Quantum GL(N): Nilpotent Parts and PBW Bases

For the Drinfeld–Jimbo quantum group $U_q(gl_N)$, let $U_q(n_+)$ and $U_q(n_-)$ denote the positive and negative nilpotent parts.

Simple generators.

$x_{(a,a+1)} := E_a, \quad y_{(a+1,a)} := F_a, \quad 1 \le a \le N-1.$

Root vectors.

For $a<b$, define recursively

$$x_{ab} = x_{ac} x_{cb} - q^{-1} x_{cb} x_{ac},$$

where $a<c<b$. The resulting element is independent of the choice of c. Similarly,

$$y_{ba} = y_{bc} y_{ca} - q^{-1} y_{ca} y_{bc}.$$

Examples.

$$x_{13} = x_{12} x_{23} - q^{-1} x_{23} x_{12},$$

$$x_{14} = x_{12} x_{24} - q^{-1} x_{24} x_{12} = x_{13} x_{34} - q^{-1} x_{34} x_{13}.$$

Convex ordering of positive roots.

A total ordering < on the positive roots Φ_+ is called *convex* if whenever $\alpha, \beta, \alpha+\beta \in \Phi_+$ and $\alpha<\beta$, then

$$\alpha < \alpha+\beta < \beta.$$

For gl_N, identifying positive roots with pairs (a,b), $a<b$, a standard convex ordering is

$$(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots < (N-1,N).$$

PBW basis.

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\prod^{\rightarrow} x_{ab}^{m_{ab}}, \quad m_{ab} \ge 0,$$

form a basis of $U_q(n_+)$. Likewise,

$$\prod^{\rightarrow} y_{ba}^{n_{ab}}$$

form a basis of $U_q(n_-)$.

Vector-space identification.

Hence, as vector spaces,

$$U_q(n_+) \cong Q(q)[x_{ab} : a<b],$$

$$U_q(n_-) \cong Q(q)[y_{ba} : a<b],$$

via ordered PBW monomials. This is not an algebra isomorphism: the multiplication is q-deformed and noncommutative.

`Standard references: Lusztig, Jantzen, Chari–Pressley.`