Pensieve header: A full implementation of gl_n^ϵ.

In[∘]:=
```
SetDirectory["C:/drorbn/AcademicPensieve/Projects/glneps"];
Once[
 << "KnotTheory`";
 << "Rot.m";
 << "PolyPlot.m";
]
```

In[∘]:=
```
Format[ξα_,β_] := ξαβ;
ξ1,2
```

$RecursionLimit: Recursion depth of 4096 exceeded.

Out[∘]=

utm for “upper triangular multiply”
lcm for “last column multiply”.

In[∘]:=
```
utm1[i_, j_, , k_] := 𝔼[i, j, , k][(ξ1,1[i] + ξ1,1[j]) x1,1[k]];
lcmn_[i_, j_, , k_] :=
```
$$\mathbb{E}[i, j, , k]\left[(\xi_{n,n}[i] + \xi_{n,n}[j])\, x_{n,n}[k] + \sum_{\nu=1}^{n-1} \left(\Xi_{n,n}[j]^{-1}\, \xi_{\nu,n}[i] + \xi_{\nu,n}[j]\right) x_{\nu,n}[k]\right];$$

In[∘]:=
```
lcm3[i, j, , k]
```

Out[∘]=
$$\mathbb{E}[i, j, \text{Null}, k]\left[x_{3,3}[k]\,(\xi_{3,3}[i] + \xi_{3,3}[j]) + x_{1,3}[k]\left(\xi_{1,3}[j] + \frac{\xi_{1,3}[i]}{\Xi_{3,3}[j]}\right) + x_{2,3}[k]\left(\xi_{2,3}[j] + \frac{\xi_{2,3}[i]}{\Xi_{3,3}[j]}\right)\right]$$

In[∘]:=
```
n = 2;
ZM = Table[0, n, n];
Eα_,β_ := ReplacePart[ZM, {α, β} → 1];
B = MatrixExp[Sum[ξα,β Eα,β, {α, 1, n}, {β, α, n}]];
MatrixForm[B]
Bi = B /. ξαβ__ ⧴ ξαβ[i];
Bj = B /. ξαβ__ ⧴ ξαβ[j];
Bk = B /. ξαβ__ ⧴ ξαβ[k];
MatrixForm[Simplify[Bi.Bj]]
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} e^{\xi_{1,1}} & \frac{\left(e^{\xi_{1,1}} - e^{\xi_{2,2}}\right)\xi_{1,2}}{\xi_{1,1} - \xi_{2,2}} \\ 0 & e^{\xi_{2,2}} \end{pmatrix}$$

Out[∘]//MatrixForm=
$$\begin{pmatrix} e^{\xi_{1,1}[i] + \xi_{1,1}[j]} & \frac{e^{\xi_{2,2}[j]}\left(e^{\xi_{1,1}[i]} - e^{\xi_{2,2}[i]}\right)\xi_{1,2}[i]}{\xi_{1,1}[i] - \xi_{2,2}[i]} + \frac{e^{\xi_{1,1}[i]}\left(e^{\xi_{1,1}[j]} - e^{\xi_{2,2}[j]}\right)\xi_{1,2}[j]}{\xi_{1,1}[j] - \xi_{2,2}[j]} \\ 0 & e^{\xi_{2,2}[i] + \xi_{2,2}[j]} \end{pmatrix}$$

In[∘]:=
```
eqns = And @@ Thread[Flatten[Simplify[(Inverse[Bk] /. ξ_{α_,α_}[k] ⧴ ξ_{α,α}[i] + ξ_{α,α}[j]).Bi.Bj]] ==
   Flatten[IdentityMatrix[n]]]
```

Out[∘]=

$$e^{-\xi_{1,1}[j]}\left(\frac{e^{-\xi_{1,1}[i]+\xi_{2,2}[j]}\left(e^{\xi_{1,1}[i]}-e^{\xi_{2,2}[i]}\right)\xi_{1,2}[i]}{\xi_{1,1}[i]-\xi_{2,2}[i]}+\frac{\left(e^{\xi_{1,1}[j]}-e^{\xi_{2,2}[j]}\right)\xi_{1,2}[j]}{\xi_{1,1}[j]-\xi_{2,2}[j]}-\frac{\left(e^{\xi_{1,1}[j]}-e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{2,2}[j]}\right)\xi_{1,2}[k]}{\xi_{1,1}[i]+\xi_{1,1}[j]-\xi_{2,2}[i]-\xi_{2,2}[j]}\right)=0$$

In[∘]:=
```
vars = Union@Cases[eqns, ξ__[k], ∞]
```

Out[∘]=

$$\{\xi_{1,2}[k]\}$$

In[∘]:=
```
{sol} = Solve[eqns, vars]
```

Out[∘]=

$$\Big\{\Big\{\xi_{1,2}[k]\to -\Big(\Big((\xi_{1,1}[i]+\xi_{1,1}[j]-\xi_{2,2}[i]-\xi_{2,2}[j])\Big(-e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,1}[j]\,\xi_{1,2}[i]+e^{\xi_{2,2}[i]+\xi_{2,2}[j]}\,\xi_{1,1}[j]\,\xi_{1,2}[i]-e^{\xi_{1,1}[i]+\xi_{1,1}[j]}\,\xi_{1,1}[i]\,\xi_{1,2}[j]+e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,1}[i]\,\xi_{1,2}[j]+e^{\xi_{1,1}[i]+\xi_{1,1}[j]}\,\xi_{1,2}[j]\,\xi_{2,2}[i]-e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[j]\,\xi_{2,2}[i]+e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[i]\,\xi_{2,2}[j]-e^{\xi_{2,2}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[i]\,\xi_{2,2}[j]\Big)\Big)\Big/\Big(\Big(e^{\xi_{1,1}[i]+\xi_{1,1}[j]}-e^{\xi_{2,2}[i]+\xi_{2,2}[j]}\Big)(\xi_{1,1}[i]-\xi_{2,2}[i])(\xi_{1,1}[j]-\xi_{2,2}[j])\Big)\Big)\Big\}\Big\}$$

In[∘]:=
```
Simplify[vars /. sol]
```

Out[∘]=

$$\Big\{\Big((\xi_{1,1}[i]+\xi_{1,1}[j]-\xi_{2,2}[i]-\xi_{2,2}[j])\Big(e^{\xi_{2,2}[j]}\left(e^{\xi_{1,1}[i]}-e^{\xi_{2,2}[i]}\right)\xi_{1,1}[j]\,\xi_{1,2}[i]+e^{\xi_{1,1}[i]}\left(e^{\xi_{1,1}[j]}-e^{\xi_{2,2}[j]}\right)\xi_{1,1}[i]\,\xi_{1,2}[j]-e^{\xi_{1,1}[i]+\xi_{1,1}[j]}\,\xi_{1,2}[j]\,\xi_{2,2}[i]+e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[j]\,\xi_{2,2}[i]-e^{\xi_{1,1}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[i]\,\xi_{2,2}[j]+e^{\xi_{2,2}[i]+\xi_{2,2}[j]}\,\xi_{1,2}[i]\,\xi_{2,2}[j]\Big)\Big)\Big/\Big(\Big(e^{\xi_{1,1}[i]+\xi_{1,1}[j]}-e^{\xi_{2,2}[i]+\xi_{2,2}[j]}\Big)(\xi_{1,1}[i]-\xi_{2,2}[i])(\xi_{1,1}[j]-\xi_{2,2}[j])\Big)\Big\}$$