

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
Once[<< KnotTheory`]
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[*]:= XFalse = 0; XTrue = 1;
```

```
In[*]:= CF[ $\mathcal{E}$ _] := Expand@Collect[ $\mathcal{E}$ , g_., F] /. F → Factor;
```

```
In[*]:= T3 = T1 T2;
```

```
In[*]:= F1[{s_, i_, j_}] := CF[
  s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
  g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj + ((T1^s - 1) g1ji (T2^s g2ji - T2^s g2jj + T2^s g3jj) +
  (T3^s - 1) g3ji (1 - T2^s g1ii + g2ij + (T2^s - 2) g2jj - (T1^s - 1) (T2^s + 1) g1ji)) / (T2^s - 1)]
```

```
In[*]:= F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] := CF[
  s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1 ( (T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0) ) ]
```

```
In[*]:= b0 = Simplify[Together[s (-1 + T2^s) F1[{s, i, j}]]] /.
  {T2 → 1, s^2 → 1, g2,α,β → Xα≤β, g3,α,β → g1,α,β} /. {Xj≤j → 1, Xj≤i → 1 - Xi≤j}
]
```

```
Out[*]= -((-1 + T1^s) g1,j,i (g1,i,i + 2(-1 + T1^s) g1,j,i - g1,j,j))
```

```
In[*]:= Simplify[b0 /. {(-1 + T1^s) g1,j,i → g1,j,j - g1,j,j^+}]
```

```
Out[*]= -(g1,i,i + g1,j,j - 2 g1,j,j^+) (g1,j,j - g1,j,j^+)
```

```
In[*]:= b1 = Simplify[Together[s1 (-1 + T2^s1) F2[{s0, i0, j0}, {s1, i1, j1}]]] /.
  {T2 → 1, s1^2 → 1, g2,α,β → Xα≤β, g3,α,β → g1,α,β}
]
```

```
Out[*]= (-1 + T1^s0) (-1 + T1^s1) (X11≤i0 - X11≤j0 - Xj1≤i0 + Xj1≤j0) g1,j0,i1 g1,j1,i0
```

```
In[*]:= Simplify[b1 /. {(-1 + T1^s0) g1,j1,i0 → g1,j1,j0 - g1,j1,j0^+, (-1 + T1^s1) g1,j0,i1 → g1,j0,j1 - g1,j0,j1^+}]
```

```
Out[*]= (X11≤i0 - X11≤j0 - Xj1≤i0 + Xj1≤j0) (g1,j0,j1 - g1,j0,j1^+) (g1,j1,j0 - g1,j1,j0^+)
```

```
In[*]:= b2 = Simplify[Together[s0 (-1 + T2^s0) F2[{s1, i1, j1}, {s0, i0, j0}]]] /.
  {T2 → 1, s0^2 → 1, g2,α,β → Xα≤β, g3,α,β → g1,α,β}
]
```

```
Out[*]= (-1 + T1^s0) (-1 + T1^s1) (X10≤i1 - X10≤j1 - Xj0≤i1 + Xj0≤j1) g1,j0,i1 g1,j1,i0
```

```

In[*]:= Simplify[(b1 + b2) / 2]
Out[*]=

$$\frac{1}{2} \left( -1 + T_1^{s_0} \right) \left( -1 + T_1^{s_1} \right) \left( \chi_{i_0 \leq i_1} - \chi_{i_0 \leq j_1} + \chi_{i_1 \leq i_0} - \chi_{i_1 \leq j_0} - \chi_{j_0 \leq i_1} + \chi_{j_0 \leq j_1} - \chi_{j_1 \leq i_0} + \chi_{j_1 \leq j_0} \right) g_{1,j_0,i_1} g_{1,j_1,i_0}$$


In[*]:= Simplify[Factor[(b1 + b2) / 2] /.
  {(-1 + T1^s0) g1,j1,i0 -> g1,j1,j0 - g1,j1,j0*, (-1 + T1^s1) g1,j0,i1 -> g1,j0,j1 - g1,j0,j1*}]
Out[*]=

$$\frac{1}{2} \left( \chi_{i_0 \leq i_1} - \chi_{i_0 \leq j_1} + \chi_{i_1 \leq i_0} - \chi_{i_1 \leq j_0} - \chi_{j_0 \leq i_1} + \chi_{j_0 \leq j_1} - \chi_{j_1 \leq i_0} + \chi_{j_1 \leq j_0} \right) \left( g_{1,j_0,j_1} - g_{1,j_0,j_1*} \right) \left( g_{1,j_1,j_0} - g_{1,j_1,j_0*} \right)$$


```

Testing

```

In[*]:= Rot[X_List] := X;
Rot[K_] := Cases[PD[K], x_X -> { {1, x[[4]], x[[1]]} PositiveQ@x
  { -1, x[[2]], x[[1]]} True }

In[*]:= Res[K_] := Module[{X, n, A, G, ev, r, k, k1, k2},
  (* 1 *) X = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  (* 2 *) Cases[X, {s_, i_, j_} -> (A[[{i, j}], {i + 1, j + 1}] += (-T^s T^s - 1))];
  (* 3 *) G = Inverse[A];
  (* 4 *) ev[s_] := Factor[s /. i_ -> i + 1 /. g_{\alpha, \beta} -> G[[\alpha, \beta]]];
  (* 5 *) r = ev[Sum[F1[X[[k]]], {k, n}]];
  (* 6 *) r += ev[Sum[F2[X[[k1]], X[[k2]]], {k1, n}, {k2, n}]];
  (* 7 *) Factor[r]
];

In[*]:= F1[{s_, i_, j_}] := -((g_{i,i} + g_{j,j} - 2 g_{j,j*}) (g_{j,j} - g_{j,j*}));
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  -(X_{i1 \leq i0} - X_{i1 \leq j0} - X_{j1 \leq i0} + X_{j1 \leq j0}) (-X_{j0 = j1*} + X_{j0 = i1*} + g_{j0,i1} - g_{j0,i1*}) (g_{j1,j0} - g_{j1,j0*});
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  (X_{i1 \leq i0} - X_{i1 \leq j0} - X_{j1 \leq i0} + X_{j1 \leq j0}) (g_{j0,j1} - g_{j0,j1*}) (g_{j1,j0} - g_{j1,j0*});
F1[{s, i, j}] // TeXForm

In[*]:= Rot[Knot[3, 1]]
Res[Knot[3, 1]]
Out[*]=
{{-1, 4, 1}, {-1, 6, 3}, {-1, 2, 5}}

Out[*]=
0

```



```

In[*]:= nRules[{s0, i0, j0}, {s1, i1, j1}]
Out[*]=

$$\left\{ \begin{aligned} &\chi_{i0 \leq \beta} \mapsto \chi_{i0^+ \leq \beta}, \chi_{i1 \leq \beta} \mapsto \chi_{i1^+ \leq \beta}, \chi_{j0 \leq \beta} \mapsto \chi_{j0^+ \leq \beta}, \chi_{j1 \leq \beta} \mapsto \chi_{j1^+ \leq \beta}, \chi_{\alpha \leq i0} \mapsto \chi_{\alpha \leq i0^+}, \\ &\chi_{\alpha \leq i1} \mapsto \chi_{\alpha \leq i1^+}, \chi_{\alpha \leq j0} \mapsto \chi_{\alpha \leq j0^+}, \chi_{\alpha \leq j1} \mapsto \chi_{\alpha \leq j1^+}, g_{i0, \beta} \mapsto T^{s0} g_{i0^+, \beta} + (1 - T^{s0}) g_{j0^+, \beta}, \\ &g_{i1, \beta} \mapsto T^{s1} g_{i1^+, \beta} + (1 - T^{s1}) g_{j1^+, \beta}, g_{j0, \beta} \mapsto g_{j0^+, \beta}, g_{j1, \beta} \mapsto g_{j1^+, \beta}, g_{\alpha, i0} \mapsto T^{s0} g_{\alpha, i0^+}, \\ &g_{\alpha, i1} \mapsto T^{s1} g_{\alpha, i1^+}, g_{\alpha, j0} \mapsto g_{\alpha, j0^+} + (1 - T^{s0}) g_{\alpha, i0^+}, g_{\alpha, j1} \mapsto g_{\alpha, j1^+} + (1 - T^{s1}) g_{\alpha, i1^+} \end{aligned} \right\}$$


In[*]:= a1 = Simplify[Together[ $\frac{s1}{(-1 + T^{s0})(-1 + T^{s1})} (-1 + T_2^{s1}) F_2[\{s0, i0, j0\}, \{s1, i1, j1\}]$ ] /.
 $\{T_2 \rightarrow 1, s1^2 \rightarrow 1, g_{2, \alpha, \beta} \mapsto \chi_{\alpha \leq \beta}, g_{1|3, \alpha, \beta} \mapsto g_{\alpha, \beta}, T_1 \rightarrow T\}$  /.
nRules[{s0, i0, j0}, {s1, i1, j1}]
]
Out[*]=
 $(\chi_{i1^+ \leq i0} - \chi_{i1^+ \leq j0} - \chi_{j1^+ \leq i0} + \chi_{j1^+ \leq j0}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= a2 = Simplify[Together[ $\frac{s0}{(-1 + T^{s0})(-1 + T^{s1})} (-1 + T_2^{s0}) F_2[\{s1, i1, j1\}, \{s0, i0, j0\}]$ ] /.
 $\{T_2 \rightarrow 1, s0^2 \rightarrow 1, g_{2, \alpha, \beta} \mapsto \chi_{\alpha \leq \beta}, g_{1|3, \alpha, \beta} \mapsto g_{\alpha, \beta}, T_1 \rightarrow T\}$  /.
nRules[{s0, i0, j0}, {s1, i1, j1}]
]
Out[*]=
 $(\chi_{i0^+ \leq i1} - \chi_{i0^+ \leq j1} - \chi_{j0^+ \leq i1} + \chi_{j0^+ \leq j1}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= Simplify[a1 + a2]
Out[*]=
 $(\chi_{i0^+ \leq i1} - \chi_{i0^+ \leq j1} + \chi_{i1^+ \leq i0} - \chi_{i1^+ \leq j0} - \chi_{j0^+ \leq i1} + \chi_{j0^+ \leq j1} - \chi_{j1^+ \leq i0} + \chi_{j1^+ \leq j0}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= D[{s_, i_, j_}][ $\mathcal{E}$ ] :=
CF[( $\mathcal{E} /. \# \rightarrow i^+$ ) + ( $\mathcal{E} /. \# \rightarrow j^+$ ) - ( $\mathcal{E} /. \# \rightarrow i$ ) - ( $\mathcal{E} /. \# \rightarrow j$ )] /. nRules[{s, i, j}]

In[*]:= D[{s, i, j}][{g $_{\alpha, \#}$ , g $_{\alpha, \#}^2$ }]
Out[*]=
 $\{0, 2 T^s (-1 + T^s) g_{\alpha, i}^2 - 2 (-1 + T^s) g_{\alpha, i} g_{\alpha, j}\}$ 

In[*]:= List @@ Expand[( $\chi_{\# \leq i0} + \chi_{\# \leq j0}$ ) (g $_{i0^+, \#} + g_{j0^+, \#}$ ) (g $_{\#, i0} + g_{\#, j0}$ )]
Out[*]=
 $\{\chi_{\#1 \leq i0} g_{\#1, i0} g_{i0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, i0} g_{i0^+, \#1}, \chi_{\#1 \leq i0} g_{\#1, j0} g_{i0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, j0} g_{i0^+, \#1},$ 
 $\chi_{\#1 \leq i0} g_{\#1, i0} g_{j0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, i0} g_{j0^+, \#1}, \chi_{\#1 \leq i0} g_{\#1, j0} g_{j0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, j0} g_{j0^+, \#1}\}$ 

```

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In[*]:= {X_{#≤i0} G_{#,i0} G_{i0^*,#}, X_{#≤j0} G_{#,i0} G_{i0^*,#}, X_{#≤i0} G_{#,j0} G_{i0^*,#}, X_{#≤j0} G_{#,j0} G_{i0^*,#},
          X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}, X_{#≤i0} G_{#,j0} G_{j0^*,#}, X_{#≤j0} G_{#,j0} G_{j0^*,#}} // D_{s1,i1,j1}

Out[*]=
{ (-1 + T^{s1}) (X_{i1^+≤i0} - X_{j1^+≤i0}) G_{i0^*,i1} G_{j1^+,i0}, (-1 + T^{s1}) (X_{i1^+≤j0} - X_{j1^+≤j0}) G_{i0^*,i1} G_{j1^+,i0},
  (-1 + T^{s1}) (X_{i1^+≤i0} - X_{j1^+≤i0}) G_{i0^*,i1} G_{j1^+,j0}, (-1 + T^{s1}) (X_{i1^+≤j0} - X_{j1^+≤j0}) G_{i0^*,i1} G_{j1^+,j0},
  (-1 + T^{s1}) (X_{i1^+≤i0} - X_{j1^+≤i0}) G_{j0^*,i1} G_{j1^+,i0}, (-1 + T^{s1}) (X_{i1^+≤j0} - X_{j1^+≤j0}) G_{j0^*,i1} G_{j1^+,i0},
  (-1 + T^{s1}) (X_{i1^+≤i0} - X_{j1^+≤i0}) G_{j0^*,i1} G_{j1^+,j0}, (-1 + T^{s1}) (X_{i1^+≤j0} - X_{j1^+≤j0}) G_{j0^*,i1} G_{j1^+,j0}}

In[*]:= {0, 0, 0, 0, 1, -1, 0, 0} . {X_{#≤i0} G_{#,i0} G_{i0^*,#}, X_{#≤j0} G_{#,i0} G_{i0^*,#}, X_{#≤i0} G_{#,j0} G_{i0^*,#}, X_{#≤j0} G_{#,j0} G_{i0^*,#},
          X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}, X_{#≤i0} G_{#,j0} G_{j0^*,#}, X_{#≤j0} G_{#,j0} G_{j0^*,#}} // D_{s1,i1,j1}

Out[*]=
(-1 + T^{s1}) (X_{i1^+≤i0} - X_{i1^+≤j0} - X_{j1^+≤i0} + X_{j1^+≤j0}) G_{j0^*,i1} G_{j1^+,i0}

In[*]:= ({1, -1} . {X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}} // D_{s1,i1,j1}) == (-1 + T^{s1}) a1

Out[*]=
True

In[*]:= List @@ Expand[(X_{i1^+≤#} + X_{j1^+≤#}) (G_{#,i1} + G_{#,j1}) (G_{i1^*,#} + G_{j1^*,#})]

Out[*]=
{X_{i1^+≤#1} G_{#1,i1} G_{i1^*,#1}, X_{j1^+≤#1} G_{#1,i1} G_{i1^*,#1}, X_{i1^+≤#1} G_{#1,j1} G_{i1^*,#1}, X_{j1^+≤#1} G_{#1,j1} G_{i1^*,#1},
  X_{i1^+≤#1} G_{#1,i1} G_{j1^*,#1}, X_{j1^+≤#1} G_{#1,i1} G_{j1^*,#1}, X_{i1^+≤#1} G_{#1,j1} G_{j1^*,#1}, X_{j1^+≤#1} G_{#1,j1} G_{j1^*,#1}}

In[*]:= {X_{i1^+≤#1} G_{#1,i1} G_{i1^*,#1}, X_{j1^+≤#1} G_{#1,i1} G_{i1^*,#1}, X_{i1^+≤#1} G_{#1,j1} G_{i1^*,#1}, X_{j1^+≤#1} G_{#1,j1} G_{i1^*,#1},
          X_{i1^+≤#1} G_{#1,i1} G_{j1^*,#1}, X_{j1^+≤#1} G_{#1,i1} G_{j1^*,#1}, X_{i1^+≤#1} G_{#1,j1} G_{j1^*,#1}, X_{j1^+≤#1} G_{#1,j1} G_{j1^*,#1}} // D_{s0,i0,j0}

Out[*]=
{ (-1 + T^{s0}) (X_{i1^+≤i0} - X_{i1^+≤j0}) G_{i1^+,i0} G_{j0^*,i1}, (-1 + T^{s0}) (X_{j1^+≤i0} - X_{j1^+≤j0}) G_{i1^+,i0} G_{j0^*,i1},
  (-1 + T^{s0}) (X_{i1^+≤i0} - X_{i1^+≤j0}) G_{i1^+,i0} G_{j0^*,j1}, (-1 + T^{s0}) (X_{j1^+≤i0} - X_{j1^+≤j0}) G_{i1^+,i0} G_{j0^*,j1},
  (-1 + T^{s0}) (X_{i1^+≤i0} - X_{i1^+≤j0}) G_{j0^*,i1} G_{j1^+,i0}, (-1 + T^{s0}) (X_{j1^+≤i0} - X_{j1^+≤j0}) G_{j0^*,i1} G_{j1^+,i0},
  (-1 + T^{s0}) (X_{i1^+≤i0} - X_{i1^+≤j0}) G_{j0^*,j1} G_{j1^+,i0}, (-1 + T^{s0}) (X_{j1^+≤i0} - X_{j1^+≤j0}) G_{j0^*,j1} G_{j1^+,i0}}

```