```
In[]:= n = 2
Out[]= 2

In[]:= ξ_{α_,α_} := Log[Ξ_α]

In[]:= A = B = IdentityMatrix[n]
Out[]= {{1, 0}, {0, 1}}

In[]:= Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β} /. ξ_{α_,α_} :→ Log[Ξ_α], {n, n}]],
         {β, 1, n}, {α, 1, β}]; (* This specifies the PBW ordering *)

In[]:= Do[B = MatrixExp[SparseArray[{β, α} → η_{α,β}, {n, n}]].B, {β, 2, n}, {α, 1, β - 1}];

In[]:= MatrixForm /@ {A, B}
```

$$\left\{\begin{pmatrix} \Xi_1 & \Xi_1 \Xi_2 \xi_{1,2} \\ 0 & \Xi_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \eta_{1,2} & 1 \end{pmatrix}\right\}$$

```
In[]:= M = (A /. {ξ_{αβ__} :→ ξ_{αβ}[i], Ξ_{α_} :→ Ξ_α[i]}).(B /. η_{αβ__} :→ η_{αβ}[j])
```

$$\{\{\Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i],\ \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i]\},\ \{\Xi_2[i]\,\eta_{1,2}[j],\ \Xi_2[i]\}\}$$

```
In[]:= LU = First@LUDecomposition[M, Pivoting → False]
```

$$\left\{\{\Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i],\ \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i]\},\ \left\{\frac{\Xi_2[i]\,\eta_{1,2}[j]}{\Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]},\ \Xi_2[i] - \frac{\Xi_1[i]\,\Xi_2[i]^2\,\eta_{1,2}[j]\,\xi_{1,2}[i]}{\Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]}\right\}\right\}$$

```
In[]:= MatrixForm[L = LowerTriangularize[LU, -1] + IdentityMatrix[n]]
```

$$\begin{pmatrix} 1 & 0 \\ \frac{\Xi_2[i]\,\eta_{1,2}[j]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]} & 1 \end{pmatrix}$$

```
In[]:= MatrixForm[U = UpperTriangularize[LU]]
```

$$\begin{pmatrix} \Xi_1[i] + \Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i] & \Xi_1[i]\,\Xi_2[i]\,\xi_{1,2}[i] \\ 0 & \Xi_2[i] - \frac{\Xi_1[i]\,\Xi_2[i]^2\,\eta_{1,2}[j]\,\xi_{1,2}[i]}{\Xi_1[i]+\Xi_1[i]\,\Xi_2[i]\,\eta_{1,2}[j]\,\xi_{1,2}[i]} \end{pmatrix}$$

```
In[]:= Simplify[L.U - M]
Out[]= {{0, 0}, {0, 0}}

In[]:= A /. {ξ → ξ1, Ξ → Ξ1} // MatrixForm
```

$$\begin{pmatrix} \Xi1_1 & \Xi1_1\,\Xi1_2\,\xi1_{1,2} & \Xi1_3\,(\Xi1_1\,\xi1_{1,3} + \Xi1_1\,\Xi1_2\,\xi1_{1,2}\,\xi1_{2,3}) \\ 0 & \Xi1_2 & \Xi1_2\,\Xi1_3\,\xi1_{2,3} \\ 0 & 0 & \Xi1_3 \end{pmatrix}$$

In[∘]:= **B /. η → η1 // MatrixForm**

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ \eta 1_{1,2} & 1 & 0 \\ \eta 1_{1,3} + \eta 1_{1,2}\, \eta 1_{2,3} & \eta 1_{2,3} & 1 \end{pmatrix}$$

In[∘]:= **eqnsU = Table[U⟦α, α + δα⟧ == (A⟦α, α + δα⟧ /. {ξ → ξ1, Ξ → Ξ1}), {δα, 0, n − 1}, {α, 1, n − δα}]**

Out[∘]=

$$\left\{\left\{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i] == \Xi 1_1,\ \Xi_2[i] - \frac{\Xi_1[i]\, \Xi_2[i]^2\, \eta_{1,2}[j]\, \xi_{1,2}[i]}{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]} == \Xi 1_2\right\},\ \{\Xi_1[i]\, \Xi_2[i]\, \xi_{1,2}[i] == \Xi 1_1\, \Xi 1_2\, \xi 1_{1,2}\}\right\}$$

In[∘]:= **varsU = Flatten@{Table[Ξ1$_{α}$, {α, 1, n}], Table[ξ1$_{α,α+δα}$, {δα, 1, n − 1}, {α, 1, n − δα}]}**

Out[∘]=

$$\{\Xi 1_1,\ \Xi 1_2,\ \xi 1_{1,2}\}$$

In[∘]:= **solU = TriangularSolve[eqnsU, varsU]**

Out[∘]=

$$\left\{\Xi 1_1 \to \Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i],\ \Xi 1_2 \to \frac{\Xi_2[i]}{1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]},\ \xi 1_{1,2} \to \xi_{1,2}[i]\right\}$$

In[∘]:= **eqnsL = Table[(L⟦α + δα, α⟧ /. solU) == (B⟦α + δα, α⟧ /. {η → η1}), {δα, 1, n − 1}, {α, 1, n − δα}]**

Out[∘]=

$$\left\{\left\{\frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]} == \eta 1_{1,2}\right\}\right\}$$

In[∘]:= **varsL = Flatten@Table[η1$_{α,α+δα}$, {δα, 1, n − 1}, {α, 1, n − δα}]**

Out[∘]=

$$\{\eta 1_{1,2}\}$$

In[∘]:= **solL = TriangularSolve[eqnsL, varsL]**

Out[∘]=

$$\left\{\eta 1_{1,2} \to \frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i]\, (1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i])}\right\}$$

In[∘]:= **vars = varsU ⋃ varsL /. Ξ1$_{α_}$:→ ξ1$_{α,α}$**

Out[∘]=

$$\{\xi 1_{1,1},\ \xi 1_{2,2},\ \eta 1_{1,2},\ \xi 1_{1,2}\}$$

In[∘]:= **sol = solU ⋃ solL /. (Ξ1$_{α_}$ → ℰ_) :→ (ξ1$_{α,α}$ → Log[ℰ])**

Out[∘]=

$$\left\{\xi 1_{1,1} \to \mathrm{Log}[\Xi_1[i] + \Xi_1[i]\, \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]],\ \xi 1_{2,2} \to \mathrm{Log}\left[\frac{\Xi_2[i]}{1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i]}\right],\ \eta 1_{1,2} \to \frac{\Xi_2[i]\, \eta_{1,2}[j]}{\Xi_1[i]\, (1 + \Xi_2[i]\, \eta_{1,2}[j]\, \xi_{1,2}[i])},\ \xi 1_{1,2} \to \xi_{1,2}[i]\right\}$$

In[∘]:= CF[(vars /. {η1$_{αβ__}$ → $y_{αβ}$[i], ξ1$_{αβ__}$ → $x_{αβ}$[j]}).(vars /. sol)]

Out[∘]=

Log[$Ξ_1$[i] + $Ξ_1$[i] $Ξ_2$[i] $η_{1,2}$[j] $ξ_{1,2}$[i]] $x_{1,1}$[j] + Log[$\frac{Ξ_2[i]}{1 + Ξ_2[i]\, η_{1,2}[j]\, ξ_{1,2}[i]}$] $x_{2,2}$[j] + $x_{1,2}$[j] $ξ_{1,2}$[i] + $\frac{Ξ_2[i]\, y_{1,2}[i]\, η_{1,2}[j]}{Ξ_1[i]\, (1 + Ξ_2[i]\, η_{1,2}[j]\, ξ_{1,2}[i])}$

In[∘]:= {$η_{1,2}$[4]}*

Out[∘]=

List*[$η_{1,2}$[4]]

In[∘]:= With[{n = 2, $k = 0},
 $sw_{ut,ln,n,\$k}$[2, 4] // Echo // $sw_{ut,ln,n,\$k}$[1, 4] // Echo // utm_n[1, 2 → 3]
]

» $\mathbb{E}_{\{2,4\}\to\{2,4\}}$[$e^{-ξ_{1,1}[2]+ξ_{2,2}[2]}$ $y_{1,2}$[4] $η_{1,2}$[4] + $x_{1,1}$[2] $ξ_{1,1}$[2] + $x_{1,2}$[2] $ξ_{1,2}$[2] + $x_{2,2}$[2] $ξ_{2,2}$[2]]

» {4}

» {$η_{1,2}$[4]}

Intersection: Heads List* and List at positions 2 and 1 are expected to be the same.

» {$y_{1,2}$[4]} ⋂ List*[$η_{1,2}$[4]]

Intersection: Heads List* and List at positions 2 and 1 are expected to be the same.

Union: Heads SuperStar and Intersection at positions 2 and 1 are expected to be the same.

Complement: Heads List and Union at positions 2 and 1 are expected to be the same.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]])*} does not have appropriate bounds.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]])*} does not have appropriate bounds.

Table: Iterator {u$181277, {$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]] ∪ ({$[$y_{1,2}$[4]]} ∩ List*[$[$η_{1,2}$[4]]])*} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

Out[∘]=

$Aborted