QUESTIONS ASKED AT A BANFF CONFERENCE, APRIL 2025.

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1. Question 1: Emergent Knotted Objects

Consider some space $\mathcal{K}$ of knotted objects in a three manifold M (knots, or links, or knotted graphs, or tangles if M has a boundary, or braids if one can specify a "vertical" direction in M). If we mod out $\mathcal{K}$ by the relation $\overcrossing = \undercrossing$, knottedness is eliminated and what remains is a space of curves (or multi-curves, or graphs, etc.) regarded up to homotopy. "Emergent knotted objects" are what you get when you mod out $\mathbb{Q}\mathcal{K}$, the space of formal linear combinations of elements of $\mathcal{K}$, by a slightly weaker version of $\overcrossing = \undercrossing$ known in the theory of finite type invariants [BN1, CDM] as the relation $\doublepoint^2 = 0$, or more precisely, $(\doublepoint\doublepoint) := (\overcrossing\overcrossing) - (\overcrossing\undercrossing) - (\undercrossing\overcrossing) + (\undercrossing\undercrossing) = 0$. Knottedness almost entirely disappears yet in the spaces we get, $\mathcal{K}^{em} := \mathbb{Q}\mathcal{K}/(\doublepoint^2 = 0)$, we see that just a bit of knot theory begins to emerge beyond the homotopy theory that is already there. If, for example, we are looking at links in $\mathbb{R}^3$, then the only non-constant invariants in $(\mathcal{K}^{em})^*$ are the linking numbers.

Questions. What can you say about emergent knots in general? How much more are they than just homotopy classes?

Emergent knots can be filtered using the Goussarov bracelet filtration [BN2] (it makes sense in an arbitrary three manifold, and it is not the same as the Vassiliev filtration as the different "detours" may differ in homotopy).

Questions. What is the associated graded $\operatorname{gr}\mathcal{K}^{em}$ of $\mathcal{K}^{em}$? Are there expansions (universal finite type invariants) $Z^{em}\colon \mathcal{K}^{em} \to \operatorname{gr}\mathcal{K}^{em}$?

In the case where M is a pole dancing studio ($D^2 \times I$ with a number of vertical lines removed), the references [BN3, BDHLS, Ku] suggest strongly that the answers to these questions are closely related to the much-studied and highly non-trivial "Kashiwara-Vergne equations" [KV, AM, AT, BD, AKKN].

The appendix of [Ku] is a more detailed version of this question.

2. Question 2: Knots and Bnots

After asking this question I learned that it is essentially contained in Polyak's lovely paper [Po] (see especially his Figure 7 and the discussion around it), and so I will keep my discussion brief. We all know that the set of knots is equivalent to the set $\mathcal{D}$ of knot diagrams, modulo the three Reidemeister moves R1, R2, and R3, taken with all possible strand orientations. The moves R2 and R3 have two variants: The "b" variant which stands for "those R2 and R3 moves that may occur in braids", and the "c" variant, which stands for "those moves in which the middle bigon/triangle is cyclically oriented". So more precisely, $\{\text{knots}\} = \mathcal{D}/(R1, R2b, R2c, R3b, R3c)$. We define $\{\text{bnots}\} := \mathcal{D}/(R1, R2b, R3b)$. Namely, bnots are knot diagrams module R1 moves and only the R2 and R3 moves that may occur in braids.

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The "obvious" (False) map $F\colon \{\text{knots}\} \to \{\text{bnots}\}$ which maps every knot diagram to itself regarded as a bnot diagram is not well defined because its domain has more relations than its range. And yet there is a more subtle (True!) embedding $T\colon \{\text{knots}\} \to \{\text{bnots}\}$: namely, for any knot K find a braid B whose closure is K and map $T\colon K \mapsto B$. This map is well defined because the moves that occur within Markov's theorem, which relates knots and braids, are only R1 and the "b" versions of R2 and R3. I expect that there are many bnots that are not in the image of the map T.

Question. What are bnots? How much more are they than knots? Do they have any significance?

In principle, there may be quantities that can be defined on all knot diagrams and are invariant only under R1, R2b, and R3b. Such quantities, even though they are defined on all knot diagrams, define knot invariants only through the use of braids. I know of only one real life example: Khovanov-Rozansky homology [KR].

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