

# Quantum $\mathfrak{gl}_N$ : Nilpotent Parts and PBW Coordinates

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This is a convention sheet for the positive and negative nilpotent parts of the Drinfeld–Jimbo quantum group. The point is that, after choosing an order of roots, these quantum nilpotent parts are *visibly polynomial as vector spaces*.

## 1. Strictly upper-triangular generators

Let  $U_q(\mathfrak{n}_+)$  denote the positive nilpotent part. We write its root vectors as

$$x_{ab} \quad (1 \leq a < b \leq N),$$

where the simple generators are

$$x_{a,a+1} \quad (1 \leq a < N).$$

For  $a < b$ , define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac} = [x_{ac}, x_{cb}]_{q^{-1}}, \quad a < c < b. \quad (1)$$

With the usual Drinfeld–Jimbo conventions, the right-hand side is independent of the choice of intermediate index  $c$ . Equivalently, one may take the adjacent recursion

$$x_{ab} = x_{a,a+1}x_{a+1,b} - q^{-1}x_{a+1,b}x_{a,a+1}.$$

For example,

$$x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12},$$

and

$$x_{14} = x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}.$$

## 2. Strictly lower-triangular generators

Similarly, let  $U_q(\mathfrak{n}_-)$  denote the negative nilpotent part. We label its root vectors by the same positive-root indices as the upper ones:

$$y_{ab} \quad (1 \leq a < b \leq N),$$

so that  $y_{ab}$  is the adjoint/lower-triangular partner of  $x_{ab}$ . Thus the simple generators are  $y_{a,a+1}$ , meaning the usual lowering generators paired with  $x_{a,a+1}$ . The analogous recursion is

$$y_{ab} = y_{cb}y_{ac} - q^{-1}y_{ac}y_{cb}, \quad a < c < b. \quad (2)$$

This is what one gets by applying an anti-involution/adjoint with  $x_{ab}^* = y_{ab}$  to (1). For example,

$$y_{13} = y_{23}y_{12} - q^{-1}y_{12}y_{23}.$$

Some authors use the opposite normalization or replace  $q^{-1}$  by  $q$ ; this changes the named root vectors but not the PBW/vector-space statement below.

### 3. Convex orderings of roots

A total ordering  $<$  on the set  $\Phi_+$  of positive roots is called *convex* if whenever

$$\alpha, \beta, \alpha + \beta \in \Phi_+ \quad \text{and} \quad \alpha < \beta,$$

then

$$\alpha < \alpha + \beta < \beta. \tag{3}$$

For type  $A_{N-1}$ , identify the positive root  $\varepsilon_a - \varepsilon_b$  with the pair  $(a, b)$ , where  $a < b$ . A standard convex ordering is

$$(1, 2) < (1, 3) < \cdots < (1, N) < (2, 3) < \cdots < (2, N) < \cdots < (N-1, N).$$

Another common convex ordering is the column order

$$(1, 2) < (2, 3) < (1, 3) < (3, 4) < (2, 4) < (1, 4) < \cdots .$$

Different convex orders give different PBW bases, but all give the same vector-space conclusion.

### 4. PBW bases

Fix a convex ordering of the positive roots. Then the ordered monomials

$$\overrightarrow{\prod}_{a < b} x_{ab}^{m_{ab}}, \quad m_{ab} \in \mathbb{Z}_{\geq 0}, \tag{4}$$

form a basis of  $U_q(\mathfrak{n}_+)$ . Here the arrow means: multiply the factors in the chosen convex order.

Likewise, the ordered monomials

$$\overrightarrow{\prod}_{a < b} y_{ab}^{n_{ab}}, \quad n_{ab} \in \mathbb{Z}_{\geq 0}, \tag{5}$$

form a basis of  $U_q(\mathfrak{n}_-)$ , with the chosen convention for ordering.

### 5. Polynomial vector-space identification

Therefore there are explicit vector-space isomorphisms

$$\mathbb{Q}(q)[X_{ab} : a < b] \xrightarrow{\sim} U_q(\mathfrak{n}_+), \quad \overrightarrow{\prod}_{a < b} X_{ab}^{m_{ab}} \mapsto \overrightarrow{\prod}_{a < b} x_{ab}^{m_{ab}},$$

and

$$\mathbb{Q}(q)[Y_{ab} : a < b] \xrightarrow{\sim} U_q(\mathfrak{n}_-), \quad \overrightarrow{\prod}_{a < b} Y_{ba}^{n_{ab}} \mapsto \overrightarrow{\prod}_{a < b} y_{ab}^{n_{ab}}.$$

These are **not** algebra isomorphisms to commutative polynomial rings. They are PBW, hence vector-space, identifications. The multiplication on the right is the noncommutative  $q$ -deformed multiplication, governed by straightening relations.

## 6. Universal $R$ -matrix in the same PBW coordinates

With the same convex ordering and the same paired root vectors  $x_{ab} \leftrightarrow y_{ab}$ , the universal  $R$ -matrix has the ordered factorization

$$\mathcal{R} = \mathcal{R}_0 \mathcal{R}_{\text{nil}}, \quad \mathcal{R}_{\text{nil}} = \prod_{a < b}^{\rightarrow} \exp_{q^{-2}}((q - q^{-1})x_{ab} \otimes y_{ab}). \quad (6)$$

Here

$$\exp_t(z) := \sum_{m \geq 0} \frac{z^m}{[m]_t!}, \quad [m]_t! := \prod_{j=1}^m \frac{1-t^j}{1-t}.$$

Equivalently, the nilpotent factor expands in the PBW basis as

$$\mathcal{R}_{\text{nil}} = \sum_{(m_{ab})} \left( \prod_{a < b}^{\rightarrow} \frac{(q - q^{-1})^{m_{ab}}}{[m_{ab}]_{q^{-2}}!} \right) \left( \prod_{a < b}^{\rightarrow} x_{ab}^{m_{ab}} \right) \otimes \left( \prod_{a < b}^{\rightarrow} y_{ab}^{m_{ab}} \right), \quad (7)$$

where the sum is over all functions  $(a, b) \mapsto m_{ab} \in \mathbb{Z}_{\geq 0}$ . The order of the factors in both tensor components is the chosen convex order; using the opposite convention for the lower PBW order moves the reversal into the displayed formula.

For  $\mathfrak{gl}_N$ , the Cartan factor may be written, in the diagonal basis  $h_1, \dots, h_N$ , as

$$\mathcal{R}_0 = q^{\sum_{i=1}^N h_i \otimes h_i}, \quad (8)$$

up to the usual completion and normalization conventions. In the  $\mathfrak{sl}_N$  normalization one replaces this by the factor determined by the inverse Cartan matrix.

## 7. Tiny case: $N = 3$

For  $N = 3$ , the upper nilpotent part has root vectors

$$x_{12}, \quad x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12}, \quad x_{23}.$$

For the convex order  $(1, 2) < (1, 3) < (2, 3)$ , the PBW basis is

$$x_{12}^a x_{13}^b x_{23}^c, \quad a, b, c \geq 0.$$

Hence, as a vector space,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[X_{12}, X_{13}, X_{23}].$$

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Standard references: Lusztig, *Introduction to Quantum Groups*; Jantzen, *Lectures on Quantum Groups*; Chari–Pressley, *A Guide to Quantum Groups*.