Quantum $\mathfrak{gl}_N$ Cheat Sheet

For the Drinfeld–Jimbo quantum group $U_q(\mathfrak{gl}_N)$, let $U_q(\mathfrak{n}_+)$ and $U_q(\mathfrak{n}_-)$ denote the positive and negative nilpotent parts.

Simple generators.

$$x_{a,a+1} := E_a, \qquad y_{a,a+1} := E_a \qquad {\scriptstyle (1\le a\le N-1).}$$

Root vectors.
For $a < b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac},$$

where $a < c < b$. The resulting element is independent of the choice of c. Similarly,

$$y_{ab} = y_{ac}y_{cb} - q^{-1}y_{cb}y_{ac}.$$

Examples.

$$x_{13} = x_{12}x_{23} - q^{-1}x_{23}x_{12},$$
$$x_{14} = x_{12}x_{24} - q^{-1}x_{24}x_{12} = x_{13}x_{34} - q^{-1}x_{34}x_{13}.$$

Convex ordering of positive roots.
A total ordering $<$ on the positive roots Φ_+ is called *convex* if whenever

$$\alpha, \beta, \alpha+\beta \in \Phi_+ \qquad \text{and} \qquad \alpha < \beta,$$

then

$$\alpha < \alpha+\beta < \beta.$$

For $\mathfrak{gl}_N$, identifying positive roots with pairs (a, b), $a < b$, a standard convex ordering is

$$(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots < (N-1,N).$$

PBW basis.
Fix a convex ordering of the positive roots. Then the ordered monomials

$$\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}} \qquad (m_{ab} \ge 0)$$

form a basis of $U_q(\mathfrak{n}_+)$. Likewise,

$$\overrightarrow{\prod_{a<b}} y_{ab}^{n_{ab}}$$

form a basis of $U_q(\mathfrak{n}_-)$.

Vector-space identification.
Hence, as vector spaces,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[x_{ab} : a < b],$$
$$U_q(\mathfrak{n}_-) \cong \mathbb{Q}(q)[y_{ab} : a < b],$$

via ordered PBW monomials.

This is not an algebra isomorphism: the multiplication is q-deformed and noncommutative.

Universal R-matrix.
With the same convex ordering,

$$\mathcal{R} = q^{\sum h_i \otimes h^i} \prod_{a<b}^{\rightarrow} \exp_q\big((q - q^{-1}) x_{ab} \otimes y_{ab}\big) .$$

The product is taken in the chosen convex order.