Quantum $\mathfrak{gl}_N$ Cheat Sheet

Fix a convex ordering of positive roots.

A total ordering $<$ on Φ_+ is called *convex* if whenever

$$\alpha, \beta, \alpha+\beta \in \Phi_+ \qquad \text{and} \qquad \alpha < \beta,$$

then

$$\alpha < \alpha+\beta < \beta.$$

For $\mathfrak{gl}_N$, identifying positive roots with pairs (a,b), $a<b$, a standard convex ordering is

$$(1,2) < (1,3) < \cdots < (1,N) < (2,3) < \cdots < (N-1,N).$$

The simple generators are

$$x_{a,a+1} := E_a, \qquad y_{a,a+1} := F_a.$$

For $a<b$, define recursively

$$x_{ab} = x_{ac}x_{cb} - q^{-1}x_{cb}x_{ac}, \qquad a<c<b.$$

The result is independent of the choice of c.

Likewise define

$$y_{ab} = y_{ac}y_{cb} - q^{-1}y_{cb}y_{ac}.$$

The ordered PBW monomials

$$\overrightarrow{\prod_{a<b}} x_{ab}^{m_{ab}}$$

form a basis of $U_q(\mathfrak{n}_+)$, while

$$\overrightarrow{\prod_{a<b}} y_{ab}^{n_{ab}}$$

form a basis of $U_q(\mathfrak{n}_-)$.

Hence, as vector spaces,

$$U_q(\mathfrak{n}_+) \cong \mathbb{Q}(q)[x_{ab} : a<b],$$
$$U_q(\mathfrak{n}_-) \cong \mathbb{Q}(q)[y_{ab} : a<b].$$

The universal R-matrix is

$$\mathcal{R} = q^{\sum h_i \otimes h^i} \overrightarrow{\prod_{a<b}} \exp_q\big((q-q^{-1})x_{ab} \otimes y_{ab}\big),$$

with the product taken in the chosen convex ordering.

The q-exponential is

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} z^n,$$

where

$$[n]_q! = \prod_{k=1}^{n} \frac{q^k - q^{-k}}{q - q^{-1}}.$$