In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
```

tex

We start by loading the package \verb$KnotTheory`$ --- it is only needed because it had many specific knots pre-defined:

pdf

In[∘]:=
```
<< KnotTheory`
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

tex

Next we quietly define the commands \verb$Rot$, used to compute rotation numbers, and \verb$PolyPlot$, used to plot polynomials as bar codes and as hexagonal QR codes. Neither is a part of the core of the computation of Θ, so neither is shown; yet we do show some usage examples.

pdf

In[∘]:=
```
(* Rot suppressed *)
```

In[∘]:=
```
Rot[pd_PD] := Module[{n, xs, x, rots, Xp, Xm, front = {1}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x⟦4⟧, x⟦1⟧]   PositiveQ@x
                         { Xm[x⟦2⟧, x⟦1⟧]   True        ];
   For[k = 1, k ≤ 2 n, ++k,
    If[FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
           Xp[k, l_] | Xm[l_, k] :> {l + 1, k + 1, -l},
           Xp[l_, k] | Xm[k, l_] :> (++rots⟦l⟧; {-l, k + 1, l + 1}),
           _Xp | _Xm :> {}
          }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots⟦k⟧;
    ]
   ];
   {xs /. {Xp[i_, j_] :> {+1, i, j}, Xm[i_, j_] :> {-1, i, j}}, rots} ];
Rot[K_] := Rot[PD[K]];
```

pdf

In[∘]:=
```
Rot[Mirror@Knot[3, 1]]
```

Out[∘]=

pdf

```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0}}
```

tex

We urge the reader to compare the above output with the knot diagram in Section~\ref{ssec:OldFormulas}.

pdf

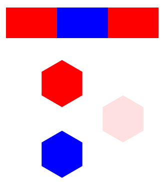
In[◦]:=
```
(* PolyPlot suppressed *)
```

In[◦]:=
```
PolyPlot[{Δ_, θ_}, opts___Rule] := Module[
  {crs, m, m1, m2, maxc, minc, s, rect, hex},
  GraphicsColumn[{
    rect = {{0, 0}, {1, 0}, {1, 1}, {0, 1}};
    hex = Table[{Cos[α], Sin[α]} / Cos[2 π / 12] / 2, {α, 2 π / 12, 2 π, 2 π / 6}];
    If[Expand[Δ] === 0, Graphics[],
     m = Max[-Exponent[Δ, T, Min], Exponent[Δ, T, Max]];
     crs = CoefficientRules[T^m Δ, {T}];
     maxc = N@Log@Max@Abs[Last /@ crs];
     minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
     If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
     Graphics[crs /. ({x_} → c_) :→ {
         Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
         Tooltip[Polygon[({x + m - 1 / 2, 0} + #) & /@ rect], c T^(x-m)]
        }, AspectRatio → Min[1 / 5, 1 / Sqrt[m + 1]],
      ImagePadding → None, PlotRangePadding → None]
    ],
    If[Expand[θ] === 0, Graphics[{White, Disk[]}],
     m1 = Max[-Exponent[θ, T1, Min], Exponent[θ, T1, Max]];
     m2 = Max[-Exponent[θ, T2, Min], Exponent[θ, T2, Max]];
     crs = CoefficientRules[T1^m1 T2^m2 θ, {T1, T2}];
     maxc = N@Log@Max@Abs[Last /@ crs];
     minc = N@Log@Min@Select[Abs[Last /@ crs], # > 0 &];
     If[minc == maxc, s[_] = 0, s[c_] := s[c] = (maxc - Log@c) / (maxc - minc)];
     Graphics[{{White, Disk[{0, 0}, 1 + Cos[2 π / 12] Norm[{m1, m2}] / Sqrt[2]]},
       crs /. ({x1_, x2_} → c_) :→ {
          Lighter[Which[c == 0, White, c > 0, Red, c < 0, Blue], 0.88 s[Abs@c]],
          Tooltip[Polygon[(({{1, -1 / 2}, {0, Sqrt[3] / 2}}).{x1 - m1, x2 - m2} + #) & /@ hex], c T1^(x1-m1) T2^(x2-m2)]
         }
      }, ImagePadding → None, PlotRangePadding → None]
    ]
   }, Spacings → Scaled@0.08, ImagePadding → None, PlotRangePadding → None, opts]];
```

pdf

In[∘]:= $\text{PolyPlot}\left[\left\{T - 1 + T^{-1},\ T_1 + 2\,T_2 - 2\,T_1^{-1}\,T_2^{-1}\right\},\ \text{ImageSize} \to \text{Tiny}\right]$

Out[∘]=
pdf



In[∘]:= $T_3 = T_1\,T_2;$

In[∘]:=

$$R_{11}[\{s\_,\ i\_,\ j\_\}] = s\left(1/2 - g_{3ii} + T_2^s\,g_{1ii}\,g_{2ji} - g_{1ii}\,g_{2jj} - \left(T_2^s - 1\right)g_{2ji}\,g_{3ii} + 2\,g_{2jj}\,g_{3ii} - \left(1 - T_3^s\right)g_{2ji}\,g_{3ji} - g_{2ii}\,g_{3jj} - T_2^s\,g_{2ji}\,g_{3jj} + g_{1ii}\,g_{3jj} + \left(\left(T_1^s - 1\right)g_{1ji}\left(T_2^{2\,s}\,g_{2ji} - T_2^s\,g_{2jj} + T_2^s\,g_{3jj}\right) + \left(T_3^s - 1\right)g_{3ji}\left(1 - T_2^s\,g_{1ii} - \left(T_1^s - 1\right)\left(T_2^s + 1\right)g_{1ji} + \left(T_2^s - 2\right)g_{2jj} + g_{2ij}\right)\right) \Big/ \left(T_2^s - 1\right)\right)$$

Out[∘]=

$$s\left(\frac{1}{2} + T_2^s\,g_{1,i,i}\,g_{2,j,i} - g_{1,i,i}\,g_{2,j,j} - g_{3,i,i} - \left(-1 + T_2^s\right)g_{2,j,i}\,g_{3,i,i} + 2\,g_{2,j,j}\,g_{3,i,i} - \left(1 - (T_1\,T_2)^s\right)g_{2,j,i}\,g_{3,j,i} + g_{1,i,i}\,g_{3,j,j} - g_{2,i,i}\,g_{3,j,j} - T_2^s\,g_{2,j,i}\,g_{3,j,j} + \frac{1}{-1 + T_2^s}\left(\left(-1 + (T_1\,T_2)^s\right)\left(1 - T_2^s\,g_{1,i,i} - \left(-1 + T_1^s\right)\left(1 + T_2^s\right)g_{1,j,i} + g_{2,i,j} + \left(-2 + T_2^s\right)g_{2,j,j}\right)g_{3,j,i} + \left(-1 + T_1^s\right)g_{1,j,i}\left(T_2^{2\,s}\,g_{2,j,i} - T_2^s\,g_{2,j,j} + T_2^s\,g_{3,j,j}\right)\right)\right)$$

In[∘]:=

$$R_{12}[\{s0\_,\ i0\_,\ j0\_\},\ \{s1\_,\ i1\_,\ j1\_\}] := s1\left(T_1^{s0} - 1\right)\left(T_2^{s1} - 1\right)^{-1}\left(T_3^{s1} - 1\right)g_{1,j1,i0}\,g_{3,j0,i1}\left(\left(T_2^{s0}\,g_{2,i1,i0} - g_{2,i1,j0}\right) - \left(T_2^{s0}\,g_{2,j1,i0} - g_{2,j1,j0}\right)\right)$$

In[∘]:= $\Gamma_1[\varphi\_,\ k\_] = -\varphi / 2 + \varphi\,g_{3kk}$

Out[∘]=

$$-\frac{\varphi}{2} + \varphi\,g_{3,k,k}$$

In[∘]:=
```
Θ[K_] :=
  Module[{Cs, φ, n, A, s, i, j, k, Δ, G, Δν, gναβ, gEval, z, t = AbsoluteTime[], out},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                                 0     -1   ))];
   Δ = Factor[T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A]];
   Δν = Product[Δ /. T → T_ν, {ν, 3}];
   Print["After Δ: ", AbsoluteTime[] - t];
   G = Factor@Inverse[A];
   Print["After G: ", AbsoluteTime[] - t];
   gναβ = Dispatch@
     Flatten@Table[g_{ν,α,β} → (G[[α, β]] /. T → T_ν), {ν, 3}, {α, 2 n + 1}, {β, 2 n + 1}];
   gEval[ℰ_] := Expand[Δν (ℰ /. gναβ)];
   Print["After gEval: ", AbsoluteTime[] - t];
   z = Sum_{k1=1}^{n} Sum_{k2=1}^{n} gEval@R_12[Cs[[k1]], Cs[[k2]]];
   z += Sum_{k=1}^{n} gEval@R_11[Cs[[k]]]; z += Sum_{k=1}^{2 n} gEval@Γ_1[φ[[k]], k];
   Print["After z: ", AbsoluteTime[] - t];
   out = {Δ, z} // Factor;
   Print["After out: ", AbsoluteTime[] - t];
   out
  ];
```

In[∘]:=
```
Expand[Θ[Knot[3, 1]]]
```

After △: 0.0019521

After G: 0.0029286

After gEval: 0.0039052

After z: 0.0132342

After out: 0.0712492

Out[∘]=

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

In[∘]:= **Expand**[θ[**Knot**[**10, 165**]]]

Out[∘]=

$$\left\{-15-\frac{2}{T^2}+\frac{10}{T}+10\,T-2\,T^2,\right.$$

$$-1404-\frac{9}{T_1^4}-\frac{178}{T_1^3}+\frac{607}{T_1^2}+\frac{624}{T_1}+624\,T_1+607\,T_1^2-178\,T_1^3-9\,T_1^4-\frac{9}{T_2^4}-\frac{9}{T_1^4\,T_2^4}+\frac{44}{T_1^3\,T_2^4}-$$

$$\frac{65}{T_1^2\,T_2^4}+\frac{44}{T_1\,T_2^4}-\frac{178}{T_2^3}+\frac{44}{T_1^4\,T_2^3}-\frac{178}{T_1^3\,T_2^3}+\frac{104}{T_1^2\,T_2^3}+\frac{104}{T_1\,T_2^3}+\frac{44\,T_1}{T_2^3}+\frac{607}{T_2^2}-\frac{65}{T_1^4\,T_2^2}+\frac{104}{T_1^3\,T_2^2}+\frac{607}{T_1^2\,T_2^2}-$$

$$\frac{1041}{T_1\,T_2^2}+\frac{104\,T_1}{T_2^2}-\frac{65\,T_1^2}{T_2^2}+\frac{624}{T_2}+\frac{44}{T_1^4\,T_2}+\frac{104}{T_1^3\,T_2}-\frac{1041}{T_1^2\,T_2}+\frac{624}{T_1\,T_2}-\frac{1041\,T_1}{T_2}+\frac{104\,T_1^2}{T_2}+\frac{44\,T_1^3}{T_2}+$$

$$624\,T_2+\frac{44\,T_2}{T_1^3}+\frac{104\,T_2}{T_1^2}-\frac{1041\,T_2}{T_1}+624\,T_1\,T_2-1041\,T_1^2\,T_2+104\,T_1^3\,T_2+44\,T_1^4\,T_2+$$

$$607\,T_2^2-\frac{65\,T_2^2}{T_1^2}+\frac{104\,T_2^2}{T_1}-1041\,T_1\,T_2^2+607\,T_1^2\,T_2^2+104\,T_1^3\,T_2^2-65\,T_1^4\,T_2^2-178\,T_2^3+\frac{44\,T_2^3}{T_1}+$$

$$\left.104\,T_1\,T_2^3+104\,T_1^2\,T_2^3-178\,T_1^3\,T_2^3+44\,T_1^4\,T_2^3-9\,T_2^4+44\,T_1\,T_2^4-65\,T_1^2\,T_2^4+44\,T_1^3\,T_2^4-9\,T_1^4\,T_2^4\right\}$$

In[∘]:= θ[**Knot**[**3, 1**]]⟦**2**⟧ // **Expand**

KnotTheory: Loading precomputed data in PD4Knots`.

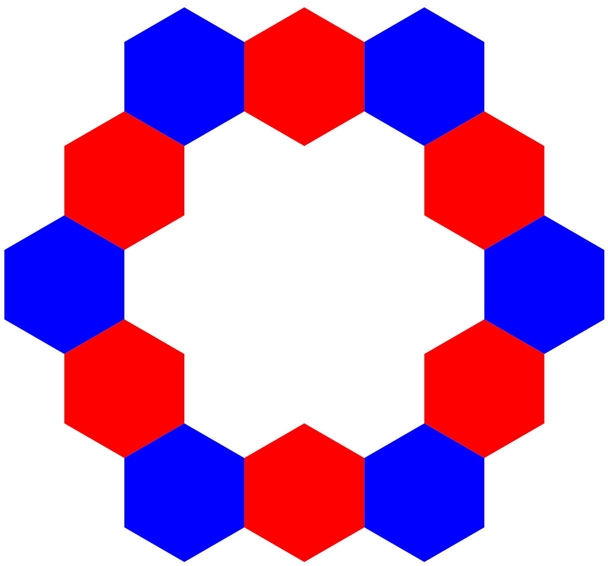
Out[∘]=

$$-\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2$$

In[∘]:= **PolyPlot**[θ[**Knot**[**3, 1**]]]

```
After △: 0.0019533

After G: 0.0035676

After gEval: 0.0045743

After z: 0.0143981

After out: 0.0405181
```

Out[ ]=

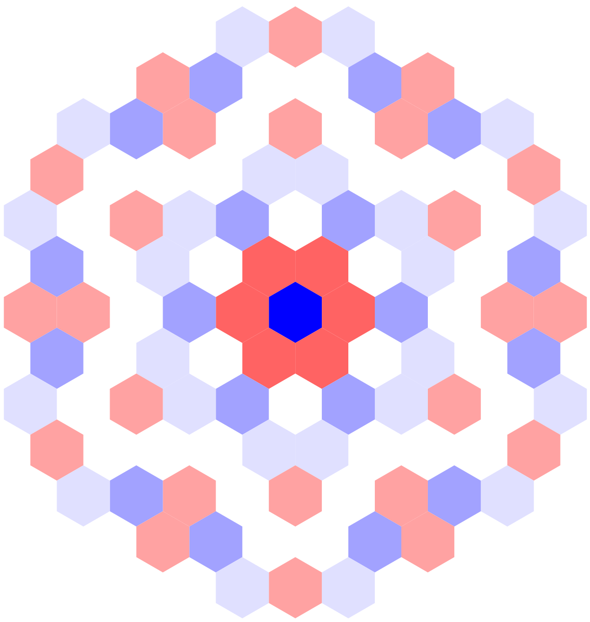


In[∘]:= `PolyPlot[θ[Knot["K11n34"]]]`

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

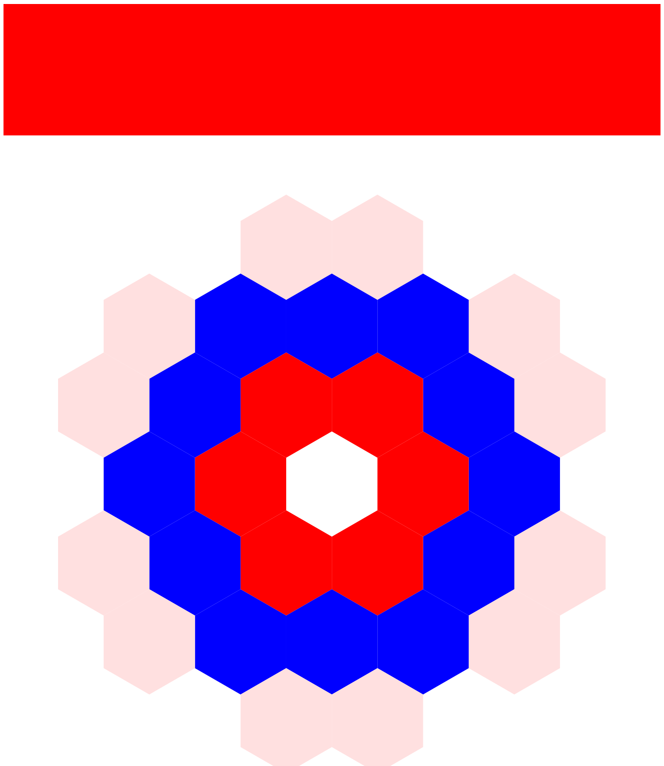
KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[∘]=

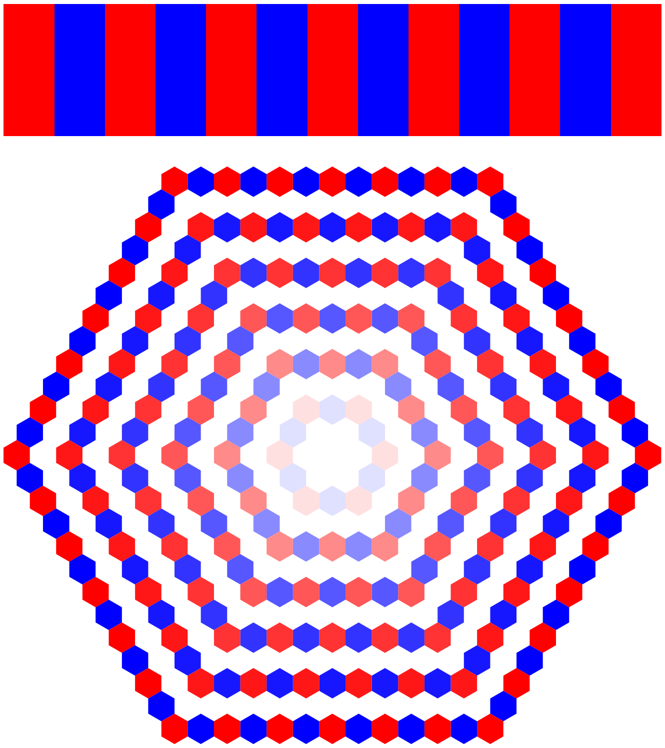


In[∘]:= PolyPlot[θ[Knot["K11n42"]]]

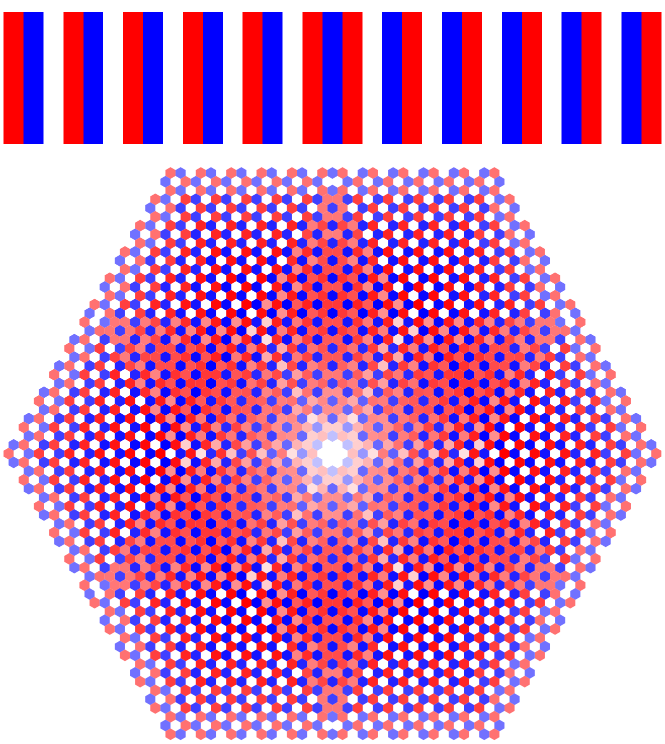
Out[∘]=



In[∘]:=
```
PolyPlot[θ[TorusKnot[13, 2]]]
```
Out[∘]=



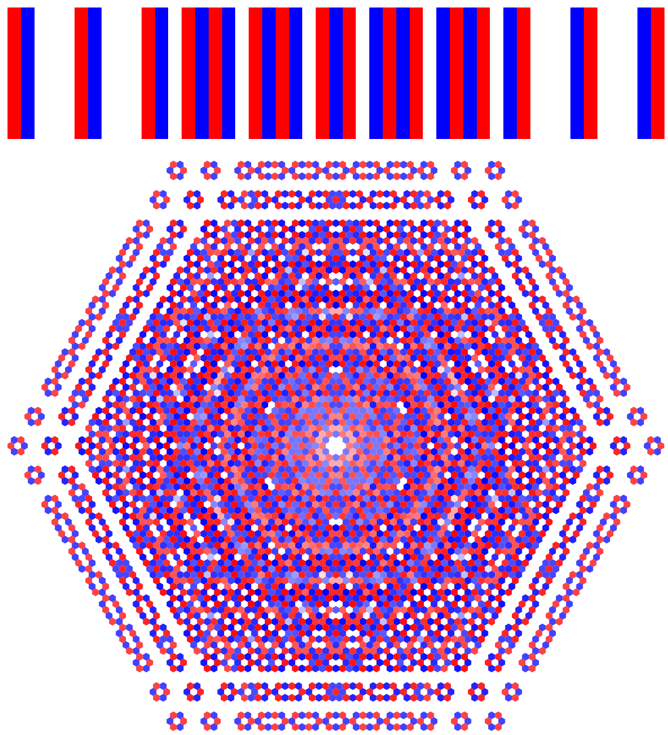
In[∘]:=
```
PolyPlot[θ[TorusKnot[17, 3]]]
```
Out[∘]=



In[ ]:= PolyPlot[θ[TorusKnot[13, 5]]]

```
After △: 0.2141511
After G: 17.9732460
After gEval: 20.1062978
After z: 20.1514453
After out: 93.0097957
```

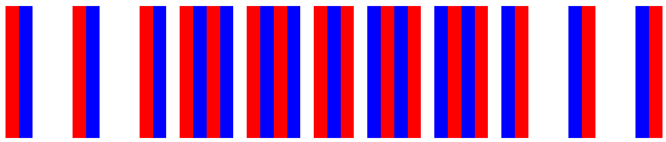
Out[ ]=

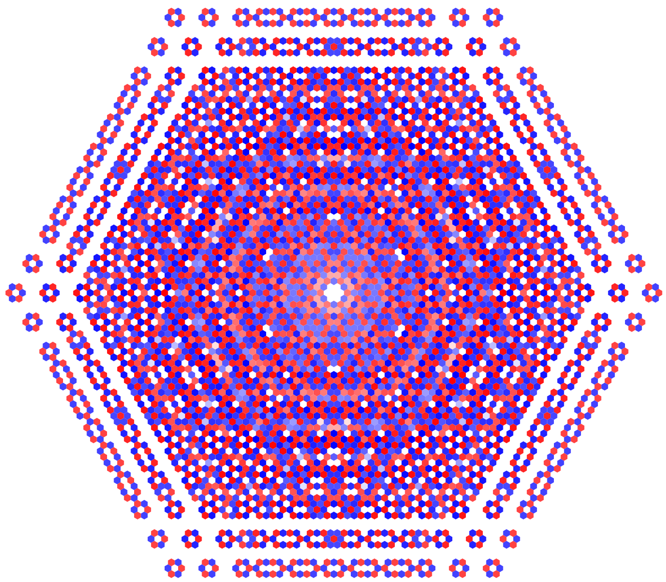


In[ ]:= PolyPlot[θ[TorusKnot[13, 5]]]

```
After △: 0.2155422

After G: 18.1539619

After gEval: 20.3798731

After z: 7366.6256760

After out: 8397.7290368
```

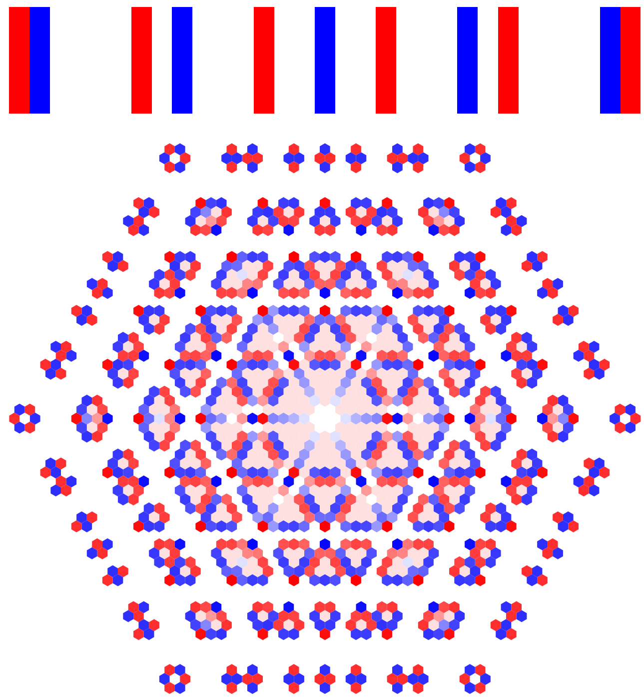
Out[∘]=





In[ ]:= **PolyPlot[θ[TorusKnot[7, 6]]]**

Out[ ]=



In[∘]:=
```
AbsoluteTiming[th = θ[TorusKnot[22, 7]]]
PolyPlot[th]
```

Out[∘]=

$$\Big\{4740.09,\ \Big\{\frac{1}{T^{63}}\left(1-T+T^2-T^3+T^4-T^5+T^6\right)\left(1-T+T^7-T^8+T^{11}-T^{12}+T^{14}-T^{15}+T^{18}-T^{19}+T^{21}-T^{23}+T^{25}-T^{26}+T^{28}-T^{30}+T^{32}-T^{34}+T^{35}-T^{37}+T^{39}-T^{41}+T^{42}-T^{45}+T^{46}-T^{48}+T^{49}-T^{52}+T^{53}-T^{59}+T^{60}\right)\left(1+T-T^7-T^8-T^{11}-T^{12}+T^{14}+T^{15}+T^{18}+T^{19}-T^{21}+T^{23}-T^{25}-T^{26}+T^{28}-T^{30}+T^{32}-T^{34}-T^{35}+T^{37}-T^{39}+T^{41}+T^{42}+T^{45}+T^{46}-T^{48}-T^{49}-T^{52}-T^{53}+T^{59}+T^{60}\right),$$

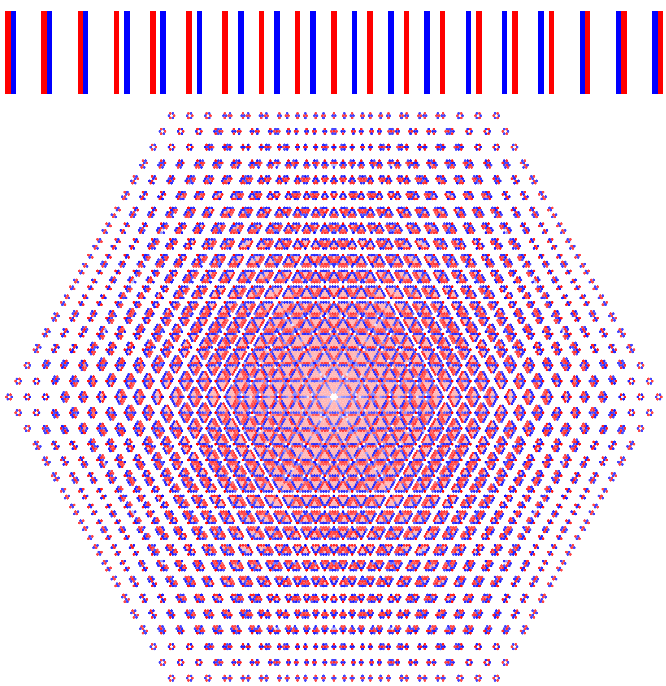
$$\frac{1}{T_1^{126}T_2^{126}}\left(63-63\,T_1+63\,T_1^7-63\,T_1^8+63\,T_1^{14}-63\,T_1^{15}+63\,T_1^{21}-63\,T_1^{23}+63\,T_1^{28}-63\,T_1^{30}+63\,T_1^{35}-63\,T_1^{37}+\right.$$

$$\cdots 30611 \cdots +63\,T_1^{210}T_2^{252}-63\,T_1^{215}T_2^{252}+63\,T_1^{217}T_2^{252}-63\,T_1^{222}T_2^{252}+63\,T_1^{224}T_2^{252}-63\,T_1^{229}T_2^{252}+$$

$$\left.63\,T_1^{231}T_2^{252}-63\,T_1^{237}T_2^{252}+63\,T_1^{238}T_2^{252}-63\,T_1^{244}T_2^{252}+63\,T_1^{245}T_2^{252}-63\,T_1^{251}T_2^{252}+63\,T_1^{252}T_2^{252}\right)\Big\}\Big\}$$

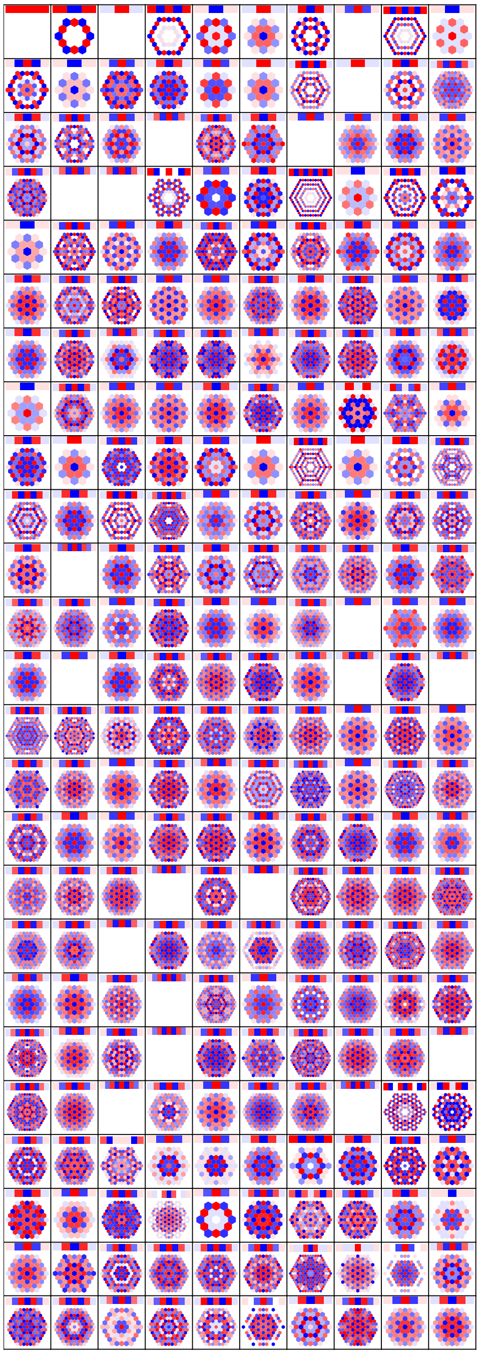
Full expression not available (original memory size: 8.1 MB)

Out[∘]=



In[∘]:=
```
tab250 = {{1, 0}}~Join~Table[θ[K], {K, AllKnots[{3, 10}]}];
```

In[∘]:=
```
g250 = GraphicsGrid[Partition[PolyPlot /@ tab250, 10],
  Spacings → 0, Dividers → All, ImagePadding → None, PlotRangePadding → None]
```
Out[∘]=



In[∘]:=
```
Export["Theta4Rolfsen.pdf", g250]
```
Out[∘]=

```
Theta4Rolfsen.pdf
```

See also https://drorbn.net/AcademicPensieve/Projects/HigherRank/DunfieldKnots/.

In[∘]:=
```
θ[BR[5, {1, 2, 3, -1, 2, 1, 3}]]
```

Out[∘]=

$$\left\{\frac{2-3\,T+2\,T^2}{T},\ \frac{1}{T_1^2\,T_2^2}\left(9-13\,T_1+9\,T_1^2-13\,T_2+6\,T_1\,T_2+6\,T_1^2\,T_2-13\,T_1^3\,T_2+9\,T_2^2+6\,T_1\,T_2^2-12\,T_1^2\,T_2^2+6\,T_1^3\,T_2^2+9\,T_1^4\,T_2^2-13\,T_1\,T_2^3+6\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3-13\,T_1^4\,T_2^3+9\,T_1^2\,T_2^4-13\,T_1^3\,T_2^4+9\,T_1^4\,T_2^4\right)\right\}$$

```
θ[BR[5, {1, 2, 3, -1, 2, 1, 3}]]
```

In[∘]:=
```
PolyPlot@Expand[
  Get["C:\\drorbn\\AcademicPensieve\\Projects\\HigherRank\\DunfieldKnots\\D300.m"][[2]] /.
   {T1 → T₁, T2 → T₂}]
```

Out[∘]=