```
In[•]:= n = 3;
        A = IdentityMatrix[n];
        Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β}, {n, n}]], {β, 1, n}, {α, 1, β}];
        A // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} e^{\xi_{1,1}} & e^{\xi_{1,1}+\xi_{2,2}}\,\xi_{1,2} & e^{\xi_{3,3}}\left(e^{\xi_{1,1}}\,\xi_{1,3}+e^{\xi_{1,1}+\xi_{2,2}}\,\xi_{1,2}\,\xi_{2,3}\right) \\ 0 & e^{\xi_{2,2}} & e^{\xi_{2,2}+\xi_{3,3}}\,\xi_{2,3} \\ 0 & 0 & e^{\xi_{3,3}} \end{pmatrix}$$

```
In[•]:= Reduce[x + y + z == 0 && x + y == 0]
```

Out[•]=

$$z == 0 \;\&\&\; x == -y$$

```
In[•]:= Solve[e^x == e^y, x, Assuming → {y ∈ Reals}]
```

Solve: Unknown option Assuming in Solve[$e^x == e^y$, x, Assuming → {y ∈ ℝ}].

Out[•]=

$$\text{Solve}\left[e^x == e^y, x, \text{Assuming} \to \{y \in \mathbb{R}\}\right]$$

```
In[•]:= utm_n[i_, j_ → k_] := Module[{A, α, β, eqns, vars, sol, ξ},
          𝔼_{i,j→{k}}[
           A = IdentityMatrix[n];
           Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β}, {n, n}]], {β, 1, n}, {α, 1, β}];
           (* This specifies the PBW ordering *)
           eqns = And @@ Thread[Flatten[CF[(Inverse[A /. ξ_{αβ__} :↦ ξ_{αβ}] /. ξ_{α_,α_} :↦ ξ_{α,α}[i] + ξ_{α,α}[j]).
                    (A /. ξ_{αβ__} :↦ ξ_{αβ}[i]).(A /. ξ_{αβ__} :↦ ξ_{αβ}[j])]] == Flatten[IdentityMatrix[n]]];
           vars = Union@Cases[eqns, ξ__, ∞];
           {sol} = Solve[eqns, vars];
           CF[(vars /. ξ_{αβ__} → x_{αβ}[k]).(vars /. sol) + Sum[x_{α,α}[k] (ξ_{α,α}[i] + ξ_{α,α}[j]), {α, n}]]
          ]
         ];
```

```
In[•]:= utm_n[i_, j_ → k_] := Module[{A, α, β, lhs, rhs, eqns, vars, sol, ξ},
          𝔼_{i,j→{k}}[
           A = IdentityMatrix[n];
           Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β}, {n, n}]], {β, 1, n}, {α, 1, β}];
           (* This specifies the PBW ordering *)
           lhs = (A /. ξ_{αβ__} :↦ ξ_{αβ}[i]).(A /. ξ_{αβ__} :↦ ξ_{αβ}[j]);
           rhs = A /. ξ_{αβ__} :↦ ξ_{αβ} /. ξ_{α_,α_} :↦ ξ_{α,α}[i] + ξ_{α,α}[j];
           eqns = And @@ Thread[Flatten /@ CF /@ (lhs == rhs)];
           vars = Union@Cases[rhs, ξ__, ∞]; {sol} = Solve[eqns, vars];
           CF[(vars /. ξ_{αβ__} → x_{αβ}[k]).(vars /. sol) + Sum[x_{α,α}[k] (ξ_{α,α}[i] + ξ_{α,α}[j]), {α, n}]]
          ]
         ];
```

```
In[•]:= utm3[1, 2 → 3]
```

```
» e^(ξ1,1[1]+ξ2,2[1]+ξ2,2[2]) ξ1,2[1] + e^(ξ1,1[1]+ξ1,1[2]+ξ2,2[2]) ξ1,2[2] == e^(ξ1,1[1]+ξ1,1[2]+ξ2,2[1]+ξ2,2[2]) ζ$15529_1,2 &&
  e^(ξ1,1[1]+ξ3,3[1]+ξ3,3[2]) ξ1,3[1] + e^(ξ1,1[1]+ξ1,1[2]+ξ3,3[2]) ξ1,3[2] + e^(ξ1,1[1]+ξ2,2[1]+ξ3,3[1]+ξ3,3[2]) ξ1,2[1] ξ2,3[1] +
    e^(ξ1,1[1]+ξ2,2[1]+ξ2,2[2]+ξ3,3[2]) ξ1,2[1] ξ2,3[2] + e^(ξ1,1[1]+ξ1,1[2]+ξ2,2[2]+ξ3,3[2]) ξ1,2[2] ξ2,3[2] ==
  e^(ξ1,1[1]+ξ1,1[2]+ξ3,3[1]+ξ3,3[2]) (ζ$15529_1,3 + e^(ξ2,2[1]+ξ2,2[2]) ζ$15529_1,2 ζ$15529_2,3) &&
  e^(ξ2,2[1]+ξ3,3[1]+ξ3,3[2]) ξ2,3[1] + e^(ξ2,2[1]+ξ2,2[2]+ξ3,3[2]) ξ2,3[2] == e^(ξ2,2[1]+ξ2,2[2]+ξ3,3[1]+ξ3,3[2]) ζ$15529_2,3
```

```
Out[•]=
𝔼{1,2}→{3} [x1,1[3] ξ1,1[1] + x1,1[3] ξ1,1[2] +
  e^(-ξ1,1[2]) x1,2[3] ξ1,2[1] + e^(-ξ2,2[1]) x1,2[3] ξ1,2[2] + e^(-ξ1,1[2]) x1,3[3] ξ1,3[1] +
  e^(-ξ3,3[1]) x1,3[3] ξ1,3[2] + x2,2[3] ξ2,2[1] + x2,2[3] ξ2,2[2] + e^(-ξ2,2[2]) x2,3[3] ξ2,3[1] -
  x1,3[3] ξ1,2[2] ξ2,3[1] + e^(-ξ3,3[1]) x2,3[3] ξ2,3[2] + x3,3[3] ξ3,3[1] + x3,3[3] ξ3,3[2]]
```

```
In[•]:= Table[First@Timing[n → utmn[1, 2 → 3]], {n, 6}]
        Table[First@Timing[n → utmn[1, 2 → 3]], {n, 7}]
```

```
Out[•]=
{0., 0., 0.03125, 0.078125, 0.796875, 7.6875}
```

```
Out[•]=
{0., 0.015625, 0.03125, 0.109375, 0.859375, 6.92188, 95.4688}
```

```
In[•]:= Short[lhs = utm5[1, 2 → 2] // utm5[2, 3 → 1]]
        rhs = utm5[2, 3 → 2] // utm5[1, 2 → 1];
        lhs ≡ rhs
```

```
Out[•]//Short=
𝔼{1,2,3}→{1} [x1,1[1] ξ1,1[1] + <<108>> + x5,5[1] ξ5,5[3]]
```

```
Out[•]=
True
```

```
In[•]:= TriangularSolve[eqns_, vars_] := Module[{sol = {}},
          MapThread[
           {e, v} ↦ sol = sol ⋃ First@Solve[e /. sol, v],
           {eqns, vars}
          ];
          sol
         ]
```

In[•]:=
```
utm_n_[i_, j_ → k_] := Module[{A, α, β, lhs, rhs, eqns, vars, sol, ξ},
   𝔼_{i,j}→{k}[
    A = IdentityMatrix[n];
    Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β}, {n, n}]], {β, 1, n}, {α, 1, β}];
    (* This specifies the PBW ordering *)
    lhs = (A /. ξ_{αβ__} :> ξ_{αβ}[i]).(A /. ξ_{αβ__} :> ξ_{αβ}[j]);
    rhs = A /. ξ_{αβ__} :> ξ_{αβ} /. ξ_{α_,α_} :> ξ_{α,α}[i] + ξ_{α,α}[j];
    sol = TriangularSolve[
      Flatten@Table[lhs⟦α, α + δα⟧ == rhs⟦α, α + δα⟧, {δα, 1, n - 1}, {α, 1, n - δα}],
      vars = Flatten@Table[ξ_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}]
     ];
    CF[(vars /. ξ_{αβ__} → x_{αβ}[k]).(vars /. sol) + Sum[x_{α,α}[k] (ξ_{α,α}[i] + ξ_{α,α}[j]), {α, n}]]
   ]
  ];
```

In[•]:=
```
utm_3[1, 2 → 3]
```

Out[•]=

$$\mathbb{E}_{\{1,2\}\to\{3\}}\Big[x_{1,1}[3]\,\xi_{1,1}[1] + x_{1,1}[3]\,\xi_{1,1}[2] + e^{-\xi_{1,1}[2]}\,x_{1,2}[3]\,\xi_{1,2}[1] + e^{-\xi_{2,2}[1]}\,x_{1,2}[3]\,\xi_{1,2}[2] + e^{-\xi_{1,1}[2]}\,x_{1,3}[3]\,\xi_{1,3}[1] + e^{-\xi_{3,3}[1]}\,x_{1,3}[3]\,\xi_{1,3}[2] + x_{2,2}[3]\,\xi_{2,2}[1] + x_{2,2}[3]\,\xi_{2,2}[2] + e^{-\xi_{2,2}[2]}\,x_{2,3}[3]\,\xi_{2,3}[1] - x_{1,3}[3]\,\xi_{1,2}[2]\,\xi_{2,3}[1] + e^{-\xi_{3,3}[1]}\,x_{2,3}[3]\,\xi_{2,3}[2] + x_{3,3}[3]\,\xi_{3,3}[1] + x_{3,3}[3]\,\xi_{3,3}[2]\Big]$$

In[•]:=
```
Table[First@Timing[n → utm_n[1, 2 → 3]], {n, 9}]
```

Out[•]=
```
{0., 0., 0.015625, 0.015625, 0.0625, 0.0625, 0.34375, 1.625, 8.875}
```

In[•]:=
```
n = 5;
lhs = utm_n[1, 2 → 2] // utm_n[2, 3 → 1]
rhs = utm_n[2, 3 → 2] // utm_n[1, 2 → 1];
lhs ≡ rhs
```

Out[•]=

$$\mathbb{E}_{\{1,2,3\}\to\{1\}}\Big[x_{1,1}[1]\,\xi_{1,1}[1]+x_{1,1}[1]\,\xi_{1,1}[2]+x_{1,1}[1]\,\xi_{1,1}[3]+e^{-\xi_{1,1}[2]-\xi_{1,1}[3]}\,x_{1,2}[1]\,\xi_{1,2}[1]+e^{-\xi_{1,1}[3]-\xi_{2,2}[1]}\,x_{1,2}[1]\,\xi_{1,2}[2]+e^{-\xi_{2,2}[1]-\xi_{2,2}[2]}\,x_{1,2}[1]\,\xi_{1,2}[3]+e^{-\xi_{1,1}[2]-\xi_{1,1}[3]}\,x_{1,3}[1]\,\xi_{1,3}[1]+e^{-\xi_{1,1}[3]-\xi_{3,3}[1]}\,x_{1,3}[1]\,\xi_{1,3}[2]+e^{-\xi_{3,3}[1]-\xi_{3,3}[2]}\,x_{1,3}[1]\,\xi_{1,3}[3]+e^{-\xi_{1,1}[2]-\xi_{1,1}[3]}\,x_{1,4}[1]\,\xi_{1,4}[1]+e^{-\xi_{1,1}[3]-\xi_{4,4}[1]}\,x_{1,4}[1]\,\xi_{1,4}[2]+e^{-\xi_{4,4}[1]-\xi_{4,4}[2]}\,x_{1,4}[1]\,\xi_{1,4}[3]+e^{-\xi_{1,1}[2]-\xi_{1,1}[3]}\,x_{1,5}[1]\,\xi_{1,5}[1]+e^{-\xi_{1,1}[3]-\xi_{5,5}[1]}\,x_{1,5}[1]\,\xi_{1,5}[2]+e^{-\xi_{5,5}[1]-\xi_{5,5}[2]}\,x_{1,5}[1]\,\xi_{1,5}[3]+x_{2,2}[1]\,\xi_{2,2}[1]+x_{2,2}[1]\,\xi_{2,2}[2]+x_{2,2}[1]\,\xi_{2,2}[3]+e^{-\xi_{2,2}[2]-\xi_{2,2}[3]}\,x_{2,3}[1]\,\xi_{2,3}[1]-e^{-\xi_{1,1}[3]}\,x_{1,3}[1]\,\xi_{1,2}[2]\,\xi_{2,3}[1]-e^{-\xi_{2,2}[2]}\,x_{1,3}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[1]+e^{-\xi_{2,2}[3]-\xi_{3,3}[1]}\,x_{2,3}[1]\,\xi_{2,3}[2]-e^{-\xi_{3,3}[1]}\,x_{1,3}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[2]+e^{-\xi_{3,3}[1]-\xi_{3,3}[2]}\,x_{2,3}[1]\,\xi_{2,3}[3]+e^{-\xi_{2,2}[2]-\xi_{2,2}[3]}\,x_{2,4}[1]\,\xi_{2,4}[1]-e^{-\xi_{1,1}[3]}\,x_{1,4}[1]\,\xi_{1,2}[2]\,\xi_{2,4}[1]-e^{-\xi_{2,2}[2]}\,x_{1,4}[1]\,\xi_{1,2}[3]\,\xi_{2,4}[1]+e^{-\xi_{2,2}[3]-\xi_{4,4}[1]}\,x_{2,4}[1]\,\xi_{2,4}[2]-e^{-\xi_{4,4}[1]}\,x_{1,4}[1]\,\xi_{1,2}[3]\,\xi_{2,4}[2]+e^{-\xi_{4,4}[1]-\xi_{4,4}[2]}\,x_{2,4}[1]\,\xi_{2,4}[3]+e^{-\xi_{2,2}[2]-\xi_{2,2}[3]}\,x_{2,5}[1]\,\xi_{2,5}[1]-e^{-\xi_{1,1}[3]}\,x_{1,5}[1]\,\xi_{1,2}[2]\,\xi_{2,5}[1]-e^{-\xi_{2,2}[2]}\,x_{1,5}[1]\,\xi_{1,2}[3]\,\xi_{2,5}[1]+e^{-\xi_{2,2}[3]-\xi_{5,5}[1]}\,x_{2,5}[1]\,\xi_{2,5}[2]-e^{-\xi_{5,5}[1]}\,x_{1,5}[1]\,\xi_{1,2}[3]\,\xi_{2,5}[2]+e^{-\xi_{5,5}[1]-\xi_{5,5}[2]}\,x_{2,5}[1]\,\xi_{2,5}[3]+x_{3,3}[1]\,\xi_{3,3}[1]+x_{3,3}[1]\,\xi_{3,3}[2]+x_{3,3}[1]\,\xi_{3,3}[3]+e^{-\xi_{3,3}[2]-\xi_{3,3}[3]}\,x_{3,4}[1]\,\xi_{3,4}[1]-e^{-\xi_{1,1}[3]}\,x_{1,4}[1]\,\xi_{1,3}[2]\,\xi_{3,4}[1]-e^{-\xi_{3,3}[2]}\,x_{1,4}[1]\,\xi_{1,3}[3]\,\xi_{3,4}[1]-e^{-\xi_{2,2}[3]}\,x_{2,4}[1]\,\xi_{2,3}[2]\,\xi_{3,4}[1]+x_{1,4}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[2]\,\xi_{3,4}[1]-e^{-\xi_{3,3}[2]}\,x_{2,4}[1]\,\xi_{2,3}[3]\,\xi_{3,4}[1]+e^{-\xi_{3,3}[3]-\xi_{4,4}[1]}\,x_{3,4}[1]\,\xi_{3,4}[2]-e^{-\xi_{4,4}[1]}\,x_{1,4}[1]\,\xi_{1,3}[3]\,\xi_{3,4}[2]-e^{-\xi_{4,4}[1]}\,x_{2,4}[1]\,\xi_{2,3}[3]\,\xi_{3,4}[2]+e^{-\xi_{4,4}[1]-\xi_{4,4}[2]}\,x_{3,4}[1]\,\xi_{3,4}[3]+e^{-\xi_{3,3}[2]-\xi_{3,3}[3]}\,x_{3,5}[1]\,\xi_{3,5}[1]-e^{-\xi_{1,1}[3]}\,x_{1,5}[1]\,\xi_{1,3}[2]\,\xi_{3,5}[1]-e^{-\xi_{3,3}[2]}\,x_{1,5}[1]\,\xi_{1,3}[3]\,\xi_{3,5}[1]-e^{-\xi_{2,2}[3]}\,x_{2,5}[1]\,\xi_{2,3}[2]\,\xi_{3,5}[1]+x_{1,5}[1]\,\xi_{1,2}[3]\,\xi_{2,3}[2]\,\xi_{3,5}[1]-e^{-\xi_{3,3}[2]}\,x_{2,5}[1]\,\xi_{2,3}[3]\,\xi_{3,5}[1]+e^{-\xi_{3,3}[3]-\xi_{5,5}[1]}\,x_{3,5}[1]\,\xi_{3,5}[2]-e^{-\xi_{5,5}[1]}\,x_{1,5}[1]\,\xi_{1,3}[3]\,\xi_{3,5}[2]-e^{-\xi_{5,5}[1]}\,x_{2,5}[1]\,\xi_{2,3}[3]\,\xi_{3,5}[2]+e^{-\xi_{5,5}[1]-\xi_{5,5}[2]}\,x_{3,5}[1]\,\xi_{3,5}[3]+x_{4,4}[1]\,\xi_{4,4}[1]+x_{4,4}[1]\,\xi_{4,4}[2]+x_{4,4}[1]\,\xi_{4,4}[3]+e^{-\xi_{4,4}[2]-\xi_{4,4}[3]}\,x_{4,5}[1]\,\xi_{4,5}[1]-e^{-\xi_{1,1}[3]}\,x_{1,5}[1]\,\xi_{1,4}[2]\,\xi_{4,5}[1]-e^{-\xi_{4,4}[2]}\,x_{1,5}[1]\,\xi_{1,4}[3]\,\xi_{4,5}[1]-e^{-\xi_{2,2}[3]}\,x_{2,5}[1]\,\xi_{2,4}[2]\,\xi_{4,5}[1]+x_{1,5}[1]\,\xi_{1,2}[3]\,\xi_{2,4}[2]\,\xi_{4,5}[1]-e^{-\xi_{4,4}[2]}\,x_{2,5}[1]\,\xi_{2,4}[3]\,\xi_{4,5}[1]-e^{-\xi_{3,3}[3]}\,x_{3,5}[1]\,\xi_{3,4}[2]\,\xi_{4,5}[1]+x_{1,5}[1]\,\xi_{1,3}[3]\,\xi_{3,4}[2]\,\xi_{4,5}[1]+x_{2,5}[1]\,\xi_{2,3}[3]\,\xi_{3,4}[2]\,\xi_{4,5}[1]-e^{-\xi_{4,4}[2]}\,x_{3,5}[1]\,\xi_{3,4}[3]\,\xi_{4,5}[1]+e^{-\xi_{4,4}[3]-\xi_{5,5}[1]}\,x_{4,5}[1]\,\xi_{4,5}[2]-e^{-\xi_{5,5}[1]}\,x_{1,5}[1]\,\xi_{1,4}[3]\,\xi_{4,5}[2]-e^{-\xi_{5,5}[1]}\,x_{2,5}[1]\,\xi_{2,4}[3]\,\xi_{4,5}[2]-e^{-\xi_{5,5}[1]}\,x_{3,5}[1]\,\xi_{3,4}[3]\,\xi_{4,5}[2]+e^{-\xi_{5,5}[1]-\xi_{5,5}[2]}\,x_{4,5}[1]\,\xi_{4,5}[3]+x_{5,5}[1]\,\xi_{5,5}[1]+x_{5,5}[1]\,\xi_{5,5}[2]+x_{5,5}[1]\,\xi_{5,5}[3]\Big]$$

Out[•]=
```
True
```