Pensieve header: A full implementation of gl_n^ϵ.

```
In[•]:= SetDirectory["C:/drorbn/AcademicPensieve/Projects/glneps"];
Once[
 << "KnotTheory`";
 << "Rot.m";
 << "PolyPlot.m";
]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m to compute rotation numbers.
Loading PolyPlot.m to plot 2-variable polynomials.
```

Conventions

$x_{\alpha,\beta}$ is always with $\alpha \le \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is always with $\beta \le \alpha$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\beta,\alpha}$.

Variable declarations:

```
In[•]:= VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)__)[_];
```

Greek - Latin duality:

```
In[•]:= {y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_αβ__)* := (u*)αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

```
In[•]:= {x1,2, y2,3, η1,2[7]}*
Out[•]= {ξ1,2, η2,3, y1,2[7]}
```

Weights:

```
In[•]:= Wt[xα_,β_] := {1, α - β}; Wt[yα_,β_] := {0, α - β};
        Wt[ξα_,β_] := {0, β - α}; Wt[ηα_,β_] := {1, β - α};
        Wt[u_[_]] := Wt[u];
```

```
In[•]:= $k = 2;
```

The Engine

Canonical Forms:

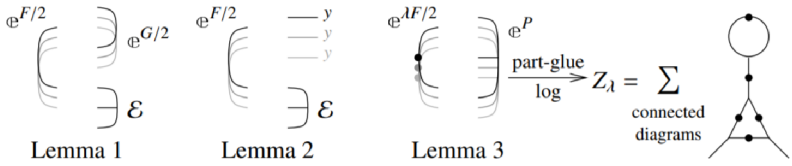
```
In[•]:= LogReduce[ℰ_] :=
          ℰ /. c_ * Log[a_] :↦ Log@Factor[a^c] //. Log[a_] + Log[b_] :↦ Log@Factor[a b];
        CF[ℰ_] := Expand@Collect[ℰ, Cases[ℰ, VarPattern, ∞], CCF] /. CCF → Factor;
        CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
        CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

𝔼 operations:

```
In[•]:= ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
        ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
        𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
        𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
        𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
        𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
```

```
In[•]:= 𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
          lvs = Cases[{ℰ1s}, (x | y)__[i_] /; MemberQ[is, i], ∞] ⋃
            Cases[{ℰ2s}, (ξ | η)__[i_] /; MemberQ[is, i], ∞]*;
          llvs = Map[$, lvs, {2}];
          𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{llvs⋃llvs*}[{(llvs)*.(llvs), Times[
                𝔼[ℰ1s] /. Thread[lvs → llvs],
                𝔼[ℰ2s] /. Thread[lvs* → llvs*]
              ]}])
        ]
```

Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight *n* variables in perturbations with the weight 0 variables in the coefficients of *Q*.

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii}’s and the y_{ii}’s. They have weights {1,0} and {0,0}.

```
In[•]:= Zip_{vs_}[{ℱ_, ℰ_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
          {ℱ, ℰ} // Zip1_{vs}(*//Zip2_{Complement[vs,dvs]}*) // Zip2_{dvs} //
            Zip3_{Complement[vs,dvs]} // Zip3_{dvs}
        ] // Last
```

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon \mathcal{E}\, \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1}\colon \mathcal{E} \right\rangle_B$$

In[•]:=
```
Zip1{} = Identity;
Zip1vs_@{ℱ_, 𝔼[Q_, P___]} := Module[{ℐ, F, G, u, v},
  ℐ = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂u,v ℱ, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂u,v Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[ℐ - G.F]).(vs) / 2],
   𝔼[CF[Q - PowerExpand@Log[Det[ℐ - G.F]] / 2 - vs.G.vs / 2], P]}
 ]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F\colon \mathcal{E}\, \mathbb{e}^{\sum_{i\in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j} \left\langle F\colon\ \mathcal{E}|_{z_B \to z_B + F y_B} \right\rangle_B$.

In[•]:=
```
Zip2{} = Identity;
Zip2vs_@{ℱ_, 𝔼[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂u,v ℱ], 0], {u, vs}, {v, vs}];
  Y = Table[∂v Q, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {ℱ, 𝔼[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
 ]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F\colon \mathbb{e}^P]_B$ satisfies
and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

```
In[•]:= Zip3vs_@{ℱ_, ℰ_, 𝔼} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = ℰ[[kk + 1]], {kk, 0, ℰ@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂u,vℱ, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂uZ[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂vZ[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ ℰ@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
       If[F[u, v] === 0, 0, F[u, v]/(2 (m + 1)) (Z[m, kk, u, v] +
           Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
       {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
      ℱ - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, ℰ@$}]
     } /. Table[v → 0, {v, vs}])
 ]
```

“Define” Code

```
In[•]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_[is__] = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op[nisp, $k_Integer], Block[{i, j, k}, op[isp, $k] = ℰ;
        op[nis, $k]]];
      SD[op[isp], op[{is}, $k]];
      SD[op[sis__], op[{sis}]];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

```
In[•]:= TriangularSolve[eqns_, vars_] := Module[{sol = {}, e, v},
          MapThread[
           {e, v} ↦ sol = sol ⋃ First@Solve[e /. sol, v],
           Flatten /@ {eqns, vars}
          ];
          sol
         ]
```

```
In[•]:= Λ2𝔼d_→r_[Λ_] := Λ2𝔼d→r[$k, Λ];
        Λ2𝔼d_→r_[$k_, Λ_] :=
          Module[{k}, 𝔼d→r @@ Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

The Objects

```
In[•]:= Table[xα,β, {β, 1, 5}, {α, 1, β}]
```

```
Out[•]= {{x1,1}, {x1,2, x2,2}, {x1,3, x2,3, x3,3}, {x1,4, x2,4, x3,4, x4,4}, {x1,5, x2,5, x3,5, x4,5, x5,5}}
```

```
In[•]:= utmn_[i_, j_ → k_] := utmn[i, j → k] = Module[{A, α, β, lhs, rhs, eqns, δα, vars, sol, ξ},
           A = IdentityMatrix[n];
           Do[A = A.MatrixExp[SparseArray[{α, β} → ξα,β, {n, n}]], {β, 1, n}, {α, 1, β}];
           (* This specifies the PBW ordering *)
           lhs = (A /. ξαβ__ :↦ ξαβ[i]).(A /. ξαβ__ :↦ ξαβ[j]);
           rhs = A /. ξαβ__ :↦ ξαβ /. ξα_,α_ :↦ ξα,α[i] + ξα,α[j];
           eqns = Table[lhs⟦α, α + δα⟧ == rhs⟦α, α + δα⟧, {δα, 1, n - 1}, {α, 1, n - δα}];
           vars = Flatten@Table[ξα,α+δα, {δα, 1, n - 1}, {α, 1, n - δα}];
           sol = TriangularSolve[eqns, vars];
           Λ2𝔼{i,j}→{k}[
            CF[(vars /. ξαβ__ → xαβ[k]).(vars /. sol) + Sum[xα,α[k] (ξα,α[i] + ξα,α[j]), {α, n}]]
           ]
          ];
```

```
In[•]:= utm4[1, 2 → 2]
```

```
Out[•]= 𝔼{1,2}→{2}[x1,1[2] ξ1,1[1] + x1,1[2] ξ1,1[2] + ℯ^-ξ1,1[2] x1,2[2] ξ1,2[1] +
          ℯ^-ξ2,2[1] x1,2[2] ξ1,2[2] + ℯ^-ξ1,1[2] x1,3[2] ξ1,3[1] + ℯ^-ξ3,3[1] x1,3[2] ξ1,3[2] +
          ℯ^-ξ1,1[2] x1,4[2] ξ1,4[1] + ℯ^-ξ4,4[1] x1,4[2] ξ1,4[2] + x2,2[2] ξ2,2[1] + x2,2[2] ξ2,2[2] +
          ℯ^-ξ2,2[2] x2,3[2] ξ2,3[1] - x1,3[2] ξ1,2[2] ξ2,3[1] + ℯ^-ξ3,3[1] x2,3[2] ξ2,3[2] +
          ℯ^-ξ2,2[2] x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] + ℯ^-ξ4,4[1] x2,4[2] ξ2,4[2] +
          x3,3[2] ξ3,3[1] + x3,3[2] ξ3,3[2] + ℯ^-ξ3,3[2] x3,4[2] ξ3,4[1] - x1,4[2] ξ1,3[2] ξ3,4[1] -
          x2,4[2] ξ2,3[2] ξ3,4[1] + ℯ^-ξ4,4[1] x3,4[2] ξ3,4[2] + x4,4[2] ξ4,4[1] + x4,4[2] ξ4,4[2], 0, 0]
```

```
In[•]:= Table[First@Timing[n → utm_n[1, 2 → 3]], {n, 8}]
```

```
Out[•]=
{0., 0., 0.015625, 0.03125, 0.0625, 0.15625, 0.40625, 1.65625}
```

```
In[•]:= Table[First@Timing[n → utm_n[1, 2 → 3]], {n, 8}]
```

```
Out[•]=
{0., 0., 0., 0., 0., 0., 0., 0.}
```

```
In[•]:= n = 4;
lhs = utm_n[1, 2 → 2] // utm_n[2, 3 → 1]
rhs = utm_n[2, 3 → 2] // utm_n[1, 2 → 1]; lhs ≡ rhs
```

```
Out[•]=
𝔼_{1,2,3}→{1}[x_{1,1}[1] ξ_{1,1}[1] + x_{1,1}[1] ξ_{1,1}[2] + x_{1,1}[1] ξ_{1,1}[3] + e^{-ξ_{1,1}[2]-ξ_{1,1}[3]} x_{1,2}[1] ξ_{1,2}[1] +
  e^{-ξ_{1,1}[3]-ξ_{2,2}[1]} x_{1,2}[1] ξ_{1,2}[2] + e^{-ξ_{2,2}[1]-ξ_{2,2}[2]} x_{1,2}[1] ξ_{1,2}[3] + e^{-ξ_{1,1}[2]-ξ_{1,1}[3]} x_{1,3}[1] ξ_{1,3}[1] +
  e^{-ξ_{1,1}[3]-ξ_{3,3}[1]} x_{1,3}[1] ξ_{1,3}[2] + e^{-ξ_{3,3}[1]-ξ_{3,3}[2]} x_{1,3}[1] ξ_{1,3}[3] + e^{-ξ_{1,1}[2]-ξ_{1,1}[3]} x_{1,4}[1] ξ_{1,4}[1] +
  e^{-ξ_{1,1}[3]-ξ_{4,4}[1]} x_{1,4}[1] ξ_{1,4}[2] + e^{-ξ_{4,4}[1]-ξ_{4,4}[2]} x_{1,4}[1] ξ_{1,4}[3] + x_{2,2}[1] ξ_{2,2}[1] + x_{2,2}[1] ξ_{2,2}[2] +
  x_{2,2}[1] ξ_{2,2}[3] + e^{-ξ_{2,2}[2]-ξ_{2,2}[3]} x_{2,3}[1] ξ_{2,3}[1] - e^{-ξ_{1,1}[3]} x_{1,3}[1] ξ_{1,2}[2] ξ_{2,3}[1] -
  e^{-ξ_{2,2}[2]} x_{1,3}[1] ξ_{1,2}[3] ξ_{2,3}[1] + e^{-ξ_{2,2}[3]-ξ_{3,3}[1]} x_{2,3}[1] ξ_{2,3}[2] - e^{-ξ_{3,3}[1]} x_{1,3}[1] ξ_{1,2}[3] ξ_{2,3}[2] +
  e^{-ξ_{3,3}[1]-ξ_{3,3}[2]} x_{2,3}[1] ξ_{2,3}[3] + e^{-ξ_{2,2}[2]-ξ_{2,2}[3]} x_{2,4}[1] ξ_{2,4}[1] - e^{-ξ_{1,1}[3]} x_{1,4}[1] ξ_{1,2}[2] ξ_{2,4}[1] -
  e^{-ξ_{2,2}[2]} x_{1,4}[1] ξ_{1,2}[3] ξ_{2,4}[1] + e^{-ξ_{2,2}[3]-ξ_{4,4}[1]} x_{2,4}[1] ξ_{2,4}[2] - e^{-ξ_{4,4}[1]} x_{1,4}[1] ξ_{1,2}[3] ξ_{2,4}[2] +
  e^{-ξ_{4,4}[1]-ξ_{4,4}[2]} x_{2,4}[1] ξ_{2,4}[3] + x_{3,3}[1] ξ_{3,3}[1] + x_{3,3}[1] ξ_{3,3}[2] + x_{3,3}[1] ξ_{3,3}[3] +
  e^{-ξ_{3,3}[2]-ξ_{3,3}[3]} x_{3,4}[1] ξ_{3,4}[1] - e^{-ξ_{1,1}[3]} x_{1,4}[1] ξ_{1,3}[2] ξ_{3,4}[1] - e^{-ξ_{3,3}[2]} x_{1,4}[1] ξ_{1,3}[3] ξ_{3,4}[1] -
  e^{-ξ_{2,2}[3]} x_{2,4}[1] ξ_{2,3}[2] ξ_{3,4}[1] + x_{1,4}[1] ξ_{1,2}[3] ξ_{2,3}[2] ξ_{3,4}[1] - e^{-ξ_{3,3}[2]} x_{2,4}[1] ξ_{2,3}[3] ξ_{3,4}[1] +
  e^{-ξ_{3,3}[3]-ξ_{4,4}[1]} x_{3,4}[1] ξ_{3,4}[2] - e^{-ξ_{4,4}[1]} x_{1,4}[1] ξ_{1,3}[3] ξ_{3,4}[2] - e^{-ξ_{4,4}[1]} x_{2,4}[1] ξ_{2,3}[3] ξ_{3,4}[2] +
  e^{-ξ_{4,4}[1]-ξ_{4,4}[2]} x_{3,4}[1] ξ_{3,4}[3] + x_{4,4}[1] ξ_{4,4}[1] + x_{4,4}[1] ξ_{4,4}[2] + x_{4,4}[1] ξ_{4,4}[3], 0, 0]
```

```
Out[•]=
True
```

```
In[•]:= lnmΛ_n[i_, j_ → k_] := lnmΛ_n[i, j → k] = Module[{B, α, β, lhs, rhs, eqns, δα, vars, sol, ξ},
    B = IdentityMatrix[n];
    Do[B = MatrixExp[SparseArray[{β, α} → η_{α,β}, {n, n}]].B, {β, 2, n}, {α, 1, β - 1}];
    (* This specifies the PBW ordering *)
    lhs = (B /. η_{αβ__} :→ η_{αβ}[i]).(B /. η_{αβ__} :→ η_{αβ}[j]);
    rhs = B /. η_{αβ__} :→ ξ_{αβ};
    eqns = Table[lhs[[α + δα, α]] == rhs[[α + δα, α]], {δα, 1, n - 1}, {α, 1, n - δα}];
    vars = Flatten@Table[ξ_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}];
    sol = TriangularSolve[eqns, vars];
    CF[(vars /. ξ_{αβ__} → y_{αβ}[k]).(vars /. sol)]
   ];
lnm_n[i_, j_ → k_] := Λ2𝔼_{i,j}→{k}[lnmΛ_n[i, j → k]];
lnmϵ_n[i_, j_ → k_] := Λ2𝔼_{i,j}→{k}[
   lnmΛ_n[i, j → k] /. {η_{αβ__}[i] :→ ϵ η_{αβ}[i], η_{αβ__}[j] :→ ϵ η_{αβ}[j], y_{αβ__}[k] :→ ϵ^{-1} y_{αβ}[k]}];
```

In[•]:=
```
lnmΛ5[1, 2 → 3]
```

Out[•]=
$$\begin{aligned}&y_{1,2}[3]\,\eta_{1,2}[1]+y_{1,2}[3]\,\eta_{1,2}[2]+y_{1,3}[3]\,\eta_{1,3}[1]+y_{1,3}[3]\,\eta_{1,3}[2]+y_{1,4}[3]\,\eta_{1,4}[1]+\\&\ y_{1,4}[3]\,\eta_{1,4}[2]+y_{1,5}[3]\,\eta_{1,5}[1]+y_{1,5}[3]\,\eta_{1,5}[2]+y_{2,3}[3]\,\eta_{2,3}[1]+y_{2,3}[3]\,\eta_{2,3}[2]-\\&\ y_{1,3}[3]\,\eta_{1,2}[1]\,\eta_{2,3}[2]+y_{2,4}[3]\,\eta_{2,4}[1]+y_{2,4}[3]\,\eta_{2,4}[2]-y_{1,4}[3]\,\eta_{1,2}[1]\,\eta_{2,4}[2]+\\&\ y_{2,5}[3]\,\eta_{2,5}[1]+y_{2,5}[3]\,\eta_{2,5}[2]-y_{1,5}[3]\,\eta_{1,2}[1]\,\eta_{2,5}[2]+y_{3,4}[3]\,\eta_{3,4}[1]+\\&\ y_{3,4}[3]\,\eta_{3,4}[2]-y_{1,4}[3]\,\eta_{1,3}[1]\,\eta_{3,4}[2]-y_{2,4}[3]\,\eta_{2,3}[1]\,\eta_{3,4}[2]+y_{3,5}[3]\,\eta_{3,5}[1]+\\&\ y_{3,5}[3]\,\eta_{3,5}[2]-y_{1,5}[3]\,\eta_{1,3}[1]\,\eta_{3,5}[2]-y_{2,5}[3]\,\eta_{2,3}[1]\,\eta_{3,5}[2]+y_{4,5}[3]\,\eta_{4,5}[1]+\\&\ y_{4,5}[3]\,\eta_{4,5}[2]-y_{1,5}[3]\,\eta_{1,4}[1]\,\eta_{4,5}[2]-y_{2,5}[3]\,\eta_{2,4}[1]\,\eta_{4,5}[2]-y_{3,5}[3]\,\eta_{3,4}[1]\,\eta_{4,5}[2]\end{aligned}$$

In[•]:=
```
lnm4[1, 2 → 3]
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2\}\to\{3\}}[y_{1,2}[3]\,\eta_{1,2}[1]+y_{1,2}[3]\,\eta_{1,2}[2]+y_{1,3}[3]\,\eta_{1,3}[1]+\\&\ y_{1,3}[3]\,\eta_{1,3}[2]+y_{1,4}[3]\,\eta_{1,4}[1]+y_{1,4}[3]\,\eta_{1,4}[2]+y_{2,3}[3]\,\eta_{2,3}[1]+y_{2,3}[3]\,\eta_{2,3}[2]-\\&\ y_{1,3}[3]\,\eta_{1,2}[1]\,\eta_{2,3}[2]+y_{2,4}[3]\,\eta_{2,4}[1]+y_{2,4}[3]\,\eta_{2,4}[2]-y_{1,4}[3]\,\eta_{1,2}[1]\,\eta_{2,4}[2]+\\&\ y_{3,4}[3]\,\eta_{3,4}[1]+y_{3,4}[3]\,\eta_{3,4}[2]-y_{1,4}[3]\,\eta_{1,3}[1]\,\eta_{3,4}[2]-y_{2,4}[3]\,\eta_{2,3}[1]\,\eta_{3,4}[2],0,0]\end{aligned}$$

In[•]:=
```
lnmϵ3[1, 2 → 3]
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2\}\to\{3\}}[y_{1,2}[3]\,\eta_{1,2}[1]+y_{1,2}[3]\,\eta_{1,2}[2]+y_{1,3}[3]\,\eta_{1,3}[1]+\\&\ y_{1,3}[3]\,\eta_{1,3}[2]+y_{2,3}[3]\,\eta_{2,3}[1]+y_{2,3}[3]\,\eta_{2,3}[2],-y_{1,3}[3]\,\eta_{1,2}[1]\,\eta_{2,3}[2],0]\end{aligned}$$

In[•]:=
```
lhs = lnm4[1, 2 → 2] // lnm4[2, 3 → 1]
rhs = lnm4[2, 3 → 2] // lnm4[1, 2 → 1]; lhs ≡ rhs
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2,3\}\to\{1\}}[y_{1,2}[1]\,\eta_{1,2}[1]+y_{1,2}[1]\,\eta_{1,2}[2]+y_{1,2}[1]\,\eta_{1,2}[3]+\\&\ y_{1,3}[1]\,\eta_{1,3}[1]+y_{1,3}[1]\,\eta_{1,3}[2]+y_{1,3}[1]\,\eta_{1,3}[3]+y_{1,4}[1]\,\eta_{1,4}[1]+y_{1,4}[1]\,\eta_{1,4}[2]+\\&\ y_{1,4}[1]\,\eta_{1,4}[3]+y_{2,3}[1]\,\eta_{2,3}[1]+y_{2,3}[1]\,\eta_{2,3}[2]-y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[2]+\\&\ y_{2,3}[1]\,\eta_{2,3}[3]-y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[3]-y_{1,3}[1]\,\eta_{1,2}[2]\,\eta_{2,3}[3]+y_{2,4}[1]\,\eta_{2,4}[1]+\\&\ y_{2,4}[1]\,\eta_{2,4}[2]-y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,4}[2]+y_{2,4}[1]\,\eta_{2,4}[3]-y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,4}[3]-\\&\ y_{1,4}[1]\,\eta_{1,2}[2]\,\eta_{2,4}[3]+y_{3,4}[1]\,\eta_{3,4}[1]+y_{3,4}[1]\,\eta_{3,4}[2]-y_{1,4}[1]\,\eta_{1,3}[1]\,\eta_{3,4}[2]-\\&\ y_{2,4}[1]\,\eta_{2,3}[1]\,\eta_{3,4}[2]+y_{3,4}[1]\,\eta_{3,4}[3]-y_{1,4}[1]\,\eta_{1,3}[1]\,\eta_{3,4}[3]-y_{1,4}[1]\,\eta_{1,3}[2]\,\eta_{3,4}[3]-\\&\ y_{2,4}[1]\,\eta_{2,3}[1]\,\eta_{3,4}[3]-y_{2,4}[1]\,\eta_{2,3}[2]\,\eta_{3,4}[3]+y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[2]\,\eta_{3,4}[3],0,0]\end{aligned}$$

Out[•]=
```
True
```

In[•]:=
```
Table[First@Timing[n → lnmn[1, 2 → 3]], {n, 9}]
```

Out[•]=
```
{0., 0., 0., 0., 0., 0.03125, 0.078125, 0.28125, 2.01563}
```

In[•]:=
```
$k = 5
```

Out[•]=
```
5
```

In[•]:=
```
lnmϵ3[1, 2 → 3]
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2\}\to\{3\}}[y_{1,2}[3]\,\eta_{1,2}[1]+y_{1,2}[3]\,\eta_{1,2}[2]+y_{1,3}[3]\,\eta_{1,3}[1]+y_{1,3}[3]\,\eta_{1,3}[2]+\\&\ y_{2,3}[3]\,\eta_{2,3}[1]+y_{2,3}[3]\,\eta_{2,3}[2],-y_{1,3}[3]\,\eta_{1,2}[1]\,\eta_{2,3}[2],0,0,0,0]\end{aligned}$$

In[•]:=
```
lhs = lnmϵ4[1, 2 → 2] // lnmϵ4[2, 3 → 1]
rhs = lnmϵ4[2, 3 → 2] // lnmϵ4[1, 2 → 1]; lhs ≡ rhs
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2,3\}\to\{1\}}[y_{1,2}[1]\,\eta_{1,2}[1]+y_{1,2}[1]\,\eta_{1,2}[2]+y_{1,2}[1]\,\eta_{1,2}[3]+\\&\quad y_{1,3}[1]\,\eta_{1,3}[1]+y_{1,3}[1]\,\eta_{1,3}[2]+y_{1,3}[1]\,\eta_{1,3}[3]+y_{1,4}[1]\,\eta_{1,4}[1]+y_{1,4}[1]\,\eta_{1,4}[2]+\\&\quad y_{1,4}[1]\,\eta_{1,4}[3]+y_{2,3}[1]\,\eta_{2,3}[1]+y_{2,3}[1]\,\eta_{2,3}[2]+y_{2,3}[1]\,\eta_{2,3}[3]+y_{2,4}[1]\,\eta_{2,4}[1]+\\&\quad y_{2,4}[1]\,\eta_{2,4}[2]+y_{2,4}[1]\,\eta_{2,4}[3]+y_{3,4}[1]\,\eta_{3,4}[1]+y_{3,4}[1]\,\eta_{3,4}[2]+y_{3,4}[1]\,\eta_{3,4}[3],\\&\quad -y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[2]-y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[3]-y_{1,3}[1]\,\eta_{1,2}[2]\,\eta_{2,3}[3]-\\&\quad y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,4}[2]-y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,4}[3]-y_{1,4}[1]\,\eta_{1,2}[2]\,\eta_{2,4}[3]-y_{1,4}[1]\,\eta_{1,3}[1]\,\eta_{3,4}[2]-\\&\quad y_{2,4}[1]\,\eta_{2,3}[1]\,\eta_{3,4}[2]-y_{1,4}[1]\,\eta_{1,3}[1]\,\eta_{3,4}[3]-y_{1,4}[1]\,\eta_{1,3}[2]\,\eta_{3,4}[3]-\\&\quad y_{2,4}[1]\,\eta_{2,3}[1]\,\eta_{3,4}[3]-y_{2,4}[1]\,\eta_{2,3}[2]\,\eta_{3,4}[3],\,y_{1,4}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[2]\,\eta_{3,4}[3],\,0,\,0,\,0]\end{aligned}$$

Out[•]=
```
True
```

In[•]:=
```
ltmn_[i_, j_ → k_] := ltmn[i, j → k] = Module[{B, α, β, lhs, rhs, eqns, δα, vars, sol, ξ},
    B = IdentityMatrix[n];
    Do[B = MatrixExp[SparseArray[{β, α} → ηα,β, {n, n}]].B, {β, 1, n}, {α, 1, β}];
    (* This specifies the PBW ordering *)
    lhs = (B /. ηαβ__ :↦ ηαβ[i]).(B /. ηαβ__ :↦ ηαβ[j]);
    rhs = B /. ηαβ__ :↦ ξαβ /. ξα_,α_ :↦ ηα,α[i] + ηα,α[j];
    eqns = Table[lhs⟦α + δα, α⟧ == rhs⟦α + δα, α⟧, {δα, 1, n - 1}, {α, 1, n - δα}];
    vars = Flatten@Table[ξα,α+δα, {δα, 1, n - 1}, {α, 1, n - δα}];
    sol = TriangularSolve[eqns, vars];
    Λ2𝔼{i,j}→{k}[
     CF[(vars /. ξαβ__ → yαβ[k]).(vars /. sol) + Sum[yα,α[k] (ηα,α[i] + ηα,α[j]), {α, n}]]]
   ];
```

In[•]:=
```
ltm2[1, 2 → 3]
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2\}\to\{3\}}\left[y_{1,1}[3]\,\eta_{1,1}[1]+y_{1,1}[3]\,\eta_{1,1}[2]+e^{-\eta_{2,2}[2]}\,y_{1,2}[3]\,\eta_{1,2}[1]+\right.\\&\quad\left.e^{-\eta_{1,1}[1]}\,y_{1,2}[3]\,\eta_{1,2}[2]+y_{2,2}[3]\,\eta_{2,2}[1]+y_{2,2}[3]\,\eta_{2,2}[2],\,0,\,0,\,0,\,0,\,0\right]\end{aligned}$$

In[•]:=
```
lhs = ltm3[1, 2 → 2] // ltm3[2, 3 → 1]
rhs = ltm3[2, 3 → 2] // ltm3[1, 2 → 1]; lhs ≡ rhs
```

Out[•]=
$$\begin{aligned}&\mathbb{E}_{\{1,2,3\}\to\{1\}}\left[y_{1,1}[1]\,\eta_{1,1}[1]+y_{1,1}[1]\,\eta_{1,1}[2]+y_{1,1}[1]\,\eta_{1,1}[3]+e^{-\eta_{2,2}[2]-\eta_{2,2}[3]}\,y_{1,2}[1]\,\eta_{1,2}[1]+\right.\\&\quad e^{-\eta_{1,1}[1]-\eta_{2,2}[3]}\,y_{1,2}[1]\,\eta_{1,2}[2]+e^{-\eta_{1,1}[1]-\eta_{1,1}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3]+e^{-\eta_{3,3}[2]-\eta_{3,3}[3]}\,y_{1,3}[1]\,\eta_{1,3}[1]+\\&\quad e^{-\eta_{1,1}[1]-\eta_{3,3}[3]}\,y_{1,3}[1]\,\eta_{1,3}[2]+e^{-\eta_{1,1}[1]-\eta_{1,1}[2]}\,y_{1,3}[1]\,\eta_{1,3}[3]+y_{2,2}[1]\,\eta_{2,2}[1]+y_{2,2}[1]\,\eta_{2,2}[2]+\\&\quad y_{2,2}[1]\,\eta_{2,2}[3]+e^{-\eta_{3,3}[2]-\eta_{3,3}[3]}\,y_{2,3}[1]\,\eta_{2,3}[1]+e^{-\eta_{2,2}[1]-\eta_{3,3}[3]}\,y_{2,3}[1]\,\eta_{2,3}[2]-\\&\quad e^{-\eta_{3,3}[3]}\,y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[2]+e^{-\eta_{2,2}[1]-\eta_{2,2}[2]}\,y_{2,3}[1]\,\eta_{2,3}[3]-e^{-\eta_{2,2}[2]}\,y_{1,3}[1]\,\eta_{1,2}[1]\,\eta_{2,3}[3]-\\&\quad\left.e^{-\eta_{1,1}[1]}\,y_{1,3}[1]\,\eta_{1,2}[2]\,\eta_{2,3}[3]+y_{3,3}[1]\,\eta_{3,3}[1]+y_{3,3}[1]\,\eta_{3,3}[2]+y_{3,3}[1]\,\eta_{3,3}[3],\,0,\,0,\,0,\,0,\,0\right]\end{aligned}$$

Out[•]=
```
True
```

In[•]:=
```
SWΛ_{ut,ln,n_}[i_, j_] :=
  Module[{A, B, M, LU, L, U, eqnsU, eqnsL, varsU, varsL, solU, solL, vars, sol},
   A = B = IdentityMatrix[n];
   Do[A = A.MatrixExp[SparseArray[{α, β} → ξ_{α,β} /. ξ_{α_,α_} :> Log[Ξ_α], {n, n}]],
    {β, 1, n}, {α, 1, β}];
   Do[B = MatrixExp[SparseArray[{β, α} → η_{α,β}, {n, n}]].B, {β, 2, n}, {α, 1, β - 1}];
   M = (A /. {ξ_{αβ__} :> ξ_{αβ}[i], Ξ_{α_} :> Ξ_α[i]}).(B /. η_{αβ__} :> η_{αβ}[j]);
   LU = First@LUDecomposition[M, Pivoting → False];
   L = LowerTriangularize[LU, -1] + IdentityMatrix[n];
   U = UpperTriangularize[LU];
   eqnsU =
    Table[U[[α, α + δα]] == (A[[α, α + δα]] /. {ξ → ξ1, Ξ → Ξ1}), {δα, 0, n - 1}, {α, 1, n - δα}];
   varsU = Flatten@{Table[Ξ1_α, {α, 1, n}], Table[ξ1_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}]};
   solU = TriangularSolve[eqnsU, varsU];
   eqnsL = Table[
     (L[[α + δα, α]] /. solU) == (B[[α + δα, α]] /. {η → η1}), {δα, 1, n - 1}, {α, 1, n - δα}];
   varsL = Flatten@Table[η1_{α,α+δα}, {δα, 1, n - 1}, {α, 1, n - δα}];
   solL = TriangularSolve[eqnsL, varsL];
   vars = varsU ⋃ varsL /. Ξ1_{α_} :> ξ1_{α,α};
   sol = solU ⋃ solL /. (Ξ1_{α_} → ℰ_) :> (ξ1_{α,α} → Log[ℰ]);
   CF[(vars /. {η1_{αβ__} → y_{αβ}[j], ξ1_{αβ__} → x_{αβ}[i]}).(vars /. sol)]
  ];
```

In[•]:=
```
SWΛ_{ut,ln,2}[1, 2]
```

Out[•]=

$$\text{Log}[\Xi_1[1]+\Xi_1[1]\,\Xi_2[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]]\,x_{1,1}[1]+\text{Log}\left[\frac{\Xi_2[1]}{1+\Xi_2[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]}\right]x_{2,2}[1]+x_{1,2}[1]\,\xi_{1,2}[1]+\frac{\Xi_2[1]\,y_{1,2}[2]\,\eta_{1,2}[2]}{\Xi_1[1]\,(1+\Xi_2[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1])}$$

In[•]:=
```
SW_{ut,ln,n_}[i_, j_] := CF[Λ2E_{{i,j}→{i,j}}[
    SWΛ_{ut,ln,n}[i, j] /. {
      Ξ_α_[i] :> e^{ξ_{α,α}[i]}, η_{αβ__}[j] :> ϵ η_{αβ}[j], y_{αβ__}[j] :> ϵ^{-1} y_{αβ}[j]
     }
   ] /. Log[e^{ℰ_}] :> ℰ]
```

In[•]:=
```
sw_{ut,ln,2}[1, 2]
```

Out[•]=

$$\mathbb{E}_{\{1,2\}\to\{1,2\}}\Big[e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]+x_{1,1}[1]\,\xi_{1,1}[1]+x_{1,2}[1]\,\xi_{1,2}[1]+x_{2,2}[1]\,\xi_{2,2}[1],\ e^{\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]-e^{\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]\,\xi_{1,2}[1]-e^{-\xi_{1,1}[1]+2\,\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]^2\,\xi_{1,2}[1],\ -\frac{1}{2}\,e^{2\,\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]^2\,\xi_{1,2}[1]^2+\frac{1}{2}\,e^{2\,\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]^2\,\xi_{1,2}[1]^2+e^{-\xi_{1,1}[1]+3\,\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]^3\,\xi_{1,2}[1]^2,\ \frac{1}{3}\,e^{3\,\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]^3\,\xi_{1,2}[1]^3-\frac{1}{3}\,e^{3\,\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]^3\,\xi_{1,2}[1]^3-e^{-\xi_{1,1}[1]+4\,\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]^4\,\xi_{1,2}[1]^3,\ -\frac{1}{4}\,e^{4\,\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]^4\,\xi_{1,2}[1]^4+\frac{1}{4}\,e^{4\,\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]^4\,\xi_{1,2}[1]^4+e^{-\xi_{1,1}[1]+5\,\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]^5\,\xi_{1,2}[1]^4,\ \frac{1}{5}\,e^{5\,\xi_{2,2}[1]}\,x_{1,1}[1]\,\eta_{1,2}[2]^5\,\xi_{1,2}[1]^5-\frac{1}{5}\,e^{5\,\xi_{2,2}[1]}\,x_{2,2}[1]\,\eta_{1,2}[2]^5\,\xi_{1,2}[1]^5-e^{-\xi_{1,1}[1]+6\,\xi_{2,2}[1]}\,y_{1,2}[2]\,\eta_{1,2}[2]^6\,\xi_{1,2}[1]^5\Big]$$

In[•]:=
```
$k = 2
```

Out[•]=
```
2
```

In[•]:=
```
lhs = With[{n = 3},
  utm_n[1, 2 → 3] // sw_{ut,ln,n}[3, 4]
 ]
```

Out[•]=

In[•]:=
```
rhs = With[{n = 3},
  sw_{ut,ln,n}[2, 4] // sw_{ut,ln,n}[1, 4] // utm_n[1, 2 → 3]
 ]
```

Out[•]=

In[•]:=
```
lhs ≡ rhs
```

Out[•]=
```
True
```

In[•]:=
```
$k = 2
```

Out[•]=
```
2
```

In[•]:=
```
lhs = With[{n = 3},
  lnmϵ_n[2, 3 → 4] // sw_{ut,ln,n}[1, 4]
 ]
```

Out[•]=

```
In[•]:= rhs = With[{n = 3},
           sw_{ut,ln,n}[1, 2] // sw_{ut,ln,n}[1, 3] // lnmε_n[2, 3 → 4]
         ]
```

Out[•]=

In[•]:= lhs ≡ rhs

Out[•]=

True

```
In[•]:= ϒ_n_[i_ → j_, k_] := Λ2𝔼_{i}→{j,k}[Sum[η_{α,β}[i] y_{α,β}[j] + ξ_{α,β}[i] x_{α,β}[k], {α, 1, n}, {β, α, n}]]
```

```
In[•]:= glmε_n_[i_, j_ → k_] := Module[{i1, i2, j1, j2},
           (ϒ_n[i → i1, i2] ϒ_n[j → j1, j2]) // sw_{ut,ln,n}[i2, j1] // (lnmε_n[i1, j1 → k] utm_n[i2, j2 → k])
         ]
```

In[•]:= glmε_2[i, j → k]

Out[•]=

$$\begin{aligned}
&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,2}[k]\,\eta_{1,2}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j] + x_{1,1}[k]\,\xi_{1,1}[i] + x_{1,1}[k]\,\xi_{1,1}[j] + \\
&\quad e^{-\xi_{1,1}[j]}\,x_{1,2}[k]\,\xi_{1,2}[i] + e^{-\xi_{2,2}[i]}\,x_{1,2}[k]\,\xi_{1,2}[j] + x_{2,2}[k]\,\xi_{2,2}[i] + x_{2,2}[k]\,\xi_{2,2}[j], \\
&\quad e^{\xi_{2,2}[i]}\,x_{1,1}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] - e^{\xi_{2,2}[i]}\,x_{2,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] - \\
&\quad e^{-\xi_{1,1}[i]+2\,\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]^2\,\xi_{1,2}[i] + x_{1,2}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i]\,\xi_{1,2}[j], \\
&\quad -\frac{1}{2}\,e^{2\,\xi_{2,2}[i]}\,x_{1,1}[k]\,\eta_{1,2}[j]^2\,\xi_{1,2}[i]^2 + \frac{1}{2}\,e^{2\,\xi_{2,2}[i]}\,x_{2,2}[k]\,\eta_{1,2}[j]^2\,\xi_{1,2}[i]^2 + \\
&\quad e^{-\xi_{1,1}[i]+3\,\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]^3\,\xi_{1,2}[i]^2\Big]
\end{aligned}$$

```
In[•]:= diff = Normal[Exp[List @@ glmε_2[i, j → k].{1, ε, ε^2} + O[ε]^3] -
             Exp[List @@ glmε_2[j, i → k].{1, ε, ε^2} + O[ε]^3]];
        bas = {η_{1,2}, ξ_{1,1}, ξ_{1,2}, ξ_{2,2}};
        MatrixForm@
         Simplify@Table[(∂_{b1[i],b2[j]} diff) /. {_[i | j] → 0, v_[k] → v}, {b1, bas}, {b2, bas}]
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
0 & y_{1,2} & \epsilon\,(-x_{1,1}+x_{2,2}) & -y_{1,2} \\
-y_{1,2} & 0 & x_{1,2} & 0 \\
\epsilon\,(x_{1,1}-x_{2,2}) & -x_{1,2} & 0 & x_{1,2} \\
y_{1,2} & 0 & -x_{1,2} & 0
\end{pmatrix}$$

In[•]:= glmε_3[i, j → k]

Out[•]=

$$\begin{aligned}
&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,2}[k]\,\eta_{1,2}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j] + y_{1,3}[k]\,\eta_{1,3}[i] + \\
&\quad e^{-\xi_{1,1}[i]+\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j] + y_{2,3}[k]\,\eta_{2,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{2,3}[j] + \\
&\quad x_{1,1}[k]\,\xi_{1,1}[i] + x_{1,1}[k]\,\xi_{1,1}[j] + e^{-\xi_{1,1}[j]}\,x_{1,2}[k]\,\xi_{1,2}[i] - e^{\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]\,\xi_{1,2}[i] + \\
&\quad e^{-\xi_{2,2}[i]}\,x_{1,2}[k]\,\xi_{1,2}[j] + e^{-\xi_{1,1}[j]}\,x_{1,3}[k]\,\xi_{1,3}[i] + e^{-\xi_{3,3}[i]}\,x_{1,3}[k]\,\xi_{1,3}[j] + x_{2,2}[k]\,\xi_{2,2}[i] + \\
&\quad x_{2,2}[k]\,\xi_{2,2}[j] + e^{-\xi_{2,2}[j]}\,x_{2,3}[k]\,\xi_{2,3}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]\,\xi_{2,3}[i] - \\
&\quad x_{1,3}[k]\,\xi_{1,2}[j]\,\xi_{2,3}[i] + e^{-\xi_{3,3}[i]}\,x_{2,3}[k]\,\xi_{2,3}[j] + x_{3,3}[k]\,\xi_{3,3}[i] + x_{3,3}[k]\,\xi_{3,3}[j], \\
&\quad -e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,2}[i]\,\eta_{2,3}[j] + e^{\xi_{2,2}[i]}\,x_{1,1}[k]\,\eta_{1,2}[j]\,\xi_{1,2}[i] -
\end{aligned}$$

$$
\begin{aligned}
&e^{\xi_{2,2}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]\, \xi_{1,2}[i] - e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^2\, \xi_{1,2}[i] + \\
&e^{\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,2}[i]\, \eta_{1,3}[j]\, \xi_{1,2}[i] + x_{1,2}[k]\, \eta_{1,2}[j]\, \xi_{1,2}[i]\, \xi_{1,2}[j] - \\
&e^{-\xi_{2,2}[j]}\, x_{2,3}[k]\, \eta_{1,2}[j]\, \xi_{1,3}[i] + e^{\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,3}[j]\, \xi_{1,3}[i] - e^{\xi_{3,3}[i]}\, x_{3,3}[k]\, \eta_{1,3}[j]\, \xi_{1,3}[i] - \\
&e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,3}[i] - e^{-\xi_{1,1}[i]+2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,3}[j]^2\, \xi_{1,3}[i] + \\
&e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{2,3}[j]\, \xi_{1,3}[i] - e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i] - \\
&e^{-\xi_{1,1}[j]+\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,3}[i] + e^{2\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]^2\, \xi_{1,2}[i]\, \xi_{1,3}[i] + \\
&x_{1,3}[k]\, \eta_{1,2}[j]\, \xi_{1,2}[j]\, \xi_{1,3}[i] + x_{1,3}[k]\, \eta_{1,3}[j]\, \xi_{1,3}[i]\, \xi_{1,3}[j] + e^{\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{2,3}[j]\, \xi_{2,3}[i] - \\
&e^{\xi_{3,3}[i]}\, x_{3,3}[k]\, \eta_{2,3}[j]\, \xi_{2,3}[i] + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{2,3}[i] - \\
&e^{-\xi_{1,1}[i]+2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{2,3}[i] - e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{2,3}[j]^2\, \xi_{2,3}[i] + \\
&e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&2\, e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] + \\
&e^{2\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{2,3}[j]\, \xi_{1,2}[j]\, \xi_{2,3}[i] + e^{\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,2}[j]\, \xi_{2,3}[i] - \\
&e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,3}[j]^2\, \xi_{1,3}[i]\, \xi_{2,3}[i] + x_{1,3}[k]\, \eta_{2,3}[j]\, \xi_{1,3}[j]\, \xi_{2,3}[i] - \\
&e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,3}[j]^2\, \xi_{1,2}[i]\, \xi_{2,3}[i]^2 + \\
&x_{2,3}[k]\, \eta_{1,3}[j]\, \xi_{1,3}[i]\, \xi_{2,3}[j] + x_{2,3}[k]\, \eta_{2,3}[j]\, \xi_{2,3}[i]\, \xi_{2,3}[j],
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\, e^{2\xi_{2,2}[i]}\, x_{1,1}[k]\, \eta_{1,2}[j]^2\, \xi_{1,2}[i]^2 + \frac{1}{2}\, e^{2\xi_{2,2}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]^2\, \xi_{1,2}[i]^2 + \\
&e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^3\, \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i] - \\
&e^{\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i] - e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^2\, \eta_{2,3}[j]\, \xi_{1,3}[i] + \\
&e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,2}[i]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i] - \\
&e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,3}[i] + \\
&e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,3}[i] + \\
&2\, e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^2\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,3}[i] - \\
&e^{2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,2}[i]\, \eta_{1,3}[j]^2\, \xi_{1,2}[i]\, \xi_{1,3}[i] + \\
&e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{1,2}[j]\, \xi_{1,3}[i] - \\
&e^{\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]\, \xi_{1,2}[j]\, \xi_{1,3}[i] - \frac{1}{2}\, e^{2\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,3}[j]^2\, \xi_{1,3}[i]^2 + \\
&\frac{1}{2}\, e^{2\xi_{3,3}[i]}\, x_{3,3}[k]\, \eta_{1,3}[j]^2\, \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]^2\, \xi_{1,3}[i]^2 + \\
&e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,3}[j]^3\, \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i]^2 + \\
&e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]^2\, \eta_{2,3}[j]\, \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{1,3}[j]^2\, \xi_{1,2}[i]\, \xi_{1,3}[i]^2 - \\
&e^{3\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]^3\, \xi_{1,2}[i]\, \xi_{1,3}[i]^2 + e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,2}[i]\, \eta_{2,3}[j]^2\, \xi_{2,3}[i] + \\
&e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]\, \eta_{2,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&2\, e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^2\, \eta_{2,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&e^{2\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,2}[i]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,2}[i]\, \xi_{2,3}[i] - \\
&e^{2\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{1,1}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]^2\, \xi_{2,3}[i] + \\
&e^{2\xi_{2,2}[i]+\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \xi_{1,2}[i]^2\, \xi_{2,3}[i] + \\
&3\, e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]^2\, \eta_{1,3}[j]\, \xi_{1,2}[i]^2\, \xi_{2,3}[i] - \\
&e^{2\xi_{3,3}[i]}\, x_{2,2}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i]\, \xi_{2,3}[i] + e^{2\xi_{3,3}[i]}\, x_{3,3}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i]\, \xi_{2,3}[i] - \\
&2\, e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2\xi_{3,3}[i]}\, y_{1,2}[k]\, \eta_{1,2}[j]\, \eta_{1,3}[j]\, \eta_{2,3}[j]\, \xi_{1,3}[i]\, \xi_{2,3}[i] + \\
&2\, e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]}\, y_{1,3}[k]\, \eta_{1,3}[j]^2\, \eta_{2,3}[j]\, \xi_{1,3}[i]\, \xi_{2,3}[i] - \\
&e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]}\, x_{1,2}[k]\, \eta_{2,3}[j]^2\, \xi_{1,3}[i]\, \xi_{2,3}[i] + \\
&2\, e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]}\, y_{2,3}[k]\, \eta_{1,3}[j]\, \eta_{2,3}[j]^2\, \xi_{1,3}[i]\, \xi_{2,3}[i] -
\end{aligned}
$$

$$
\begin{aligned}
&e^{\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{1,1}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i] +\\
&e^{\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{2,2}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i] +\\
&4\,e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i] +\\
&e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i] -\\
&2\,e^{3\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]^2\,\eta_{2,3}[j]\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i] +\\
&e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[j]\,\xi_{1,3}[i]\,\xi_{2,3}[i] -\\
&e^{2\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]\,\xi_{1,2}[j]\,\xi_{1,3}[i]\,\xi_{2,3}[i] +\\
&e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+3\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]^3\,\xi_{1,3}[i]^2\,\xi_{2,3}[i] -\\
&\frac{1}{2}\,e^{2\xi_{3,3}[i]}\,x_{2,2}[k]\,\eta_{2,3}[j]^2\,\xi_{2,3}[i]^2+\frac{1}{2}\,e^{2\xi_{3,3}[i]}\,x_{3,3}[k]\,\eta_{2,3}[j]^2\,\xi_{2,3}[i]^2 +\\
&e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]}\,y_{1,3}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]^2\,\xi_{2,3}[i]^2+e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{2,3}[j]^3\,\xi_{2,3}[i]^2 -\\
&2\,e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[i]\,\xi_{2,3}[i]^2 -\\
&e^{3\xi_{3,3}[i]}\,y_{2,3}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]^2\,\xi_{1,2}[i]\,\xi_{2,3}[i]^2-\frac{1}{2}\,e^{2\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{1,1}[k]\\
&\quad\eta_{1,3}[j]^2\,\xi_{1,2}[i]^2\,\xi_{2,3}[i]^2+\frac{1}{2}\,e^{2\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{2,2}[k]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]^2\,\xi_{2,3}[i]^2 +\\
&3\,e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+2\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,2}[j]\,\eta_{1,3}[j]^2\,\xi_{1,2}[i]^2\,\xi_{2,3}[i]^2 +\\
&e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{2,3}[j]^2\,\xi_{1,2}[j]\,\xi_{2,3}[i]^2 -\\
&e^{2\xi_{3,3}[i]}\,x_{1,2}[k]\,\eta_{1,3}[j]\,\eta_{2,3}[j]\,\xi_{1,2}[i]\,\xi_{1,2}[j]\,\xi_{2,3}[i]^2 +\\
&2\,e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]^3\,\xi_{1,2}[i]\,\xi_{1,3}[i]\,\xi_{2,3}[i]^2 +\\
&e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+3\xi_{3,3}[i]}\,y_{1,2}[k]\,\eta_{1,3}[j]^3\,\xi_{1,2}[i]^2\,\xi_{2,3}[i]^3\Big]
\end{aligned}
$$

In[•]:=
```
n = 3;
Short[lhs = glmeₙ[1, 2 → 2] // glmeₙ[2, 3 → 1], 5]
rhs = glmeₙ[2, 3 → 2] // glmeₙ[1, 2 → 1]; lhs ≡ rhs
```

Out[•]//Short=

$$
\begin{aligned}
&\mathbb{E}_{\{1,2,3\}\to\{1\}}\Big[y_{1,2}[1]\,\eta_{1,2}[1]+e^{-\xi_{1,1}[1]+\xi_{2,2}[1]}\,y_{1,2}[1]\,\eta_{1,2}[2] +\\
&\quad e^{-\xi_{1,1}[1]-\xi_{1,1}[2]+\xi_{2,2}[1]+\xi_{2,2}[2]}\,y_{1,2}[1]\,\eta_{1,2}[3]+y_{1,3}[1]\,\eta_{1,3}[1]+\ll 34\gg +\\
&\quad e^{-\xi_{3,3}[1]-\xi_{3,3}[2]}\,x_{2,3}[1]\,\xi_{2,3}[3]+x_{3,3}[1]\,\xi_{3,3}[1]+x_{3,3}[1]\,\xi_{3,3}[2]+x_{3,3}[1]\,\xi_{3,3}[3],\ \ll 1\gg,\ \ll 1\gg\Big]
\end{aligned}
$$

Out[•]=
```
True
```