

Quantum \mathfrak{gl}_N : Nilpotent Parts and PBW Coordinates

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This is a convention sheet for the positive and negative nilpotent parts of the Drinfeld–Jimbo quantum group. The point is that, after choosing an order of roots, these quantum nilpotent parts are *visibly polynomial as vector spaces*.

1. Strictly upper-triangular generators

Let $U_q(\mathfrak{n}_+)$ denote the positive nilpotent part. We write its root vectors as

$$x_{ab} \quad (1 \leq a < b \leq N),$$

where the simple generators are

$$x_{a,a+1} \quad (1 \leq a < N).$$

For $a < b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac} = [x_{ac}, x_{cb}]_{q^{-1}}, \quad a < c < b. \quad (1)$$

With the usual Drinfeld–Jimbo conventions, the right-hand side is independent of the choice of intermediate index c . Equivalently, one may take the adjacent recursion

$$x_{ab} = x_{a,a+1}x_{a+1,b} - q^{-1}x_{a+1,b}x_{a,a+1}.$$

For example,

$$x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12},$$

and

$$x_{14} = x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}.$$

2. Strictly lower-triangular generators

Similarly, let $U_q(\mathfrak{n}_-)$ denote the negative nilpotent part. Write its root vectors as

$$y_{ba} \quad (1 \leq a < b \leq N),$$

with simple generators $y_{a+1,a}$. The analogous recursion is

$$y_{ba} = y_{bc}y_{ca} - q^{-1}y_{ca}y_{bc}, \quad a < c < b. \quad (2)$$

Thus $y_{31} = y_{32}y_{21} - q^{-1}y_{21}y_{32}$, etc. Some authors use the opposite normalization or replace q^{-1} by q ; this changes the named root vectors but not the PBW/vector-space statement below.

3. Convex orderings of roots

A total ordering $<$ on the set Φ_+ of positive roots is called *convex* if whenever

$$\alpha, \beta, \alpha + \beta \in \Phi_+ \quad \text{and} \quad \alpha < \beta,$$

then

$$\alpha < \alpha + \beta < \beta. \quad (3)$$

For type A_{N-1} , identify the positive root $\varepsilon_a - \varepsilon_b$ with the pair (a, b) , where $a < b$. A standard convex ordering is

$$(1, 2) < (1, 3) < \cdots < (1, N) < (2, 3) < \cdots < (2, N) < \cdots < (N-1, N).$$

Another common convex ordering is the column order

$$(1, 2) < (2, 3) < (1, 3) < (3, 4) < (2, 4) < (1, 4) < \cdots.$$

Different convex orders give different PBW bases, but all give the same vector-space conclusion.

4. PBW bases

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\prod_{a < b}^{\rightarrow} x_{ab}^{m_{ab}}, \quad m_{ab} \in \mathbb{Z}_{\geq 0}, \quad (4)$$

form a basis of $U_q(\mathfrak{n}_+)$. Here the arrow means: multiply the factors in the chosen convex order.

Likewise, the ordered monomials

$$\prod_{a < b}^{\rightarrow} y_{ba}^{n_{ab}}, \quad n_{ab} \in \mathbb{Z}_{\geq 0}, \quad (5)$$

form a basis of $U_q(\mathfrak{n}_-)$, with the same or opposite chosen convention for ordering.

5. Polynomial vector-space identification

Therefore there are explicit vector-space isomorphisms

$$\mathbb{Q}(q)[X_{ab} : a < b] \xrightarrow{\sim} U_q(\mathfrak{n}_+), \quad \prod_{a < b}^{\rightarrow} X_{ab}^{m_{ab}} \mapsto \prod_{a < b}^{\rightarrow} x_{ab}^{m_{ab}},$$

and

$$\mathbb{Q}(q)[Y_{ba} : a < b] \xrightarrow{\sim} U_q(\mathfrak{n}_-), \quad \prod_{a < b}^{\rightarrow} Y_{ba}^{n_{ab}} \mapsto \prod_{a < b}^{\rightarrow} y_{ba}^{n_{ab}}.$$

These are **not** algebra isomorphisms to commutative polynomial rings. They are PBW, hence vector-space, identifications. The multiplication on the right is the noncommutative q -deformed multiplication, governed by straightening relations.

6. Tiny case: $N = 3$

For $N = 3$, the upper nilpotent part has root vectors

$$x_{12}, \quad x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12}, \quad x_{23}.$$

For the convex order $(1, 2) < (1, 3) < (2, 3)$, the PBW basis is

$$x_{12}^a x_{13}^b x_{23}^c, \quad a, b, c \geq 0.$$

Hence, as a vector space,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[X_{12}, X_{13}, X_{23}].$$

Standard references: Lusztig, *Introduction to Quantum Groups*; Jantzen, *Lectures on Quantum Groups*; Chari–Pressley, *A Guide to Quantum Groups*.