).
[BBHS] —, I. Bar-Natan, I. Halacheva, N. Scherich, *Yarn Ball Knots and Faster Computations*, J. Appl. Comp. Top. **8** (2024) 175–192, [arXiv:2108.10923](#).
[BD] —, Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: W-Knots and the Alexander Polynomial*, Alg. and Geom. Top. **16-2** (2016) 1063–1133, [arXiv:1405.1956](#), and *II: Tangles and the Kashiwara-Vergne Problem*, Math. Ann. **367** (2017) 1517–1586, [arXiv:1405.1955](#).
[BV1] —, R. van der Veen, *A Polynomial Time Knot Polynomial*, Proc. Amer. Math. Soc. **147** (2019) 377–397, [arXiv:1708.04853](#).
[BV2] —, —, *A Perturbed-Alexander Invariant*, Quantum Topology **15** (2024) 449–472, [oeqf/APAI](#).
[BV3] —, —, *A Fast, Strong, Topologically Meaningful, and Fun Knot Invariant*, [oeqf/Theta](#) and [arXiv:2509.18456](#).
[DHOEBL] N. Dunfield, A. Hirani, M. Obeidin, A. Ehrenberg, S. Bhattacharyya, D. Lei, and others, *Random Knots: A Preliminary Report*, lecture notes at [oeqf/DHOEBL](#). Also a data file at [oeqf/DD](#).
[GR] S. Garoufalidis and L. Rozansky, *The Loop Expansion of the Kontsevich Integral, the Null-Move, and S-Equivalence*, [arXiv:math.GT/0003187](#).
[Kr] A. Kricker, *The Lines of the Kontsevich Integral and Rozansky's Rationality Conjecture*, [arXiv:math/0005284](#).
[Oh] T. Ohtsuki, *On the 2-Loop Polynomial of Knots*, Geom. Topol. **11-3** (2007) 1357–1475.
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, Ph.D. thesis, University of North Carolina, Aug. 2013, [oeqf/Ov](#).
[Ro1] L. Rozansky, *A Contribution of the Trivial Flat Connection to the Jones Polynomial and Witten's Invariant of 3D Manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, [arXiv:hep-th/9401061](#).
[Ro2] —, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, [arXiv:q-alg/9604005](#).
[Ro3] —, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, [arXiv:math/0201139](#).

References.

A FAST, STRONG, TOPOLOGICALLY MEANINGFUL, AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. In this paper we discuss a pair of polynomial knot invariants $\Theta = (\Delta, \theta)$ which are:

- Theoretically and practically fast: Θ can be computed in polynomial time. We can compute it in full on random knots with over 300 crossings, and its evaluation at simple rational numbers on random knots with over 600 crossings.
- Strong: Its separation power is much greater than the hyperbolic volume, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings (which being computable on much larger knots).
- Topologically meaningful: It gives a genus bound, and there are reasons to hope that it would do more.

Fun: Send to Figures 1.1–1.4, 3.1, and 6.2. Δ is mostly the Alexander polynomial. It is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oht02], containing Rozansky, Kricker, and Garoufalidis [Bla01, Kri02, Roz03, Kr08]. Yet our formulas, proofs, and programs are much simpler and enable its computation even on very large knots.

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6. Fun
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DATE: First edition September 22, 2025. This edition May 6, 2026.
2020 Mathematics Subject Classification. Primary 57M24; secondary 57M99.
KEY WORDS AND PHRASES. Alexander polynomial, loop expansion, solvable approximation, knot genus, framed knots, ribbon knots, polynomial time computations, Feynman diagrams, perturbed Gaussian integration, Seifert formulae.
This paper is available in electronic form, along with source files and a demo Mathematica notebook at [http://oeqf.berkeley.edu/Theta](#) and at [oeqf/2509.18456](#).

1. FUN

The word “fun” rarely appears in the title of a math paper, so let us start with a brief justification (more can be found in a Quanta Magazine article, [Kri]).

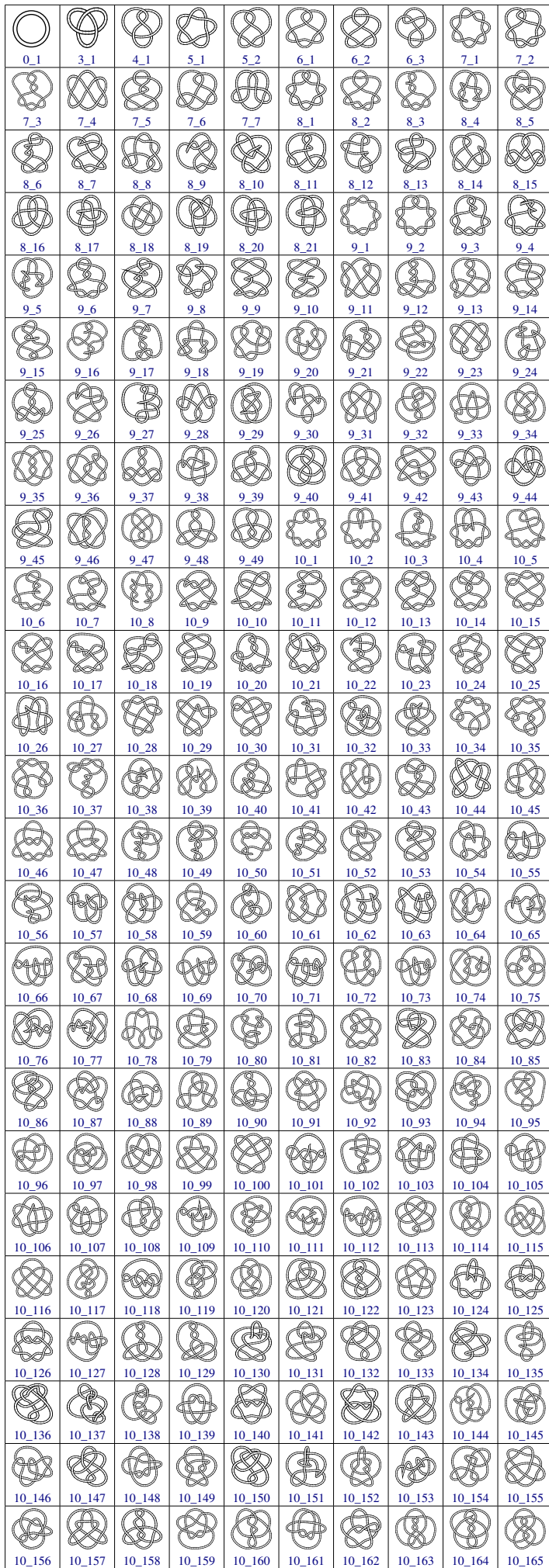
Θ is a pair of polynomials. The first, Δ , is old news, the Alexander polynomial [Al1]. It is a one-variable Laurent polynomial in a variable T . For example, $\Delta(5) = T^{-1} - 1 + T$. We turn such a polynomial into a list of coefficients (for Δ , it is $(1, -1, 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a “bar code”, and for the trefoil it is:

Similarly, θ is a 2-variable Laurent polynomial in variables T_1 and T_2 . We can turn such a polynomial into a 2D array of colours, namely into a picture. To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a shear transformation to the plane before printing. So a monomial $cT_1^aT_2^b$ gets printed at position $(n_1 - n_2/2, \sqrt{3}n_2/2)$ instead of the more straightforward (n_1, n_2) . On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^2 + T_1^2T_2 - T_2^2$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a “hexagonal QR code”. For the knots in the Rolfsen table (with the smallest prepended at the stars), they are in Figure 1.1. For some alternating square weave knots, they are in Figure 1.2, and for a random square weave, in Figure 1.3. In addition, the hexagonal QR codes of 15 knots with ≥ 300 crossings are in Figure 1.4, and Θ of a 122-crossing torus knot is in Figure 3.1. Some further computations and figures, also highlighting the parity of coefficients rather than just their signs, are at [Lal].

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagonal (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.4, there seem to be large-scale patterns perhaps reminiscent of the “Chladni figures” formed by powders atop vibrating plates (on right). We can’t prove any of these things, and the last one, we can’t even formulate properly. Yet they are clearly there, too clear to be the result of chance alone. We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

A (2, 41, −41) pretzel for dessert



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