

```

In[*]:= n = 4
In[*]:= mat = Table[ai,j, {i, n}, {j, n}]
In[*]:= {lu, p, c} = Simplify@LUdecomposition[mat]
In[*]:= MatrixForm[lu]
In[*]:= MatrixForm[l = LowerTriangularize[lu, -1] + IdentityMatrix[3]]
In[*]:= MatrixForm[u = UpperTriangularize[lu]]
In[*]:= Simplify[l.u]

```

## The Swap

```

In[*]:= n = 2;
In[*]:= A = B = IdentityMatrix[n];
Do[A = A.MatrixExp[SparseArray[{α, β} → ξα,β, {n, n}]], {β, 1, n}, {α, 1, β}];
(* This specifies the PBW ordering *)
Do[B = MatrixExp[SparseArray[{α, β} → ηα,β, {n, n}]].B, {α, 2, n}, {β, 1, α - 1}];
MatrixForm /@ {A, B}
In[*]:= mat = CF[
  (A /. ξαβ → ξ1αβ) . (B /. ηαβ → η1αβ)
]
In[*]:= mat // MatrixForm
In[*]:= Options[LUdecomposition]
In[*]:= LUdecomposition[mat, Pivoting → False]
In[*]:= {lu, p, c} = Simplify@LUdecomposition[mat, Pivoting → False]
In[*]:= lu // MatrixForm
In[*]:= MatrixForm[l = LowerTriangularize[lu, -1] + IdentityMatrix[n]]
In[*]:= MatrixForm[u = UpperTriangularize[lu]]
In[*]:= Simplify[l.u]
In[*]:= Simplify[l.u - mat]

```

## The Swap with $\epsilon$

```

In[*]:= n = 2;

```

```
In[*]:= A = B = IdentityMatrix[n];
Do[A = A.MatrixExp[SparseArray[{α, β} → ξα,β, {n, n}]], {β, 1, n}, {α, 1, β}];
(* This specifies the PBW ordering *)
Do[B = MatrixExp[SparseArray[{α, β} → εβ,α, {n, n}]].B, {α, 2, n}, {β, 1, α - 1}];
MatrixForm /@ {A, B}
```

```
Out[*]=
```

$$\left\{ \left( \begin{array}{cc} e^{\xi_{1,1}} & e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \\ 0 & e^{\xi_{2,2}} \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ \epsilon \eta_{1,2} & 1 \end{array} \right) \right\}$$

```
In[*]:= mat = CF[
  (A /. ξαβ → ξ1αβ) . (B /. ηαβ → η1αβ)
]
```

```
Out[*]=
```

$$\left\{ \left\{ e^{\xi_{1,1}} + e^{\xi_{1,1} + \xi_{2,2}} \in \eta_{1,2} \xi_{1,2}, e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \right\}, \left\{ e^{\xi_{2,2}} \in \eta_{1,2}, e^{\xi_{2,2}} \right\} \right\}$$

```
In[*]:= mat // MatrixForm
```

```
Out[*]//MatrixForm=
```

$$\left( \begin{array}{cc} e^{\xi_{1,1}} + e^{\xi_{1,1} + \xi_{2,2}} \in \eta_{1,2} \xi_{1,2} & e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \\ e^{\xi_{2,2}} \in \eta_{1,2} & e^{\xi_{2,2}} \end{array} \right)$$

```
In[*]:= {LU, p, c} = Simplify@LUdecomposition[mat, Pivoting → False]
```

```
Out[*]=
```

$$\left\{ \left\{ \left\{ e^{\xi_{1,1}} \left( 1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2} \right), e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \right\}, \left\{ \frac{e^{-\xi_{1,1}} \in \eta_{1,2}}{e^{-\xi_{2,2}} + \in \eta_{1,2} \xi_{1,2}}, \frac{e^{\xi_{2,2}}}{1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2}} \right\} \right\}, \{1, 2\}, \emptyset \right\}$$

```
In[*]:= LU // MatrixForm
```

```
Out[*]//MatrixForm=
```

$$\left( \begin{array}{cc} e^{\xi_{1,1}} \left( 1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2} \right) & e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \\ \frac{e^{-\xi_{1,1}} \in \eta_{1,2}}{e^{-\xi_{2,2}} + \in \eta_{1,2} \xi_{1,2}} & \frac{e^{\xi_{2,2}}}{1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2}} \end{array} \right)$$

```
In[*]:= MatrixForm[L = LowerTriangularize[LU, -1] + IdentityMatrix[n]]
```

```
Out[*]//MatrixForm=
```

$$\left( \begin{array}{cc} 1 & 0 \\ \frac{e^{-\xi_{1,1}} \in \eta_{1,2}}{e^{-\xi_{2,2}} + \in \eta_{1,2} \xi_{1,2}} & 1 \end{array} \right)$$

```
In[*]:= MatrixForm[U = UpperTriangularize[LU]]
```

```
Out[*]//MatrixForm=
```

$$\left( \begin{array}{cc} e^{\xi_{1,1}} \left( 1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2} \right) & e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \\ 0 & \frac{e^{\xi_{2,2}}}{1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2}} \end{array} \right)$$

```
In[*]:= Simplify[L.U]
```

```
Out[*]=
```

$$\left\{ \left\{ e^{\xi_{1,1}} \left( 1 + e^{\xi_{2,2}} \in \eta_{1,2} \xi_{1,2} \right), e^{\xi_{1,1} + \xi_{2,2}} \xi_{1,2} \right\}, \left\{ e^{\xi_{2,2}} \in \eta_{1,2}, e^{\xi_{2,2}} \right\} \right\}$$

In[\*]:= **Simplify[L.U - mat]**

Out[\*]=

$\{\{0, 0\}, \{0, 0\}\}$

In[\*]:= **Series[L, {ε, 0, 3}] // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ e^{-\xi_{1,1} + \xi_{1,2}} \eta_{1,2} \epsilon - e^{-\xi_{1,1} + 2\xi_{1,2}} \eta_{1,2}^2 \xi_{1,2} \epsilon^2 + e^{-\xi_{1,1} + 3\xi_{1,2}} \eta_{1,2}^3 \xi_{1,2}^2 \epsilon^3 + 0[\epsilon]^4 & 1 \end{pmatrix}$$

In[\*]:= **Series[U, {ε, 0, 3}] // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} e^{\xi_{1,1}} + e^{\xi_{1,1} + \xi_{1,2}} \eta_{1,2} \xi_{1,2} \epsilon + 0[\epsilon]^4 & e^{\xi_{1,1} + \xi_{1,2}} \xi_{1,2} \\ 0 & e^{\xi_{1,2}} - e^{2\xi_{1,2}} \eta_{1,2} \xi_{1,2} \epsilon + e^{3\xi_{1,2}} \eta_{1,2}^2 \xi_{1,2}^2 \epsilon^2 - e^{4\xi_{1,2}} \eta_{1,2}^3 \xi_{1,2}^3 \epsilon^3 \end{pmatrix}$$

In[\*]:= **l = Simplify[MatrixLog[L]]**

Out[\*]=

$$\{\{0, 0\}, \left\{ \frac{e^{-\xi_{1,1}} \epsilon \eta_{1,2}}{e^{-\xi_{1,2}} + \epsilon \eta_{1,2} \xi_{1,2}}, 0 \right\}\}$$

In[\*]:= **MatrixLog[U]**

Out[\*]=

$$\left\{ \left\{ \text{Log} \left[ e^{\xi_{1,1}} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right) \right], - \frac{e^{\xi_{1,1} + \xi_{1,2}} \text{Log} \left[ \frac{e^{\xi_{1,2}}}{1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2}} \right] \xi_{1,2} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right)}{-e^{\xi_{1,2}} + e^{\xi_{1,1}} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right)^2} + \frac{e^{\xi_{1,1} + \xi_{1,2}} \text{Log} \left[ e^{\xi_{1,1}} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right) \right] \xi_{1,2} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right)}{-e^{\xi_{1,2}} + e^{\xi_{1,1}} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right)^2} \right\}, \left\{ 0, \text{Log} \left[ \frac{e^{\xi_{1,2}}}{1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2}} \right] \right\} \right\}$$

In[\*]:= **LogReduce[ε\_] := ε /. c\_ \* Log[a\_] => Log@Factor[a^c] // . Log[a\_] + Log[b\_] => Log@Factor[a b];**

In[\*]:= **u = Simplify[**

**MatrixLog[U] // . {Log[a\_ b\_] => Log[a] + Log[b], Log[e^a\_] => a, Log[1/a\_] => -Log[a]}**

Out[\*]=

$$\left\{ \left\{ \text{Log} \left[ 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right] + \xi_{1,1}, \frac{e^{\xi_{1,1} + \xi_{1,2}} \xi_{1,2} \left( 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right) \left( 2 \text{Log} \left[ 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right] + \xi_{1,1} - \xi_{1,2} \right)}{e^{\xi_{1,1}} - e^{\xi_{1,2}} + 2 e^{\xi_{1,1} + \xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} + e^{\xi_{1,1} + 2\xi_{1,2}} \epsilon^2 \eta_{1,2}^2 \xi_{1,2}^2} \right\}, \left\{ 0, -\text{Log} \left[ 1 + e^{\xi_{1,2}} \epsilon \eta_{1,2} \xi_{1,2} \right] + \xi_{1,2} \right\} \right\}$$

In[\*]:= **Tr[l.Table[yα,β, {α, n}, {β, n}]]**

Out[\*]=

$$\frac{e^{-\xi_{1,1}} \epsilon \eta_{1,2}}{e^{-\xi_{1,2}} + \epsilon \eta_{1,2} \xi_{1,2}}$$

```

In[*]:= Tr[u.Table[xα,β, {α, n}, {β, n}]]
Out[*]=
x1,1 (Log[1 + eξ1,2,2 η1,2 ξ1,2] + ξ1,1) +

$$\frac{e^{\xi_{1,1} + \xi_{1,2,2}} x_{2,1} \xi_{1,2} (1 + e^{\xi_{1,2,2}} \eta_{1,2} \xi_{1,2}) (2 \text{Log}[1 + e^{\xi_{1,2,2}} \eta_{1,2} \xi_{1,2}] + \xi_{1,1} - \xi_{1,2,2})}{e^{\xi_{1,1}} - e^{\xi_{1,2,2}} + 2 e^{\xi_{1,1} + \xi_{1,2,2}} \eta_{1,2} \xi_{1,2} + e^{\xi_{1,1} + 2 \xi_{1,2,2}} \eta_{1,2}^2 \xi_{1,2}^2}$$
 +
x2,2 (-Log[1 + eξ1,2,2 η1,2 ξ1,2] + ξ1,2,2)

In[*]:= $k = 0
Out[*]=
0

In[*]:= Δ2Ed→r[A_] := Module[{k}, Ed→r @@ Table[SeriesCoefficient[A, {ε, 0, k}], {k, 0, $k}]];
In[*]:= sw[i_, j_] :=
Δ2E{i,j}→{i,j}[Tr[l.Table[yα,β[i], {α, n}, {β, n}] / ε + u.Table[xβ,α[j], {α, n}, {β, n}]] / .
{ξ1αβ → ξαβ[i], η1αβ → ηαβ[j]}]

In[*]:= sw[3, 4]
Out[*]=
E{3,4}→{3,4} [e-ξ1,1[3] + ξ2,2[3] y1,2[3] η1,2[4] + x1,1[4] ξ1,1[3] +

$$\frac{e^{\xi_{1,1}[3] + \xi_{2,2}[3]} x_{1,2}[4] \xi_{1,2}[3] (\xi_{1,1}[3] - \xi_{2,2}[3])}{e^{\xi_{1,1}[3]} - e^{\xi_{2,2}[3]}}$$
 + x2,2[4] ξ2,2[3]]

In[*]:= utm2[1, 2 → 3]
Out[*]=
E{1,2}→{3} [x1,1[3] ξ1,1[1] + x1,1[3] ξ1,1[2] +
e-ξ1,1[2] x1,2[3] ξ1,2[1] + e-ξ2,2[1] x1,2[3] ξ1,2[2] + x2,2[3] ξ2,2[1] + x2,2[3] ξ2,2[2]]

In[*]:= utm2[1, 2 → 3] // sw[3, 4]

```

» Input  $\left\{ x_{1,1}[\$[3]] \xi_{1,1}[\$[3]] + x_{1,2}[\$[3]] \xi_{1,2}[\$[3]] + x_{2,2}[\$[3]] \xi_{2,2}[\$[3]], \right.$   

$$\mathbb{E} \left[ \frac{e^{-\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} y_{1,2}[3] \eta_{1,2}[4] + x_{1,1}[\$[3]] \xi_{1,1}[1] + x_{1,1}[\$[3]] \xi_{1,1}[2] + x_{1,1}[4] \xi_{1,1}[\$[3]] + e^{-\xi_{1,1}[2]} x_{1,2}[\$[3]] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[\$[3]] \xi_{1,2}[2] + e^{\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} x_{1,2}[4] \xi_{1,1}[\$[3]] \xi_{1,2}[\$[3]]}{e^{\xi_{1,1}[\$[3]]} - e^{\xi_{2,2}[\$[3]]}} + x_{2,2}[\$[3]] \xi_{2,2}[1] + x_{2,2}[\$[3]] \xi_{2,2}[2] + x_{2,2}[4] \xi_{2,2}[\$[3]] - \frac{e^{\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} x_{1,2}[4] \xi_{1,2}[\$[3]] \xi_{2,2}[\$[3]]}{e^{\xi_{1,1}[\$[3]]} - e^{\xi_{2,2}[\$[3]]}} \right\}$$

» After Zip1  $\left\{ x_{1,1}[\$[3]] \xi_{1,1}[\$[3]] + x_{1,2}[\$[3]] \xi_{1,2}[\$[3]] + x_{2,2}[\$[3]] \xi_{2,2}[\$[3]], \right.$   

$$\mathbb{E} \left[ \frac{e^{-\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} y_{1,2}[3] \eta_{1,2}[4] + x_{1,1}[\$[3]] \xi_{1,1}[1] + x_{1,1}[\$[3]] \xi_{1,1}[2] + x_{1,1}[4] \xi_{1,1}[\$[3]] + e^{-\xi_{1,1}[2]} x_{1,2}[\$[3]] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[\$[3]] \xi_{1,2}[2] + e^{\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} x_{1,2}[4] \xi_{1,1}[\$[3]] \xi_{1,2}[\$[3]]}{e^{\xi_{1,1}[\$[3]]} - e^{\xi_{2,2}[\$[3]]}} + x_{2,2}[\$[3]] \xi_{2,2}[1] + x_{2,2}[\$[3]] \xi_{2,2}[2] + x_{2,2}[4] \xi_{2,2}[\$[3]] - \frac{e^{\xi_{1,1}[\$[3]] + \xi_{2,2}[\$[3]]} x_{1,2}[4] \xi_{1,2}[\$[3]] \xi_{2,2}[\$[3]]}{e^{\xi_{1,1}[\$[3]]} - e^{\xi_{2,2}[\$[3]]}} \right\}$$

» vs  $\{x_{1,1}[\$[3]], x_{2,2}[\$[3]], \xi_{1,1}[\$[3]], \xi_{2,2}[\$[3]]\}$

» F  $\{\{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{1, 0, 0, 0\}, \{0, 1, 0, 0\}\}$

⋮ Power: Infinite expression  $\frac{1}{0}$  encountered.

⋮ Power: Infinite expression  $\frac{1}{0^2}$  encountered.

⋮ Infinity: Indeterminate expression  $0 \text{ ComplexInfinity } x_{1,2}[4] \xi_{1,2}[\$[3]]$  encountered.

⋮ Power: Infinite expression  $\frac{1}{0}$  encountered.

⋮ General: Further output of Power::infty will be suppressed during this calculation.

⋮ Infinity: Indeterminate expression  $0 \text{ ComplexInfinity } x_{1,2}[4] \xi_{1,2}[\$[3]]$  encountered.

⋮ Infinity: Indeterminate expression  $0 \text{ ComplexInfinity } x_{1,2}[4] \xi_{1,2}[\$[3]]$  encountered.

⋮ General: Further output of Infinity::indet will be suppressed during this calculation.

» Y  $\{\xi_{1,1}[1] + \xi_{1,1}[2], \xi_{2,2}[1] + \xi_{2,2}[2], \text{Indeterminate}, \text{Indeterminate}\}$

» After Zip2  $\{x_{1,1}[\$[3]] \xi_{1,1}[\$[3]] + x_{1,2}[\$[3]] \xi_{1,2}[\$[3]] + x_{2,2}[\$[3]] \xi_{2,2}[\$[3]], \mathbb{E}[\text{Indeterminate}]\}$

Out[\*]=

\$Aborted