

Pensieve header: A full implementation of gl_n^ϵ .

```
In[*]:= SetDirectory["C:/drorbn/AcademicPensieve/Projects/glneps"];
Once[
  << "KnotTheory`";
  << "Rot.m";
  << "PolyPlot.m";
]
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m to compute rotation numbers.

Loading PolyPlot.m to plot 2-variable polynomials.

```
In[*]:= (*Format[ $\xi_{\alpha, \beta}$ ] :=  $\xi_{\alpha\beta}$  ; *)
```

$\xi_{1,2}$

Out[*]=

$\xi_{1,2}$

utm for “upper triangular multiply”

lcm for “last column multiply”.

```
In[*]:= utm1[i_, j_, , k_] :=  $\mathbb{E}[i, j, , k] [(\xi_{1,1}[i] + \xi_{1,1}[j]) x_{1,1}[k]]$ ;
```

```
lcmn[i_, j_, , k_] :=
```

$$\mathbb{E}[i, j, , k] \left[(\xi_{n,n}[i] + \xi_{n,n}[j]) x_{n,n}[k] + \sum_{v=1}^{n-1} (\xi_{n,n}[j]^{-1} \xi_{v,n}[i] + \xi_{v,n}[j]) x_{v,n}[k] \right];$$

```
In[*]:= lcm3[i, j, , k]
```

Out[*]=

$$\mathbb{E}[i, j, \text{Null}, k] \left[x_{3,3}[k] (\xi_{3,3}[i] + \xi_{3,3}[j]) + x_{1,3}[k] \left(\xi_{1,3}[j] + \frac{\xi_{1,3}[i]}{\xi_{3,3}[j]} \right) + x_{2,3}[k] \left(\xi_{2,3}[j] + \frac{\xi_{2,3}[i]}{\xi_{3,3}[j]} \right) \right]$$

```

In[ ]:= n = 5;
ZM = Table[0, n, n];
Eα,β := ReplacePart[ZM, {α, β} → 1];
B = IdentityMatrix[n];
Do[B = B.MatrixExp[ξα,α+δα Eα,α+δα], {δα, 0, n - 1}, {α, 1, n - δα}];
Bi = B /. ξαβ → ξαβ[i];
Bj = B /. ξαβ → ξαβ[j];
Bk = B /. ξαβ → ξαβ[k];
MatrixForm[Bk]
MatrixForm[Simplify[Bi.Bj]]

```

Out[]//MatrixForm=

$$\begin{pmatrix}
 e^{\xi_{1,1}[k]} & e^{\xi_{1,1}[k]} \xi_{1,2}[k] & e^{\xi_{1,1}[k]} \xi_{1,3}[k] + e^{\xi_{1,1}[k]} \xi_{1,2}[k] \xi_{2,3}[k] & e^{\xi_{1,1}[k]} \xi_{1,4}[k] + e^{\xi_{1,1}[k]} \xi_{1,2}[k] \xi_{2,4}[k] & 0 \\
 0 & e^{\xi_{2,2}[k]} & e^{\xi_{2,2}[k]} \xi_{2,3}[k] & e^{\xi_{2,2}[k]} \xi_{2,4}[k] + e^{\xi_{2,2}[k]} \xi_{3,3}[k] \xi_3 & 0 \\
 0 & 0 & e^{\xi_{3,3}[k]} & e^{\xi_{3,3}[k]} \xi_3 & 0 \\
 0 & 0 & 0 & e^{\xi_{4,4}[k]} & 0 \\
 0 & 0 & 0 & 0 & 0
 \end{pmatrix}$$

Out[]//MatrixForm=

$$\begin{pmatrix}
 e^{\xi_{1,1}[i] + \xi_{1,1}[j]} & e^{\xi_{1,1}[i]} (e^{\xi_{2,2}[j]} \xi_{1,2}[i] + e^{\xi_{1,1}[j]} \xi_{1,2}[j]) & e^{\xi_{1,1}[i]} (e^{\xi_{3,3}[j]} \xi_{1,3}[i] + e^{\xi_{1,1}[j]} \xi_{1,3}[j]) + e^{\xi_{3,3}[i]} e^{\xi_{2,2}[i]} & 0 \\
 0 & e^{\xi_{2,2}[i] + \xi_{2,2}[j]} & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{pmatrix}$$

In[*]:= eqns = And @@ Thread[Flatten[Simplify[(Inverse[Bk] /. $\xi_{\alpha, \alpha}[k] \Rightarrow \xi_{\alpha, \alpha}[i] + \xi_{\alpha, \alpha}[j]$].Bi.Bj]] == Flatten[IdentityMatrix[n]]]

Out[*]=

$$\begin{aligned}
 & e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] + \xi_{1,2}[j] - \xi_{1,2}[k] = 0 \&\& \\
 & e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] + \xi_{1,3}[j] - \xi_{1,3}[k] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,2}[i] \xi_{2,3}[i] - \\
 & e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,3}[j] + \xi_{1,2}[j] \xi_{2,3}[j] - \xi_{1,2}[k] \xi_{2,3}[j] = \\
 & 0 \&\& e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,4}[i] + \xi_{1,4}[j] - \xi_{1,4}[k] + e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,2}[i] \xi_{2,4}[i] - \\
 & e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[k] \xi_{2,4}[i] + e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,4}[j] + \\
 & \xi_{1,2}[j] \xi_{2,4}[j] - \xi_{1,2}[k] \xi_{2,4}[j] - e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{1,3}[k] \xi_{3,4}[i] + \\
 & e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[i] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[i] + \\
 & e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] - \xi_{1,3}[k] \xi_{3,4}[j] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[j] - \\
 & e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[j] + e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,3}[j] \xi_{3,4}[j] + \\
 & \xi_{1,2}[j] \xi_{2,3}[j] \xi_{3,4}[j] - \xi_{1,2}[k] \xi_{2,3}[j] \xi_{3,4}[j] + \xi_{1,3}[k] \xi_{3,4}[k] = 0 \&\& \\
 & e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,5}[i] + \xi_{1,5}[j] - \xi_{1,5}[k] + e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,2}[i] \xi_{2,5}[i] - \\
 & e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[k] \xi_{2,5}[i] + e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,5}[j] + \\
 & \xi_{1,2}[j] \xi_{2,5}[j] - \xi_{1,2}[k] \xi_{2,5}[j] + e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,3}[i] \xi_{3,5}[i] - \\
 & e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{1,3}[k] \xi_{3,5}[i] + e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,5}[i] - \\
 & e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,5}[i] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,5}[j] + \\
 & \xi_{1,3}[j] \xi_{3,5}[j] - \xi_{1,3}[k] \xi_{3,5}[j] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,5}[j] - \\
 & e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,5}[j] + e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,3}[j] \xi_{3,5}[j] + \\
 & \xi_{1,2}[j] \xi_{2,3}[j] \xi_{3,5}[j] - \xi_{1,2}[k] \xi_{2,3}[j] \xi_{3,5}[j] - e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,4}[k] \xi_{4,5}[i] - \\
 & e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{1,3}[k] \xi_{3,4}[i] \xi_{4,5}[i] + e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] - \\
 & e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] + e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,3}[k] \xi_{3,4}[k] \xi_{4,5}[i] + \\
 & e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,4}[i] \xi_{4,5}[j] - \xi_{1,4}[k] \xi_{4,5}[j] + e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,2}[i] \xi_{2,4}[i] \xi_{4,5}[j] - \\
 & e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[k] \xi_{2,4}[i] \xi_{4,5}[j] - e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{1,3}[k] \xi_{3,4}[i] \xi_{4,5}[j] + \\
 & e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[j] + \\
 & e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] - \xi_{1,3}[k] \xi_{3,4}[j] \xi_{4,5}[j] + \\
 & e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] + \\
 & e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,3}[j] \xi_{3,4}[j] \xi_{4,5}[j] + \xi_{1,2}[j] \xi_{2,3}[j] \xi_{3,4}[j] \xi_{4,5}[j] - \\
 & \xi_{1,2}[k] \xi_{2,3}[j] \xi_{3,4}[j] \xi_{4,5}[j] + \xi_{1,3}[k] \xi_{3,4}[k] \xi_{4,5}[j] + \xi_{1,4}[k] \xi_{4,5}[k] = 0 \&\& \\
 & e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] + \xi_{2,3}[j] - \xi_{2,3}[k] = 0 \&\& \\
 & e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] + \xi_{2,4}[j] - \xi_{2,4}[k] + e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,3}[i] \xi_{3,4}[i] - \\
 & e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[k] \xi_{3,4}[i] + e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] \xi_{3,4}[j] + \xi_{2,3}[j] \xi_{3,4}[j] - \xi_{2,3}[k] \xi_{3,4}[j] = \\
 & 0 \&\& e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{2,5}[i] + \xi_{2,5}[j] - \xi_{2,5}[k] + e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{2,3}[i] \xi_{3,5}[i] - \\
 & e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[k] \xi_{3,5}[i] + e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] \xi_{3,5}[j] + \\
 & \xi_{2,3}[j] \xi_{3,5}[j] - \xi_{2,3}[k] \xi_{3,5}[j] - e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{2,4}[k] \xi_{4,5}[i] + \\
 & e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] - e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[k] \xi_{3,4}[i] \xi_{4,5}[i] + \\
 & e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] \xi_{4,5}[j] - \xi_{2,4}[k] \xi_{4,5}[j] + e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[j] - \\
 & e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[k] \xi_{3,4}[i] \xi_{4,5}[j] + e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] + \\
 & \xi_{2,3}[j] \xi_{3,4}[j] \xi_{4,5}[j] - \xi_{2,3}[k] \xi_{3,4}[j] \xi_{4,5}[j] + \xi_{2,4}[k] \xi_{4,5}[k] = 0 \&\& \\
 & e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{3,4}[i] + \xi_{3,4}[j] - \xi_{3,4}[k] = 0 \&\& \\
 & e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{3,5}[i] + \xi_{3,5}[j] - \xi_{3,5}[k] + e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{3,4}[i] \xi_{4,5}[i] - \\
 & e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{3,4}[k] \xi_{4,5}[i] + e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{3,4}[i] \xi_{4,5}[j] + \xi_{3,4}[j] \xi_{4,5}[j] - \xi_{3,4}[k] \xi_{4,5}[j] = \\
 & 0 \&\& e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{4,5}[i] + \xi_{4,5}[j] - \xi_{4,5}[k] = 0
 \end{aligned}$$

```

In[*]:= vars = Union@Cases[eqns,  $\xi_{\_}[k]$ ,  $\infty$ ]
Out[*]=
{ $\xi_{1,2}[k]$ ,  $\xi_{1,3}[k]$ ,  $\xi_{1,4}[k]$ ,  $\xi_{1,5}[k]$ ,  $\xi_{2,3}[k]$ ,  $\xi_{2,4}[k]$ ,  $\xi_{2,5}[k]$ ,  $\xi_{3,4}[k]$ ,  $\xi_{3,5}[k]$ ,  $\xi_{4,5}[k]$ }

In[*]:= {sol} = Solve[eqns, vars]
Out[*]=
{ {  $\xi_{1,2}[k] \rightarrow e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] + \xi_{1,2}[j]$ ,
 $\xi_{1,3}[k] \rightarrow e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] + \xi_{1,3}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i]$ ,
 $\xi_{1,4}[k] \rightarrow e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,4}[i] + \xi_{1,4}[j] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,4}[i] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j]$ 
 $\xi_{2,3}[i] \xi_{3,4}[i] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j]$ ,
 $\xi_{1,5}[k] \rightarrow e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,5}[i] + \xi_{1,5}[j] - e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,5}[i] -$ 
 $e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{1,3}[j] \xi_{3,5}[i] - e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] +$ 
 $e^{-\xi_{1,1}[j] + \xi_{3,3}[j] - \xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,3}[i] \xi_{3,4}[j] \xi_{4,5}[i] + e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,3}[j] \xi_{3,4}[j] \xi_{4,5}[i] -$ 
 $e^{-\xi_{2,2}[j] + \xi_{3,3}[j] - \xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[i] + e^{-\xi_{1,1}[j] + \xi_{4,4}[j]}$ 
 $\xi_{1,4}[i] \xi_{4,5}[j] -$ 
 $e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,4}[i] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[j] +$ 
 $e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[j]$ ,
 $\xi_{2,3}[k] \rightarrow e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] + \xi_{2,3}[j]$ ,
 $\xi_{2,4}[k] \rightarrow e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] + \xi_{2,4}[j] - e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[j] \xi_{3,4}[i]$ ,
 $\xi_{2,5}[k] \rightarrow e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{2,5}[i] + \xi_{2,5}[j] - e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[j] \xi_{3,5}[i] -$ 
 $e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[j] \xi_{3,4}[i] \xi_{4,5}[i] + e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] \xi_{4,5}[j] -$ 
 $e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[j] \xi_{3,4}[i] \xi_{4,5}[j]$ ,  $\xi_{3,4}[k] \rightarrow e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{3,4}[i] + \xi_{3,4}[j]$ ,
 $\xi_{3,5}[k] \rightarrow e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{3,5}[i] + \xi_{3,5}[j] - e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{3,4}[j] \xi_{4,5}[i]$ ,
 $\xi_{4,5}[k] \rightarrow e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{4,5}[i] + \xi_{4,5}[j]$  } }

In[*]:= Simplify[(vars /.  $\xi \rightarrow x$ ). (vars /. sol)]
Out[*]=
 $x_{1,2}[k] (e^{-\xi_{1,1}[j] + \xi_{2,2}[j]} \xi_{1,2}[i] + \xi_{1,2}[j]) +$ 
 $x_{1,3}[k] (e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] + \xi_{1,3}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i]) +$ 
 $x_{2,3}[k] (e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[i] + \xi_{2,3}[j]) +$ 
 $x_{2,4}[k] (e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] + \xi_{2,4}[j] - e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[j] \xi_{3,4}[i]) +$ 
 $x_{3,4}[k] (e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{3,4}[i] + \xi_{3,4}[j]) +$ 
 $x_{1,4}[k] (e^{-\xi_{1,1}[j] + \xi_{4,4}[j]} \xi_{1,4}[i] + \xi_{1,4}[j] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,4}[i] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j]$ 
 $\xi_{2,3}[i] \xi_{3,4}[i] + e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j]) +$ 
 $x_{3,5}[k] (e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{3,5}[i] + \xi_{3,5}[j] - e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{3,4}[j] \xi_{4,5}[i]) +$ 
 $x_{4,5}[k] (e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{4,5}[i] + \xi_{4,5}[j]) +$ 
 $x_{2,5}[k] (e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{2,5}[i] + \xi_{2,5}[j] - e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[j] \xi_{3,5}[i] - e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{2,3}[j]$ 
 $\xi_{3,4}[i] \xi_{4,5}[i] + e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{2,4}[i] \xi_{4,5}[j] - e^{-\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{2,3}[j] \xi_{3,4}[i] \xi_{4,5}[j]) +$ 
 $x_{1,5}[k] (e^{-\xi_{1,1}[j] + \xi_{5,5}[j]} \xi_{1,5}[i] + \xi_{1,5}[j] - e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,5}[i] -$ 
 $e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} \xi_{1,3}[j] \xi_{3,5}[i] - e^{-\xi_{2,2}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] +$ 
 $e^{-\xi_{1,1}[j] + \xi_{3,3}[j] - \xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,3}[i] \xi_{3,4}[j] \xi_{4,5}[i] + e^{-\xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,3}[j] \xi_{3,4}[j] \xi_{4,5}[i] -$ 
 $e^{-\xi_{2,2}[j] + \xi_{3,3}[j] - \xi_{4,4}[j] + \xi_{5,5}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[i] + e^{-\xi_{1,1}[j] + \xi_{4,4}[j]}$ 
 $\xi_{1,4}[i] \xi_{4,5}[j] -$ 
 $e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,4}[i] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[i] \xi_{4,5}[j] +$ 
 $e^{-\xi_{1,1}[j] + \xi_{3,3}[j]} \xi_{1,3}[i] \xi_{3,4}[j] \xi_{4,5}[j] - e^{-\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{1,2}[j] \xi_{2,3}[i] \xi_{3,4}[j] \xi_{4,5}[j])$ 

```

```

In[*]:= n = 5;
ZM = Table[0, n, n];
E $\alpha, \beta$  := ReplacePart[ZM, { $\alpha, \beta$ }  $\rightarrow$  1];
B = IdentityMatrix[n];
Do[B = B.MatrixExp[ $\xi_{\alpha, \beta}$  E $\alpha, \beta$ ], { $\beta$ , 1, n}, { $\alpha$ , 1,  $\beta$ ]];
Bi = B /.  $\xi_{\alpha \beta}$   $\rightarrow$   $\xi_{\alpha \beta}$ [i];
Bj = B /.  $\xi_{\alpha \beta}$   $\rightarrow$   $\xi_{\alpha \beta}$ [j];
Bk = B /.  $\xi_{\alpha \beta}$   $\rightarrow$   $\xi_{\alpha \beta}$ [k];
MatrixForm[Bk]
MatrixForm[Simplify[Bi.Bj]]

```

```

Out[*]//MatrixForm=

$$\begin{pmatrix} e^{\xi_{1,1}[k]} & e^{\xi_{1,1}[k]+\xi_{2,2}[k]} \xi_{1,2}[k] & e^{\xi_{3,3}[k]} (e^{\xi_{1,1}[k]} \xi_{1,3}[k] + e^{\xi_{1,1}[k]+\xi_{2,2}[k]} \xi_{1,2}[k] \xi_{2,3}[k]) & e^{\xi_{4,4}[k]} (e^{\xi_{1,1}[k]} \xi_{1,4}[k] + e^{\xi_{1,1}[k]+\xi_{2,2}[k]} \xi_{1,2}[k] \xi_{2,4}[k] + e^{\xi_{1,1}[k]+\xi_{2,2}[k]+\xi_{3,3}[k]} \xi_{1,3}[k] \xi_{3,4}[k]) \\ 0 & e^{\xi_{2,2}[k]} & e^{\xi_{2,2}[k]+\xi_{3,3}[k]} \xi_{2,3}[k] & e^{\xi_{2,2}[k]+\xi_{3,3}[k]+\xi_{4,4}[k]} \xi_{2,4}[k] \\ 0 & 0 & e^{\xi_{3,3}[k]} & e^{\xi_{3,3}[k]+\xi_{4,4}[k]} \xi_{3,4}[k] \\ 0 & 0 & 0 & e^{\xi_{4,4}[k]} \end{pmatrix}$$


```

```

Out[*]//MatrixForm=

$$\begin{pmatrix} e^{\xi_{1,1}[i]+\xi_{1,1}[j]} & e^{\xi_{1,1}[i]+\xi_{2,2}[j]} (e^{\xi_{2,2}[i]} \xi_{1,2}[i] + e^{\xi_{1,1}[j]} \xi_{1,2}[j]) & e^{\xi_{1,1}[i]+\xi_{3,3}[j]} (e^{\xi_{3,3}[i]} \xi_{1,3}[i] + e^{\xi_{1,1}[j]} \xi_{1,3}[j]) & e^{\xi_{1,1}[i]+\xi_{4,4}[j]} (e^{\xi_{4,4}[i]} \xi_{1,4}[i] + e^{\xi_{1,1}[j]} \xi_{1,4}[j]) \\ 0 & e^{\xi_{2,2}[i]+\xi_{2,2}[j]} & e^{\xi_{2,2}[i]+\xi_{3,3}[j]} \xi_{2,3}[j] & e^{\xi_{2,2}[i]+\xi_{3,3}[j]+\xi_{4,4}[j]} \xi_{2,4}[j] \\ 0 & 0 & e^{\xi_{3,3}[i]} & e^{\xi_{3,3}[i]+\xi_{4,4}[j]} \xi_{3,4}[j] \\ 0 & 0 & 0 & e^{\xi_{4,4}[i]} \end{pmatrix}$$


```

```

In[*]:= eqns = And @@ Thread[Flatten[Simplify[(Inverse[Bk] /.  $\xi_{\alpha, \alpha}$ [k]  $\rightarrow$   $\xi_{\alpha, \alpha}$ [i] +  $\xi_{\alpha, \alpha}$ [j]).Bi.Bj]] == Flatten[IdentityMatrix[n]]]

```

```

Out[*]=

$$\begin{aligned} & e^{\xi_{2,2}[j]} (e^{-\xi_{1,1}[j]+\xi_{2,2}[i]} \xi_{1,2}[i] + \xi_{1,2}[j] - e^{\xi_{2,2}[i]} \xi_{1,2}[k]) = 0 \&\& \\ & e^{-\xi_{1,1}[j]+\xi_{3,3}[j]} (e^{\xi_{3,3}[i]} \xi_{1,3}[i] + e^{\xi_{1,1}[j]} \xi_{1,3}[j] - e^{\xi_{1,1}[j]+\xi_{3,3}[i]} \xi_{1,3}[k] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{\xi_{1,1}[j]+\xi_{2,2}[i]+\xi_{3,3}[i]} \xi_{1,2}[k] \xi_{2,3}[i] + e^{\xi_{2,2}[i]+\xi_{2,2}[j]} \xi_{1,2}[i] \xi_{2,3}[j] + \\ & e^{\xi_{1,1}[j]+\xi_{2,2}[j]} \xi_{1,2}[j] \xi_{2,3}[j] - e^{\xi_{1,1}[j]+\xi_{2,2}[i]+\xi_{2,2}[j]} \xi_{1,2}[k] \xi_{2,3}[j]) = 0 \&\& \\ & e^{\xi_{4,4}[j]} (\xi_{1,4}[j] - e^{\xi_{4,4}[i]} \xi_{1,4}[k] - e^{\xi_{3,3}[i]+\xi_{4,4}[i]} \xi_{1,3}[k] \xi_{3,4}[i] - \\ & e^{\xi_{2,2}[i]+\xi_{4,4}[i]} \xi_{1,2}[k] (\xi_{2,4}[i] + e^{\xi_{3,3}[i]} \xi_{2,3}[i] \xi_{3,4}[i]) + e^{-\xi_{1,1}[j]+\xi_{4,4}[i]} \\ & (\xi_{1,4}[i] + e^{\xi_{3,3}[i]} \xi_{1,3}[i] \xi_{3,4}[i] + e^{\xi_{2,2}[i]} \xi_{1,2}[i] (\xi_{2,4}[i] + e^{\xi_{3,3}[i]} \xi_{2,3}[i] \xi_{3,4}[i])) + \\ & e^{\xi_{3,3}[j]} \xi_{1,3}[j] \xi_{3,4}[j] + e^{-\xi_{1,1}[j]+\xi_{3,3}[i]+\xi_{3,3}[j]} \\ & (\xi_{1,3}[i] - e^{\xi_{1,1}[j]} \xi_{1,3}[k] + e^{\xi_{2,2}[i]} (\xi_{1,2}[i] - e^{\xi_{1,1}[j]} \xi_{1,2}[k]) \xi_{2,3}[i]) \xi_{3,4}[j] + \\ & e^{\xi_{2,2}[j]} \xi_{1,2}[j] (\xi_{2,4}[j] + e^{\xi_{3,3}[j]} \xi_{2,3}[j] \xi_{3,4}[j]) + \\ & e^{-\xi_{1,1}[j]+\xi_{2,2}[i]+\xi_{2,2}[j]} (\xi_{1,2}[i] - e^{\xi_{1,1}[j]} \xi_{1,2}[k]) (\xi_{2,4}[j] + e^{\xi_{3,3}[j]} \xi_{2,3}[j] \xi_{3,4}[j]) = 0 \&\& \\ & e^{\xi_{5,5}[j]} (\xi_{1,5}[j] - e^{\xi_{5,5}[i]} \xi_{1,5}[k] + e^{\xi_{3,3}[j]} \xi_{1,3}[j] \xi_{3,5}[j] - e^{\xi_{4,4}[i]+\xi_{5,5}[i]} \xi_{1,4}[k] \xi_{4,5}[i] - \\ & e^{\xi_{3,3}[i]+\xi_{5,5}[i]} \xi_{1,3}[k] (\xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{3,4}[i] \xi_{4,5}[i]) - e^{\xi_{2,2}[i]+\xi_{5,5}[i]} \xi_{1,2}[k] \\ & (\xi_{2,5}[i] + e^{\xi_{4,4}[i]} \xi_{2,4}[i] \xi_{4,5}[i] + e^{\xi_{3,3}[i]} \xi_{2,3}[i] (\xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{3,4}[i] \xi_{4,5}[i])) + \\ & e^{-\xi_{1,1}[j]+\xi_{5,5}[i]} (\xi_{1,5}[i] + e^{\xi_{3,3}[i]} \xi_{1,3}[i] \xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{1,4}[i] \xi_{4,5}[i] + \\ & e^{\xi_{3,3}[i]+\xi_{4,4}[i]} \xi_{1,3}[i] \xi_{3,4}[i] \xi_{4,5}[i] + e^{\xi_{2,2}[i]} \xi_{1,2}[i] \\ & (\xi_{2,5}[i] + e^{\xi_{4,4}[i]} \xi_{2,4}[i] \xi_{4,5}[i] + e^{\xi_{3,3}[i]} \xi_{2,3}[i] (\xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{3,4}[i] \xi_{4,5}[i]))) + \end{aligned}$$


```

$$\begin{aligned}
& e^{\xi_{4,4}[j]} \xi_{1,4}[j] \xi_{4,5}[j] + e^{-\xi_{1,1}[j] + \xi_{4,4}[i] + \xi_{4,4}[j]} (\xi_{1,4}[i] - e^{\xi_{1,1}[j]} \xi_{1,4}[k] + e^{\xi_{2,2}[i]} \xi_{1,2}[i] \xi_{2,4}[i] - \\
& e^{\xi_{1,1}[j] + \xi_{2,2}[i]} \xi_{1,2}[k] \xi_{2,4}[i] + e^{\xi_{3,3}[i]} \xi_{1,3}[i] \xi_{3,4}[i] - e^{\xi_{1,1}[j] + \xi_{3,3}[i]} \xi_{1,3}[k] \xi_{3,4}[i] + \\
& e^{\xi_{2,2}[i] + \xi_{3,3}[i]} \xi_{1,2}[i] \xi_{2,3}[i] \xi_{3,4}[i] - e^{\xi_{1,1}[j] + \xi_{2,2}[i] + \xi_{3,3}[i]} \xi_{1,2}[k] \xi_{2,3}[i] \xi_{3,4}[i]) \xi_{4,5}[j] + \\
& e^{\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{1,3}[j] \xi_{3,4}[j] \xi_{4,5}[j] + e^{-\xi_{1,1}[j] + \xi_{3,3}[i] + \xi_{3,3}[j]} \\
& (\xi_{1,3}[i] - e^{\xi_{1,1}[j]} \xi_{1,3}[k] + e^{\xi_{2,2}[i]} (\xi_{1,2}[i] - e^{\xi_{1,1}[j]} \xi_{1,2}[k]) \xi_{2,3}[i]) \\
& (\xi_{3,5}[j] + e^{\xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j]) + e^{\xi_{2,2}[j]} \xi_{1,2}[j] \\
& (\xi_{2,5}[j] + e^{\xi_{4,4}[j]} \xi_{2,4}[j] \xi_{4,5}[j] + e^{\xi_{3,3}[j]} \xi_{2,3}[j] (\xi_{3,5}[j] + e^{\xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j])) + \\
& e^{-\xi_{1,1}[j] + \xi_{2,2}[i] + \xi_{2,2}[j]} (\xi_{1,2}[i] - e^{\xi_{1,1}[j]} \xi_{1,2}[k]) \\
& (\xi_{2,5}[j] + e^{\xi_{4,4}[j]} \xi_{2,4}[j] \xi_{4,5}[j] + e^{\xi_{3,3}[j]} \xi_{2,3}[j] (\xi_{3,5}[j] + e^{\xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j])) = 0 \&\& \\
& e^{\xi_{3,3}[j]} (e^{-\xi_{2,2}[j] + \xi_{3,3}[i]} \xi_{2,3}[i] + \xi_{2,3}[j] - e^{\xi_{3,3}[i]} \xi_{2,3}[k]) = \\
& 0 \&\& \\
& e^{-\xi_{2,2}[j] + \xi_{4,4}[j]} (e^{\xi_{4,4}[i]} \xi_{2,4}[i] + e^{\xi_{2,2}[j]} \xi_{2,4}[j] - e^{\xi_{2,2}[j] + \xi_{4,4}[i]} \xi_{2,4}[k] + e^{\xi_{3,3}[i] + \xi_{4,4}[i]} \xi_{2,3}[i] \xi_{3,4}[i] - \\
& e^{\xi_{2,2}[j] + \xi_{3,3}[i] + \xi_{4,4}[i]} \xi_{2,3}[k] \xi_{3,4}[i] + e^{\xi_{3,3}[i] + \xi_{3,3}[j]} \xi_{2,3}[i] \xi_{3,4}[j] + \\
& e^{\xi_{2,2}[j] + \xi_{3,3}[j]} \xi_{2,3}[j] \xi_{3,4}[j] - e^{\xi_{2,2}[j] + \xi_{3,3}[i] + \xi_{3,3}[j]} \xi_{2,3}[k] \xi_{3,4}[j]) = 0 \&\& \\
& e^{\xi_{5,5}[i]} (\xi_{2,5}[j] - e^{\xi_{5,5}[i]} \xi_{2,5}[k] - e^{\xi_{4,4}[i] + \xi_{5,5}[i]} \xi_{2,4}[k] \xi_{4,5}[i] - \\
& e^{\xi_{3,3}[i] + \xi_{5,5}[i]} \xi_{2,3}[k] (\xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{3,4}[i] \xi_{4,5}[i]) + e^{-\xi_{2,2}[j] + \xi_{5,5}[i]} \\
& (\xi_{2,5}[i] + e^{\xi_{4,4}[i]} \xi_{2,4}[i] \xi_{4,5}[i] + e^{\xi_{3,3}[i]} \xi_{2,3}[i] (\xi_{3,5}[i] + e^{\xi_{4,4}[i]} \xi_{3,4}[i] \xi_{4,5}[i])) + \\
& e^{\xi_{4,4}[j]} \xi_{2,4}[j] \xi_{4,5}[j] + e^{-\xi_{2,2}[j] + \xi_{4,4}[i] + \xi_{4,4}[j]} \\
& (\xi_{2,4}[i] - e^{\xi_{2,2}[j]} \xi_{2,4}[k] + e^{\xi_{3,3}[i]} (\xi_{2,3}[i] - e^{\xi_{2,2}[j]} \xi_{2,3}[k]) \xi_{3,4}[i]) \xi_{4,5}[j] + \\
& e^{\xi_{3,3}[j]} \xi_{2,3}[j] (\xi_{3,5}[j] + e^{\xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j]) + \\
& e^{-\xi_{2,2}[j] + \xi_{3,3}[i] + \xi_{3,3}[j]} (\xi_{2,3}[i] - e^{\xi_{2,2}[j]} \xi_{2,3}[k]) (\xi_{3,5}[j] + e^{\xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j]) = 0 \&\& \\
& e^{\xi_{4,4}[j]} (e^{-\xi_{3,3}[j] + \xi_{4,4}[i]} \xi_{3,4}[i] + \xi_{3,4}[j] - e^{\xi_{4,4}[i]} \xi_{3,4}[k]) = \\
& 0 \&\& \\
& e^{-\xi_{3,3}[j] + \xi_{5,5}[j]} (e^{\xi_{5,5}[i]} \xi_{3,5}[i] + e^{\xi_{3,3}[j]} \xi_{3,5}[j] - e^{\xi_{3,3}[j] + \xi_{5,5}[i]} \xi_{3,5}[k] + e^{\xi_{4,4}[i] + \xi_{5,5}[i]} \xi_{3,4}[i] \xi_{4,5}[i] - \\
& e^{\xi_{3,3}[j] + \xi_{4,4}[i] + \xi_{5,5}[i]} \xi_{3,4}[k] \xi_{4,5}[i] + e^{\xi_{4,4}[i] + \xi_{4,4}[j]} \xi_{3,4}[i] \xi_{4,5}[j] + \\
& e^{\xi_{3,3}[j] + \xi_{4,4}[j]} \xi_{3,4}[j] \xi_{4,5}[j] - e^{\xi_{3,3}[j] + \xi_{4,4}[i] + \xi_{4,4}[j]} \xi_{3,4}[k] \xi_{4,5}[j]) = 0 \&\& \\
& e^{\xi_{5,5}[j]} (e^{-\xi_{4,4}[j] + \xi_{5,5}[i]} \xi_{4,5}[i] + \xi_{4,5}[j] - e^{\xi_{5,5}[i]} \xi_{4,5}[k]) = \\
& 0
\end{aligned}$$

In[*]:= vars = Union@Cases[eqns, $\xi_{__}[k]$, ∞]

Out[*]=

{ $\xi_{1,2}[k]$, $\xi_{1,3}[k]$, $\xi_{1,4}[k]$, $\xi_{1,5}[k]$, $\xi_{2,3}[k]$, $\xi_{2,4}[k]$, $\xi_{2,5}[k]$, $\xi_{3,4}[k]$, $\xi_{3,5}[k]$, $\xi_{4,5}[k]$ }

In[*]:= **{sol} = Solve[eqns, vars]**

Out[*]=

$$\left\{ \left\{ \begin{aligned} \xi_{1,2}[k] &\rightarrow e^{-\xi_{1,1}[j]-\xi_{2,2}[i]} \left(e^{\xi_{2,2}[i]} \xi_{1,2}[i] + e^{\xi_{1,1}[j]} \xi_{1,2}[j] \right), \\ \xi_{1,3}[k] &\rightarrow -e^{-\xi_{1,1}[j]-\xi_{3,3}[i]} \left(-e^{\xi_{3,3}[i]} \xi_{1,3}[i] - e^{\xi_{1,1}[j]} \xi_{1,3}[j] + e^{\xi_{1,1}[j]+\xi_{3,3}[i]} \xi_{1,2}[j] \xi_{2,3}[i] \right), \\ \xi_{1,4}[k] &\rightarrow -e^{-\xi_{1,1}[j]-\xi_{4,4}[i]} \\ &\quad \left(-e^{\xi_{4,4}[i]} \xi_{1,4}[i] - e^{\xi_{1,1}[j]} \xi_{1,4}[j] + e^{\xi_{1,1}[j]+\xi_{4,4}[i]} \xi_{1,2}[j] \xi_{2,4}[i] + e^{\xi_{1,1}[j]+\xi_{4,4}[i]} \xi_{1,3}[j] \xi_{3,4}[i] \right), \\ \xi_{1,5}[k] &\rightarrow -e^{-\xi_{1,1}[j]-\xi_{5,5}[i]} \left(-e^{\xi_{5,5}[i]} \xi_{1,5}[i] - e^{\xi_{1,1}[j]} \xi_{1,5}[j] + e^{\xi_{1,1}[j]+\xi_{5,5}[i]} \xi_{1,2}[j] \xi_{2,5}[i] + \right. \\ &\quad \left. e^{\xi_{1,1}[j]+\xi_{5,5}[i]} \xi_{1,3}[j] \xi_{3,5}[i] + e^{\xi_{1,1}[j]+\xi_{5,5}[i]} \xi_{1,4}[j] \xi_{4,5}[i] \right), \\ \xi_{2,3}[k] &\rightarrow e^{-\xi_{2,2}[j]-\xi_{3,3}[i]} \left(e^{\xi_{3,3}[i]} \xi_{2,3}[i] + e^{\xi_{2,2}[j]} \xi_{2,3}[j] \right), \\ \xi_{2,4}[k] &\rightarrow -e^{-\xi_{2,2}[j]-\xi_{4,4}[i]} \left(-e^{\xi_{4,4}[i]} \xi_{2,4}[i] - e^{\xi_{2,2}[j]} \xi_{2,4}[j] + e^{\xi_{2,2}[j]+\xi_{4,4}[i]} \xi_{2,3}[j] \xi_{3,4}[i] \right), \\ \xi_{2,5}[k] &\rightarrow -e^{-\xi_{2,2}[j]-\xi_{5,5}[i]} \\ &\quad \left(-e^{\xi_{5,5}[i]} \xi_{2,5}[i] - e^{\xi_{2,2}[j]} \xi_{2,5}[j] + e^{\xi_{2,2}[j]+\xi_{5,5}[i]} \xi_{2,3}[j] \xi_{3,5}[i] + e^{\xi_{2,2}[j]+\xi_{5,5}[i]} \xi_{2,4}[j] \xi_{4,5}[i] \right), \\ \xi_{3,4}[k] &\rightarrow e^{-\xi_{3,3}[j]-\xi_{4,4}[i]} \left(e^{\xi_{4,4}[i]} \xi_{3,4}[i] + e^{\xi_{3,3}[j]} \xi_{3,4}[j] \right), \\ \xi_{3,5}[k] &\rightarrow -e^{-\xi_{3,3}[j]-\xi_{5,5}[i]} \left(-e^{\xi_{5,5}[i]} \xi_{3,5}[i] - e^{\xi_{3,3}[j]} \xi_{3,5}[j] + e^{\xi_{3,3}[j]+\xi_{5,5}[i]} \xi_{3,4}[j] \xi_{4,5}[i] \right), \\ \xi_{4,5}[k] &\rightarrow e^{-\xi_{4,4}[j]-\xi_{5,5}[i]} \left(e^{\xi_{5,5}[i]} \xi_{4,5}[i] + e^{\xi_{4,4}[j]} \xi_{4,5}[j] \right) \end{aligned} \right\} \right\}$$

In[*]:= **Simplify[(vars /. $\xi \rightarrow x$). (vars /. sol)]**

Out[*]=

$$\begin{aligned} &x_{1,2}[k] \left(e^{-\xi_{1,1}[j]} \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} \xi_{1,2}[j] \right) + e^{-\xi_{1,1}[j]} x_{1,4}[k] \xi_{1,4}[i] + e^{-\xi_{4,4}[i]} x_{1,4}[k] \xi_{1,4}[j] + \\ &e^{-\xi_{1,1}[j]} x_{1,5}[k] \xi_{1,5}[i] + e^{-\xi_{5,5}[i]} x_{1,5}[k] \xi_{1,5}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + \\ &x_{1,3}[k] \left(e^{-\xi_{1,1}[j]} \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} \xi_{1,3}[j] - \xi_{1,2}[j] \xi_{2,3}[i] \right) + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + \\ &e^{-\xi_{2,2}[j]} x_{2,4}[k] \xi_{2,4}[i] - x_{1,4}[k] \xi_{1,2}[j] \xi_{2,4}[i] + e^{-\xi_{4,4}[i]} x_{2,4}[k] \xi_{2,4}[j] + e^{-\xi_{2,2}[j]} x_{2,5}[k] \xi_{2,5}[i] - \\ &x_{1,5}[k] \xi_{1,2}[j] \xi_{2,5}[i] + e^{-\xi_{5,5}[i]} x_{2,5}[k] \xi_{2,5}[j] + e^{-\xi_{3,3}[j]} x_{3,4}[k] \xi_{3,4}[i] - x_{1,4}[k] \xi_{1,3}[j] \xi_{3,4}[i] - \\ &x_{2,4}[k] \xi_{2,3}[j] \xi_{3,4}[i] + e^{-\xi_{4,4}[i]} x_{3,4}[k] \xi_{3,4}[j] + e^{-\xi_{3,3}[j]} x_{3,5}[k] \xi_{3,5}[i] - x_{1,5}[k] \xi_{1,3}[j] \xi_{3,5}[i] - \\ &x_{2,5}[k] \xi_{2,3}[j] \xi_{3,5}[i] + e^{-\xi_{5,5}[i]} x_{3,5}[k] \xi_{3,5}[j] + e^{-\xi_{4,4}[j]} x_{4,5}[k] \xi_{4,5}[i] - \\ &x_{1,5}[k] \xi_{1,4}[j] \xi_{4,5}[i] - x_{2,5}[k] \xi_{2,4}[j] \xi_{4,5}[i] - x_{3,5}[k] \xi_{3,4}[j] \xi_{4,5}[i] + e^{-\xi_{5,5}[i]} x_{4,5}[k] \xi_{4,5}[j] \end{aligned}$$