```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
        Once[<< Theta.m]
        SetOptions[PolyPlot, ImageSize → Tiny];
        Clear[Θ]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= RandomVK[n_] := {
          Prepend[#, 2 RandomInteger[1] - 1] & /@
           Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
          Table[RandomInteger[{-1, 1}], 2 n + 1]
         };
```

```
In[•]:= RandomVK[5]
```

```
Out[•]=
{{{1, 7, 3}, {-1, 6, 2}, {1, 4, 9}, {-1, 10, 1}, {1, 5, 8}},
 {-1, -1, 0, 1, 0, -1, 1, 1, -1, 1, 0}}
```

```
In[•]:= CF[ℰ_] := Expand@Collect[ℰ, g__ | χ__, F] /. F → Factor@*PowerExpand;
```

```
In[•]:= Short[Options[Θ] = {F1 → (F1i = F1[{s0, i0, j0}]),
          F2 → (F2i = F2[{s0, i0, j0}, {s1, i1, j1}]), F3 → (F3i = F3[φ, k])}]
```

```
Out[•]//Short=
{F1 → s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + <<19>> + <<1>>/<<1>> - s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0,
 F2 → <<1>>, F3 → -φ/2 + φ g<<1>>}
```

In[•]:=
```
Θ[K_, opts___Rule] := Module[{X, φφ, n, A, Δ, G, ev, θ, kk, k0, k1, f1, f2, f3},
  f1 = F1 /. {opts} /. Options[Θ];
  f2 = F2 /. {opts} /. Options[Θ];
  f3 = F3 /. {opts} /. Options[Θ];
  {X, φφ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  Cases[X, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += {{-T^s, T^s - 1}, {0, -1}})];
  Δ = T^(-Total[φφ]-Total[X[[All,1]]])/2 Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[
    ℰ /. {k_^+ :> k + 1, $ → 2 n + 1} /. {g_ν_,α_,β_ :> (G[[α, β]] /. T → T_ν), χ_True → 1, χ_False → 0}];
  θ = ev@Sum[f1 /. Thread[{s0, i0, j0} → X[[kk]]], {kk, n}];
  θ += ev@Sum[f2 /. Thread[{s0, i0, j0} → X[[k0]]] //.
      Thread[{s1, i1, j1} → X[[k1]]], {k0, n}, {k1, n}];
  θ += ev@Sum[f3 /. {φ → φφ[[kk]], k → kk}, {kk, Length@φφ}];
  Factor@{Δ, (Δ /. T → T_1) (Δ /. T → T_2) (Δ /. T → T_3) θ}
 ];
```

In[•]:=
```
Θ[Knot[7, 3]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

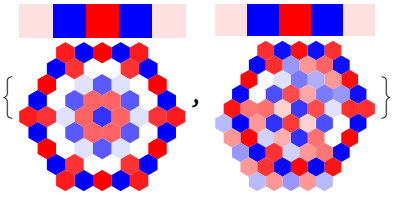
Out[•]=

$$\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\right.$$

$$\frac{1}{T_1^4\,T_2^4}\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-\right.$$
$$T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+$$
$$17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-$$
$$7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-$$
$$\left.\left.25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\}$$

In[•]:=
```
PolyPlot /@ {Θ[Knot[7, 3]], Θ[Knot[7, 3], F3 → 0]}
```

Out[•]=



```
In[•]:= δ_i_,j_ := χ_i==j; χ_ph_^p_ /; p > 1 ^:= χ_ph;
bRules[{s_, i_, j_}] := { (* b for "push indices backwards" *)
    g_ν_,j^,β_ :> g_ν,j,β - δ_j,β, g_ν_,i^,β_ :> T_ν^-s g_ν,i,β + (1 - T_ν^-s) g_ν,j,β - T_ν^-s δ_i,β - (1 - T_ν^-s) δ_j,β,
    g_ν_,α_,i^ :> T_ν^s g_ν,α,i + δ_α,i^, g_ν_,α_,j^ :> g_ν,α,j + (1 - T_ν^s) g_ν,α,i + δ_α,j^
   };
bRules[X___List] := Union @@ Table[bRules[c], {c, {X}}]
```

```
In[•]:= {g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}]
Out[•]=
{χ_i==β + T_ν^s g_ν,i^,β + (1 - T_ν^s) g_ν,j^,β, χ_j==β + g_ν,j^,β}
```

```
In[•]:= Expand[{g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}] /. bRules[{s, i, j}]]
Out[•]=
{g_ν,i,β, g_ν,j,β}
```

```
In[•]:= D_{s_,i_,j_}[ℰ_] := CF[Expand[Plus[
            ℰ /. {g_ν_,#,β_ :> g_ν,i^,β, g_ν_,α_,# :> g_ν,α,i^},
            ℰ /. {g_ν_,#,β_ :> g_ν,j^,β, g_ν_,α_,# :> g_ν,α,j^},
            -ℰ /. {g_ν_,#,β_ :> g_ν,i,β, g_ν_,α_,# :> g_ν,α,i},
            -ℰ /. {g_ν_,#,β_ :> g_ν,j,β, g_ν_,α_,# :> g_ν,α,j}
           ] /. bRules[{s, i, j}] /. bRules[{s0, i0, j0}]] /. χ_eq_Equal :> χ_Sort@eq //. {
      χ_j0==j1 → χ_i0==i1, χ_j0^==j1^ → χ_j0==j1,
      χ_i0==j1 → 0, χ_i1==j0 → 0, χ_i0^==j1^ → 0, χ_i0^==i1^ → χ_i0==i1, χ_i1^==j0^ → 0, χ_i0^==j0^ → 0,
      χ_i0==i1 A_ :> χ_i0==i1 (A /. {s1 → s0, i1 → i0, j1 → j0})
     } /. {χ_i0==j0 → 0, χ_i0^==j0^ → 0, χ_True → 1}
  ];
B[ℰ_] := CF[
   (ℰ /. {g_ν_,#,β_ :> g_ν,$,β, g_ν_,α_,# :> g_ν,α,$}) - (ℰ /. {g_ν_,#,β_ :> g_ν,1,β, g_ν_,α_,# :> g_ν,α,1}) /. {
      g_,$,β_ → χ_$==β, g_,_,$ → 1, g_,α_,1 → χ_α==1} /. {χ_$==i0 → 0, χ_i0^==1 → 0, χ_$==j0 → 0, χ_j0^==1 → 0}
  ];
```

```
In[•]:= D_{s1,i1,j1}[g_1,#,i0]
Out[•]=
-T_1^-s0 χ_i0==i1 - T_1^-s1 (-1 + T_1^s1) g_1,i1,i0 + T_1^-s1 (-1 + T_1^s1) g_1,j1,i0
```

```
In[•]:= D_{s1,i1,j1}[g_3,j0,#]
Out[•]=
χ_j0==i1^ + χ_j0==j1^
```

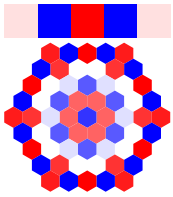
```
In[•]:= B[g_3,j0,#]
Out[•]=
1 - χ_j0==1
```

In[•]:=
```
tw = g1,#,i0 ;
⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$

$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ 0\right\}$

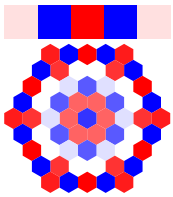
Out[•]=

In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$

$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



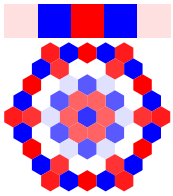
In[•]:=
```
tw = g1,#1,j0 g2,#1,j0 g3,i0,#1;
Θ[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ 0\right\}$

Out[•]=

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$
$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-\right.$
$25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-$
$T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-$
$\left.\left.T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\}$

Out[•]=



In[•]:=
```
bas = List @@ Expand[(g1,#,i0 + g1,#,j0) (g2,#,i0 + g2,#,j0) (g3,i0⁺,# + g3,j0⁺,#)]
```

Out[•]=

$\{g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\}$

In[•]:=
```
Column[(# → Θ[Knot[7, 6], F1 → -B[#], F2 → D{s1,i1,j1}[#], F3 → 0][[2]]) & /@ bas]
Column[(# → {B[#], D{s1,i1,j1}[#]}) & /@ bas]
```

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to$
$\{0,\ T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,i0,i0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}+$
$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,i0,i0}-$
$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,i0,i0}-$
$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,i0,i1}+$
$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}+$
$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}-$
$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)$
$\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1}\to$$
$$\Big\{0,\ T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,i0,i0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}+$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,i0,i0}-$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,i0,i0}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,i0,i1}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,j0}+T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)$$
$$\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)$$
$$\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to$$
$$\Big\{0,\ T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,i0,j0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,i0,i0}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,i0,i0}-$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,i0,i1}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,j0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}-$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)$$
$$\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}+$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)$$
$$\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,j0}\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to$$
$$\Big\{0,\ T_1^{-s0}T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,i0,j0}+T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0}-$$
$$T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,i0,i0}+$$
$$T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,i0,i0}+$$
$$T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,i0,i1}+$$
$$T_1^{-s0}T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}-T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}+$$
$$T_1^{-s0}T_2^{-s0}\chi_{i0==i1}\,g_{3,i0,j0}-T_1^{-s0}T_2^{-s0}\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,j0}-$$
$$T_1^{-s0}T_2^{-s0}\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,j0}-T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-$$
$$T_1^{-s0}T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}-$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0}+$$
$$T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1}+$$
$$T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}-$$
$$T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,j0}-$$
$$T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0}-T_1^{-s0}T_2^{-s0}\left(-1+T_1^{s0}T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to$$
$$\Big\{0,\ \chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}+\chi_{i0==i1}\,g_{3,j0,i0}-\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}-\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}-$$
$$\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}-\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to\Big\{0,\ \chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}+\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-$$
$$\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}-\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}-\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}+$$
$$\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+\left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}-\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\to$$
$$\Big\{0,\ \chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0}+\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}+$$
$$\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}-\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0}-\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0}+$$
$$\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}-\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,j0}\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\to$$
$$\Big\{0,\ \chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0}-\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}-\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}+$$
$$\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}-\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0}+$$
$$\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0}+\left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1}+$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$\chi_{i0==i1}\,g_{3,j0,j0}-\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0}-\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}\Big\}$$

In[•]:=
```
eq = T2 - 1 == s0 T2^((1-s0)/2) (T2^s0 - 1); Simplify[{eq /. s0 → 1, eq /. s0 → -1}]
```

Out[•]=
```
{True, True}
```

In[•]:=
```
tw = Table[a_i, {i, 8}].bas / (T_2 - 1)
{D_{s1,i1,j1}[tw], B[Tw]}
Θ[Knot[7, 6], F1 → -B[tw], F2 → D_{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot
```

Out[•]=
$$\frac{1}{-1+T_2}\left(a_1\, g_{1,\#1,i0}\, g_{2,\#1,i0}\, g_{3,i0^+,\#1} + a_2\, g_{1,\#1,j0}\, g_{2,\#1,i0}\, g_{3,i0^+,\#1} + a_3\, g_{1,\#1,i0}\, g_{2,\#1,j0}\, g_{3,i0^+,\#1} + a_4\, g_{1,\#1,j0}\, g_{2,\#1,j0}\, g_{3,i0^+,\#1} + a_5\, g_{1,\#1,i0}\, g_{2,\#1,i0}\, g_{3,j0^+,\#1} + a_6\, g_{1,\#1,j0}\, g_{2,\#1,i0}\, g_{3,j0^+,\#1} + a_7\, g_{1,\#1,i0}\, g_{2,\#1,j0}\, g_{3,j0^+,\#1} + a_8\, g_{1,\#1,j0}\, g_{2,\#1,j0}\, g_{3,j0^+,\#1}\right)$$

Out[•]=
$$\left\{\frac{a_1\, T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,i0,i0}}{-1+T_2} + \frac{a_2\, T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,i0,i0}}{-1+T_2} + \frac{a_3\, T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,i0,j0}}{-1+T_2} + \frac{a_4\, T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,i0,j0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(-a_1 + a_1\, T_1^{s0}\, T_2^{s0} + a_5\, T_1^{s0}\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(-a_2 + a_2\, T_1^{s0}\, T_2^{s0} + a_6\, T_1^{s0}\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(-a_3 + a_3\, T_1^{s0}\, T_2^{s0} + a_7\, T_1^{s0}\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,j0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(-a_4 + a_4\, T_1^{s0}\, T_2^{s0} + a_8\, T_1^{s0}\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,j0}}{-1+T_2} - \frac{T_1^{-s0}\, T_2^{-s0}\left(-a_1 + a_2 + a_3 - 2\,a_4 - a_2\, T_1^{s0} + a_4\, T_1^{s0} - a_3\, T_2^{s0} + a_4\, T_2^{s0}\right)\chi_{i0==i1}\, g_{3,i0,i0}}{-1+T_2} - \frac{T_1^{-s0}\, T_2^{-s0}\left(a_1 - a_3 + a_3\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,i0,i0}\, g_{3,i0,i0}}{-1+T_2} - \frac{T_1^{-s0}\, T_2^{-s0}\left(a_2 - a_4 + a_4\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,i0,j0}\, g_{3,i0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(a_1 - 2\,a_3 - a_1\, T_1^{s0} + a_3\, T_1^{s0} + a_3\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,i0}\, g_{3,i0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(a_2 - 2\,a_4 - a_2\, T_1^{s0} + a_4\, T_1^{s0} + a_4\, T_2^{s0}\right)\chi_{i0==i1}\, g_{1,j0,j0}\, g_{3,i0,i0}}{-1+T_2} - \frac{T_1^{-s0}\left(a_1 - a_2 + a_2\, T_1^{s0}\right)T_2^{-s0}\, \chi_{i0==i1}\, g_{2,i0,i0}\, g_{3,i0,i0}}{-1+T_2} - \frac{T_1^{-s0}\left(a_3 - a_4 + a_4\, T_1^{s0}\right)T_2^{-s0}\, \chi_{i0==i1}\, g_{2,i0,j0}\, g_{3,i0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(a_1 - 2\,a_2 + a_2\, T_1^{s0} - a_1\, T_2^{s0} + a_2\, T_2^{s0}\right)\chi_{i0==i1}\, g_{2,j0,i0}\, g_{3,i0,i0}}{-1+T_2} + \frac{T_1^{-s0}\, T_2^{-s0}\left(a_3 - 2\,a_4 + a_4\, T_1^{s0} - a_3\, T_2^{s0} + a_4\, T_2^{s0}\right)\chi_{i0==i1}\, g_{2,j0,j0}\, g_{3,i0,i0}}{-1+T_2} + \frac{a_1\, T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,i0,i1}}{-1+T_2} + \frac{a_2\, T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\, g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,i0,i1}}{-1+T_2} +$$

$$\frac{a_3\,T_1^{-s0}\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,i0,i1}}{-1+T_2}+\frac{a_4\,T_1^{-s0}\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,i0,i1}}{-1+T_2}+$$
$$\frac{a_1\,T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)\,g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}}{-1+T_2}+\frac{a_2\,T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)\,g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}}{-1+T_2}-$$
$$\frac{a_1\,T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}}{-1+T_2}-$$
$$\frac{a_2\,T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}}{-1+T_2}+\frac{a_3\,T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)\,g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}}{-1+T_2}+$$
$$\frac{a_4\,T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)\,g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}}{-1+T_2}-\frac{a_3\,T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}}{-1+T_2}-$$
$$\frac{a_4\,T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}}{-1+T_2}+\frac{a_4\,T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,j0}}{-1+T_2}-$$
$$\frac{a_3\,T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,j0}}{-1+T_2}-\frac{a_4\,T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,j0}}{-1+T_2}-$$
$$\frac{a_2\,T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,j0}}{-1+T_2}-\frac{a_4\,T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,j0}}{-1+T_2}+$$
$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(-a_1+a_2+a_3-2\,a_4-a_2\,T_1^{s0}+a_4\,T_1^{s0}-a_3\,T_2^{s0}+a_4\,T_2^{s0}+a_1\,T_1^{s0}\,T_2^{s0}-a_2\,T_1^{s0}\,T_2^{s0}-a_3\,T_1^{s0}\,T_2^{s0}+\right.$$
$$\left.2\,a_4\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}-a_6\,T_1^{s0}\,T_2^{s0}-a_7\,T_1^{s0}\,T_2^{s0}+2\,a_8\,T_1^{s0}\,T_2^{s0}+a_2\,T_1^{2\,s0}\,T_2^{s0}-a_4\,T_1^{2\,s0}\,T_2^{s0}+a_6\,T_1^{2\,s0}\,T_2^{s0}-\right.$$
$$\left.a_8\,T_1^{2\,s0}\,T_2^{s0}+a_3\,T_1^{s0}\,T_2^{2\,s0}-a_4\,T_1^{s0}\,T_2^{2\,s0}+a_7\,T_1^{s0}\,T_2^{2\,s0}-a_8\,T_1^{s0}\,T_2^{2\,s0}\right)\,\chi_{i0==i1}\,g_{3,j0,i0}-\frac{1}{-1+T_2}$$
$$T_1^{-s0}\,T_2^{-s0}\left(-a_1+a_3-a_3\,T_2^{s0}+a_1\,T_1^{s0}\,T_2^{s0}-a_3\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}-a_7\,T_1^{s0}\,T_2^{s0}+a_3\,T_1^{s0}\,T_2^{2\,s0}+a_7\,T_1^{s0}\,T_2^{2\,s0}\right)$$
$$\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}-\frac{1}{-1+T_2}$$
$$T_1^{-s0}\,T_2^{-s0}\left(-a_2+a_4-a_4\,T_2^{s0}+a_2\,T_1^{s0}\,T_2^{s0}-a_4\,T_1^{s0}\,T_2^{s0}+a_6\,T_1^{s0}\,T_2^{s0}-a_8\,T_1^{s0}\,T_2^{s0}+a_4\,T_1^{s0}\,T_2^{2\,s0}+a_8\,T_1^{s0}\,T_2^{2\,s0}\right)$$
$$\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}-$$
$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(a_1-2\,a_3-a_1\,T_1^{s0}+a_3\,T_1^{s0}+a_3\,T_2^{s0}-a_1\,T_1^{s0}\,T_2^{s0}+2\,a_3\,T_1^{s0}\,T_2^{s0}-a_5\,T_1^{s0}\,T_2^{s0}+2\,a_7\,T_1^{s0}\,T_2^{s0}+\right.$$
$$\left.a_1\,T_1^{2\,s0}\,T_2^{s0}-a_3\,T_1^{2\,s0}\,T_2^{s0}+a_5\,T_1^{2\,s0}\,T_2^{s0}-a_7\,T_1^{2\,s0}\,T_2^{s0}-a_3\,T_1^{s0}\,T_2^{2\,s0}-a_7\,T_1^{s0}\,T_2^{2\,s0}\right)\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}-$$
$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(a_2-2\,a_4-a_2\,T_1^{s0}+a_4\,T_1^{s0}+a_4\,T_2^{s0}-a_2\,T_1^{s0}\,T_2^{s0}+2\,a_4\,T_1^{s0}\,T_2^{s0}-a_6\,T_1^{s0}\,T_2^{s0}+2\,a_8\,T_1^{s0}\,T_2^{s0}+\right.$$
$$\left.a_2\,T_1^{2\,s0}\,T_2^{s0}-a_4\,T_1^{2\,s0}\,T_2^{s0}+a_6\,T_1^{2\,s0}\,T_2^{s0}-a_8\,T_1^{2\,s0}\,T_2^{s0}-a_4\,T_1^{s0}\,T_2^{2\,s0}-a_8\,T_1^{s0}\,T_2^{2\,s0}\right)\,\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}-$$
$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(-a_1+a_2-a_2\,T_1^{s0}+a_1\,T_1^{s0}\,T_2^{s0}-a_2\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}-a_6\,T_1^{s0}\,T_2^{s0}+\right.$$
$$\left.a_2\,T_1^{2\,s0}\,T_2^{s0}+a_6\,T_1^{2\,s0}\,T_2^{s0}\right)\,\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}-\frac{1}{-1+T_2}$$
$$T_1^{-s0}\,T_2^{-s0}\left(-a_3+a_4-a_4\,T_1^{s0}+a_3\,T_1^{s0}\,T_2^{s0}-a_4\,T_1^{s0}\,T_2^{s0}+a_7\,T_1^{s0}\,T_2^{s0}-a_8\,T_1^{s0}\,T_2^{s0}+a_4\,T_1^{2\,s0}\,T_2^{s0}+a_8\,T_1^{2\,s0}\,T_2^{s0}\right)$$
$$\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0}-$$
$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(a_1-2\,a_2+a_2\,T_1^{s0}-a_1\,T_2^{s0}+a_2\,T_2^{s0}-a_1\,T_1^{s0}\,T_2^{s0}+2\,a_2\,T_1^{s0}\,T_2^{s0}-a_5\,T_1^{s0}\,T_2^{s0}+2\,a_6\,T_1^{s0}\,T_2^{s0}-\right.$$

$$\left(a_2\,T_1^{2\,s0}\,T_2^{s0}-a_6\,T_1^{2\,s0}\,T_2^{s0}+a_1\,T_1^{s0}\,T_2^{2\,s0}-a_2\,T_1^{s0}\,T_2^{2\,s0}+a_5\,T_1^{s0}\,T_2^{2\,s0}-a_6\,T_1^{s0}\,T_2^{2\,s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}-$$

$$\frac{1}{-1+T_2}\,T_1^{-s0}\,T_2^{-s0}\left(a_3-2\,a_4+a_4\,T_1^{s0}-a_3\,T_2^{s0}+a_4\,T_2^{s0}-a_3\,T_1^{s0}\,T_2^{s0}+2\,a_4\,T_1^{s0}\,T_2^{s0}-a_7\,T_1^{s0}\,T_2^{s0}+2\,a_8\,T_1^{s0}\,T_2^{s0}-\right.$$

$$\left.a_4\,T_1^{2\,s0}\,T_2^{s0}-a_8\,T_1^{2\,s0}\,T_2^{s0}+a_3\,T_1^{s0}\,T_2^{2\,s0}-a_4\,T_1^{s0}\,T_2^{2\,s0}+a_7\,T_1^{s0}\,T_2^{2\,s0}-a_8\,T_1^{s0}\,T_2^{2\,s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0}+$$

$$\frac{T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-a_1+a_1\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-a_2+a_2\,T_1^{s0}\,T_2^{s0}+a_6\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-a_3+a_3\,T_1^{s0}\,T_2^{s0}+a_7\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_1+a_1\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_2+a_2\,T_1^{s0}\,T_2^{s0}+a_6\,T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_1+a_1\,T_1^{s0}\,T_2^{s0}+a_5\,T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_2+a_2\,T_1^{s0}\,T_2^{s0}+a_6\,T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_3+a_3\,T_1^{s0}\,T_2^{s0}+a_7\,T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_3+a_3\,T_1^{s0}\,T_2^{s0}+a_7\,T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}}{-1+T_2}+$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,j0}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_3+a_3\,T_1^{s0}\,T_2^{s0}+a_7\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,j0}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\left(-a_2+a_2\,T_1^{s0}\,T_2^{s0}+a_6\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}}{-1+T_2}-$$

$$\frac{T_1^{-s0}\,T_2^{-s0}\,\left(-a_4+a_4\,T_1^{s0}\,T_2^{s0}+a_8\,T_1^{s0}\,T_2^{s0}\right)\,\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}}{-1+T_2}\,,\,0\Big\}$$

KnotTheory: Loading precomputed data in PD4Knots`.

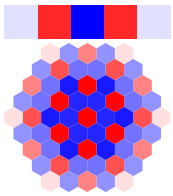
» $\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2}\,,\,0\right\}$

Out[•]=

» $\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2}\,,\right.$

$$\frac{1}{T_1^4\,T_2^4}\,\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-\right.$$
$$10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-$$
$$50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-$$
$$64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-$$
$$10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+7\,T_1^2\,T_2^6-10\,T_1^3\,T_2^6-64\,T_1^4\,T_2^6+98\,T_1^5\,T_2^6-64\,T_1^6\,T_2^6-10\,T_1^7\,T_2^6+7\,T_1^8\,T_2^6-5\,T_1^3\,T_2^7+$$
$$\left.20\,T_1^4\,T_2^7-10\,T_1^5\,T_2^7-10\,T_1^6\,T_2^7+20\,T_1^7\,T_2^7-5\,T_1^8\,T_2^7+T_1^4\,T_2^8-5\,T_1^5\,T_2^8+7\,T_1^6\,T_2^8-5\,T_1^7\,T_2^8+T_1^8\,T_2^8\right)\Big\}$$

Out[•]=



In[•]:=

```
χTrue = 1; χFalse = 0;
χα_≤α_ = 1; χ_ph^p_ /; p > 1 ^:= χph; χα_==β_ /; OrderedQ[{β, α}] := χβ==α;
χα_^+==β_^+ := χα==β;
δi_,j_ := χi==j;
```

```
In[•]:= sRules[ℰ_] := FixedPoint[CF[# /. bRules[{s0, i0, j0}] ⋃ bRules[{s1, i1, j1}] ⋃ {
          s0^2 → 1,
          g_{3,α_,β_} :→ g_{1,α,β}, g_{2,α_,β_} :→ χ_{α≤β},
          χ_{i0==j1} → 0, χ_{i1==j0} → 0, χ_{i0==j0} → 0,
          χ_{j0==j1} → χ_{i0==i1},
          χ_{i1^+≤i0} → χ_{i1≤i0} - χ_{i0==i1}, χ_{i0^+==i1^+} → χ_{i0==i1},
          χ_{i0==i1} χ_{i1≤i0} → χ_{i0==i1}, χ_{i0==i1} χ_{j1≤j0} → χ_{i0==i1},
          χ_{j0≤i0} → 1 - χ_{i0≤j0}, χ_{i1≤i0} → 1 - χ_{i0≤i1} + χ_{i1==i0},
          χ_{j1≤i0} → 1 - χ_{i0≤j1} + χ_{j1==i0}, χ_{i1≤i0} → 1 - χ_{i0≤i1} + χ_{i1==i0}, χ_{j1≤j0} → 1 - χ_{j0≤j1} + χ_{j1==j0},
          χ_{i0==i1} γ_ /; ! FreeQ[γ, s1 | i1 | j1] :→ χ_{i0==i1} (γ /. {s1 → s0, i1 → i0, j1 → j0}),
          χ_{α_==β_} g_{β_,γ_} :→ χ_{α==β} g_{α,γ}, χ_{α_==β_} g_{γ_,β_} :→ χ_{α==β} g_{γ,α}
        }] &,
  ℰ]

In[•]:= sRules[sRules[Residue[CF[χ_{i0==i1} F1i + F2i + D_{s1,i1,j1}[tw]], {T_2, 1}] ] /.
    {a_{1|2|3|4|6|8} → 0, a_7 → -a_5} /. a_5 → 1 - T_1^{s0}]

Out[•]=
0

In[•]:= eq = T_2 - 1 == s0 T_2^{(1-s0)/2} (T_2^{s0} - 1); Simplify[{eq /. s0 → 1, eq /. s0 → -1}]

Out[•]=
{True, True}

In[•]:= nF2 = CF[CF[ (χ_{i0==i1} F1i + F2i + D_{s1,i1,j1}[tw]) /. {a_{1|2|3|4|6|8} → 0, a_7 → -a_5} /. a_5 → 1 - T_1^{s0}]];
        {nF1, nF2} = Simplify@{Coefficient[nF2, χ_{i0==i1}], nF2 /. χ_{i0==i1} → 0}
        Table[Θ[K] == Θ[K, F1 → nF1, F2 → nF2], {K, AllKnots[{3, 8}]}]
```

Out[•]=

$$\Big\{\frac{s0}{2}+s0\,T_2^{s0}\,g_{1,i0,i0}\,g_{2,j0,i0}+\frac{\left(-1+T_1^{s0}\right)\,\left(1-T_2^{s0}-s0\,T_2^{2\,s0}+s0\,T_2^{1+2\,s0}\right)\,g_{1,j0,i0}\,g_{2,j0,i0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}-$$

$$s0\,g_{1,i0,i0}\,g_{2,j0,j0}-\frac{\left(-1+T_1^{s0}\right)\,\left(1-(1+s0)\,T_2^{s0}+s0\,T_2^{1+s0}\right)\,g_{1,j0,i0}\,g_{2,j0,j0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}-$$

$$s0\,g_{3,i0,i0}-s0\,\left(-1+T_2^{s0}\right)\,g_{2,j0,i0}\,g_{3,i0,i0}+2\,s0\,g_{2,j0,j0}\,g_{3,i0,i0}+$$

$$\frac{\left(-2+s0-s0\,T_2+3\,T_2^{s0}-T_2^{2\,s0}+T_1^{s0}\,\left(2-(3+s0)\,T_2^{s0}+T_2^{2\,s0}+s0\,T_2^{1+s0}\right)\right)\,g_{3,j0,i0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}-$$

$$\frac{\left(-2+(3+s0)\,T_2^{s0}-T_2^{2\,s0}-s0\,T_2^{1+s0}+T_1^{s0}\,\left(2-3\,T_2^{s0}-(-1+s0)\,T_2^{2\,s0}+s0\,T_2^{1+2\,s0}\right)\right)\,g_{1,i0,i0}\,g_{3,j0,i0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}-$$

$$\frac{1}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}\,\left(-1+T_1^{s0}\right)$$

$$\left(-3+s0-s0\,T_2+(4+s0)\,T_2^{s0}-T_2^{2\,s0}-s0\,T_2^{1+s0}+T_1^{s0}\,\left(2-(2+s0)\,T_2^{s0}-s0\,T_2^{2\,s0}+s0\,T_2^{1+s0}+s0\,T_2^{1+2\,s0}\right)\right)$$

$$g_{1,j0,i0}\,g_{3,j0,i0}+\frac{\left(-1+T_1^{s0}\right)\,g_{2,i0,i0}\,g_{3,j0,i0}}{-1+T_2}+$$

$$\frac{\left(-1+s0-s0\,T_2+T_2^{s0}+T_1^{s0}\,\left(1-(1+s0)\,T_2^{s0}+s0\,T_2^{1+s0}\right)\right)\,g_{2,i0,j0}\,g_{3,j0,i0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}+$$

$$\frac{\left(1+s0-s0\,T_2-T_2^{s0}+T_1^{s0}\,\left(-1-(-1+s0)\,T_2^{s0}+s0\,T_2^{1+s0}\right)\right)\,g_{2,j0,i0}\,g_{3,j0,i0}}{-1+T_2}+$$

$$\frac{1}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}\,\left(1-2\,s0+2\,s0\,T_2+(-2+s0)\,T_2^{s0}+T_2^{2\,s0}-s0\,T_2^{1+s0}+\right.$$

$$\left.T_1^{s0}\,\left(-1+2\,(1+s0)\,T_2^{s0}-(1+s0)\,T_2^{2\,s0}-2\,s0\,T_2^{1+s0}+s0\,T_2^{1+2\,s0}\right)\right)\,g_{2,j0,j0}\,g_{3,j0,i0}+$$

$$s0\,g_{1,i0,i0}\,g_{3,j0,j0}+\frac{\left(-1+T_1^{s0}\right)\,\left(1-(1+s0)\,T_2^{s0}+s0\,T_2^{1+s0}\right)\,g_{1,j0,i0}\,g_{3,j0,j0}}{(-1+T_2)\,\left(-1+T_2^{s0}\right)}-$$

$$s0\,g_{2,i0,i0}\,g_{3,j0,j0}-s0\,T_2^{s0}\,g_{2,j0,i0}\,g_{3,j0,j0},\ \frac{1}{(-1+T_2)\,\left(-1+T_2^{s1}\right)}$$

$$\left(-1+T_1^{s0}\right)\,\left(\left(-1+s1\,T_2^{s0}-s1\,T_2^{1+s0}+T_2^{s1}+T_1^{s1}\,\left(1-T_2^{s1}-s1\,T_2^{s0+s1}+s1\,T_2^{1+s0+s1}\right)\right)\,g_{1,j1,i0}\,g_{2,i1,i0}-\right.$$

$$\left(-1+s1-s1\,T_2+T_2^{s1}+T_1^{s1}\,\left(1-(1+s1)\,T_2^{s1}+s1\,T_2^{1+s1}\right)\right)\,g_{1,j1,i0}\,g_{2,i1,j0}-\left(-1+T_2^{s1}\right)^2\,g_{1,i1,i0}$$

$$g_{2,j1,i0}+\left(2-s1\,T_2^{s0}+s1\,T_2^{1+s0}-3\,T_2^{s1}+T_2^{2\,s1}+T_1^{s1}\,\left(-1+T_2^{s1}+s1\,T_2^{s0+s1}-s1\,T_2^{1+s0+s1}\right)\right)$$

$$g_{1,j1,i0}\,g_{2,j1,i0}+\left(-1+T_2^{s1}\right)^2\,g_{1,i1,i0}\,g_{2,j1,j0}+$$

$$\left.\left(-2+s1-s1\,T_2+3\,T_2^{s1}-T_2^{2\,s1}+T_1^{s1}\,\left(1-(1+s1)\,T_2^{s1}+s1\,T_2^{1+s1}\right)\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\right)\,g_{3,j0,i1}\Big\}$$

Out[•]=

{True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True}

In[•]:=
```
Table[Θ[K] == Θ[K,
  F1 → s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + ((-1 + T1^s0) (1 - T2^s0 - s0 T2^(2 s0) + s0 T2^(1+2 s0)) g1,j0,i0 g2,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
    s0 g1,i0,i0 g2,j0,j0 - ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g2,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
    s0 g3,i0,i0 - s0 (-1 + T2^s0) g2,j0,i0 g3,i0,i0 + 2 s0 g2,j0,j0 g3,i0,i0 +
    ((-2 + s0 - s0 T2 + 3 T2^s0 - T2^(2 s0) + T1^s0 (2 - (3 + s0) T2^s0 + T2^(2 s0) + s0 T2^(1+s0))) g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
    ((-2 + (3 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) + T1^s0 (2 - 3 T2^s0 - (-1 + s0) T2^(2 s0) + s0 T2^(1+2 s0))) g1,i0,i0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
    1/((-1 + T2) (-1 + T2^s0)) (-1 + T1^s0) (-3 + s0 - s0 T2 + (4 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) + T1^s0
      (2 - (2 + s0) T2^s0 - s0 T2^(2 s0) + s0 T2^(1+s0) + s0 T2^(1+2 s0))) g1,j0,i0 g3,j0,i0 + ((-1 + T1^s0) g2,i0,i0 g3,j0,i0)/(-1 + T2) +
    ((-1 + s0 - s0 T2 + T2^s0 + T1^s0 (1 - (1 + s0) T2^s0 + s0 T2^(1+s0))) g2,i0,j0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) +
    ((1 + s0 - s0 T2 - T2^s0 + T1^s0 (-1 - (-1 + s0) T2^s0 + s0 T2^(1+s0))) g2,j0,i0 g3,j0,i0)/(-1 + T2) +
    1/((-1 + T2) (-1 + T2^s0)) (1 - 2 s0 + 2 s0 T2 + (-2 + s0) T2^s0 + T2^(2 s0) - s0 T2^(1+s0) +
      T1^s0 (-1 + 2 (1 + s0) T2^s0 - (1 + s0) T2^(2 s0) - 2 s0 T2^(1+s0) + s0 T2^(1+2 s0))) g2,j0,j0 g3,j0,i0 +
    s0 g1,i0,i0 g3,j0,j0 + ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g3,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
    s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0,
  F2 → 1/((-1 + T2) (-1 + T2^s1)) (-1 + T1^s0)
    ((-1 + s1 T2^s0 - s1 T2^(1+s0) + T2^s1 + T1^s1 (1 - T2^s1 - s1 T2^(s0+s1) + s1 T2^(1+s0+s1))) g1,j1,i0 g2,i1,i0 -
      (-1 + s1 - s1 T2 + T2^s1 + T1^s1 (1 - (1 + s1) T2^s1 + s1 T2^(1+s1))) g1,j1,i0 g2,i1,j0 - (-1 + T2^s1)^2 g1,i1,i0
      g2,j1,i0 + (2 - s1 T2^s0 + s1 T2^(1+s0) - 3 T2^s1 + T2^(2 s1) + T1^s1 (-1 + T2^s1 + s1 T2^(s0+s1) - s1 T2^(1+s0+s1)))
      g1,j1,i0 g2,j1,i0 + (-1 + T2^s1)^2 g1,i1,i0 g2,j1,j0 +
      (-2 + s1 - s1 T2 + 3 T2^s1 - T2^(2 s1) + T1^s1 (1 - (1 + s1) T2^s1 + s1 T2^(1+s1))) g1,j1,i0 g2,j1,j0) g3,j0,i1
  ], {K, AllKnots[{3, 8}]}]
```

Out[•]=
```
{True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True, True, True, True}
```

In[•]:=
```
CF[ s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + ((-1 + T1^s0) (1 - T2^s0 - s0 T2^(2 s0) + s0 T2^(1+2 s0)) g1,j0,i0 g2,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g1,i0,i0 g2,j0,j0 - ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g2,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g3,i0,i0 - s0 (-1 + T2^s0) g2,j0,i0 g3,i0,i0 + 2 s0 g2,j0,j0 g3,i0,i0 +
 ((-2 + s0 - s0 T2 + 3 T2^s0 - T2^(2 s0) + T1^s0 (2 - (3 + s0) T2^s0 + T2^(2 s0) + s0 T2^(1+s0))) g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 ((-2 + (3 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) + T1^s0 (2 - 3 T2^s0 - (-1 + s0) T2^(2 s0) + s0 T2^(1+2 s0))) g1,i0,i0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 1/((-1 + T2) (-1 + T2^s0)) (-1 + T1^s0) (-3 + s0 - s0 T2 + (4 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) +
    T1^s0 (2 - (2 + s0) T2^s0 - s0 T2^(2 s0) + s0 T2^(1+s0) + s0 T2^(1+2 s0))) g1,j0,i0 g3,j0,i0 + ((-1 + T1^s0) g2,i0,i0 g3,j0,i0)/(-1 + T2) +
 ((-1 + s0 - s0 T2 + T2^s0 + T1^s0 (1 - (1 + s0) T2^s0 + s0 T2^(1+s0))) g2,i0,j0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) +
 ((1 + s0 - s0 T2 - T2^s0 + T1^s0 (-1 - (-1 + s0) T2^s0 + s0 T2^(1+s0))) g2,j0,i0 g3,j0,i0)/(-1 + T2) +
 1/((-1 + T2) (-1 + T2^s0)) (1 - 2 s0 + 2 s0 T2 + (-2 + s0) T2^s0 + T2^(2 s0) - s0 T2^(1+s0) +
    T1^s0 (-1 + 2 (1 + s0) T2^s0 - (1 + s0) T2^(2 s0) - 2 s0 T2^(1+s0) + s0 T2^(1+2 s0))) g2,j0,j0 g3,j0,i0 +
 s0 g1,i0,i0 g3,j0,j0 + ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g3,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0 /. s0 -> 1]
```

Out[•]=
$$\frac{1}{2} + T_2\, g_{1,i0,i0}\, g_{2,j0,i0} + (-1+T_1)(1+T_2)\, g_{1,j0,i0}\, g_{2,j0,i0} - g_{1,i0,i0}\, g_{2,j0,j0} + (1-T_1)\, g_{1,j0,i0}\, g_{2,j0,j0} - g_{3,i0,i0} + (1-T_2)\, g_{2,j0,i0}\, g_{3,i0,i0} + 2\, g_{2,j0,j0}\, g_{3,i0,i0} + (-1+2T_1)\, g_{3,j0,i0} + (2-2T_1-T_1T_2)\, g_{1,i0,i0}\, g_{3,j0,i0} - (-1+T_1)(-2+2T_1+T_1T_2)\, g_{1,j0,i0}\, g_{3,j0,i0} + \frac{(-1+T_1)\, g_{2,i0,i0}\, g_{3,j0,i0}}{-1+T_2} + T_1\, g_{2,i0,j0}\, g_{3,j0,i0} + (-2+T_1+T_1T_2)\, g_{2,j0,i0}\, g_{3,j0,i0} + \frac{\left(1+T_1-3T_1T_2+T_1T_2^2\right) g_{2,j0,j0}\, g_{3,j0,i0}}{-1+T_2} + g_{1,i0,i0}\, g_{3,j0,j0} + (-1+T_1)\, g_{1,j0,i0}\, g_{3,j0,j0} - g_{2,i0,i0}\, g_{3,j0,j0} - T_2\, g_{2,j0,i0}\, g_{3,j0,j0}$$

In[•]:=
```
Factor[ (-1 + T1) + (1 + T1 - 3 T1 T2 + T1 T2^2) ]
```

Out[•]=
$$T_1 (-2+T_2)(-1+T_2)$$

In[•]:=
```
CF[s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + ((-1 + T1^s0) (1 - T2^s0 - s0 T2^(2 s0) + s0 T2^(1+2 s0)) g1,j0,i0 g2,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g1,i0,i0 g2,j0,j0 - ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g2,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g3,i0,i0 - s0 (-1 + T2^s0) g2,j0,i0 g3,i0,i0 + 2 s0 g2,j0,j0 g3,i0,i0 +
 ((-2 + s0 - s0 T2 + 3 T2^s0 - T2^(2 s0) + T1^s0 (2 - (3 + s0) T2^s0 + T2^(2 s0) + s0 T2^(1+s0))) g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 ((-2 + (3 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) + T1^s0 (2 - 3 T2^s0 - (-1 + s0) T2^(2 s0) + s0 T2^(1+2 s0))) g1,i0,i0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) -
 1/((-1 + T2) (-1 + T2^s0)) (-1 + T1^s0) (-3 + s0 - s0 T2 + (4 + s0) T2^s0 - T2^(2 s0) - s0 T2^(1+s0) +
    T1^s0 (2 - (2 + s0) T2^s0 - s0 T2^(2 s0) + s0 T2^(1+s0) + s0 T2^(1+2 s0))) g1,j0,i0 g3,j0,i0 + ((-1 + T1^s0) g2,i0,i0 g3,j0,i0)/(-1 + T2) +
 ((-1 + s0 - s0 T2 + T2^s0 + T1^s0 (1 - (1 + s0) T2^s0 + s0 T2^(1+s0))) g2,i0,j0 g3,j0,i0)/((-1 + T2) (-1 + T2^s0)) +
 ((1 + s0 - s0 T2 - T2^s0 + T1^s0 (-1 - (-1 + s0) T2^s0 + s0 T2^(1+s0))) g2,j0,i0 g3,j0,i0)/(-1 + T2) +
 1/((-1 + T2) (-1 + T2^s0)) (1 - 2 s0 + 2 s0 T2 + (-2 + s0) T2^s0 + T2^(2 s0) - s0 T2^(1+s0) +
    T1^s0 (-1 + 2 (1 + s0) T2^s0 - (1 + s0) T2^(2 s0) - 2 s0 T2^(1+s0) + s0 T2^(1+2 s0))) g2,j0,j0 g3,j0,i0 +
 s0 g1,i0,i0 g3,j0,j0 + ((-1 + T1^s0) (1 - (1 + s0) T2^s0 + s0 T2^(1+s0)) g1,j0,i0 g3,j0,j0)/((-1 + T2) (-1 + T2^s0)) -
 s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0 /. s0 → -1]
```

Out[•]=
$$-\frac{1}{2}-\frac{g_{1,i0,i0}\,g_{2,j0,i0}}{T_2}+\frac{(-1+T_1)\,g_{1,j0,i0}\,g_{2,j0,i0}}{T_1\,T_2}+g_{1,i0,i0}\,g_{2,j0,j0}+g_{3,i0,i0}-\frac{(-1+T_2)\,g_{2,j0,i0}\,g_{3,i0,i0}}{T_2}-2\,g_{2,j0,j0}\,g_{3,i0,i0}-\frac{(1-T_1+T_1\,T_2)\,g_{3,j0,i0}}{T_1\,T_2}-\frac{(-2+T_1)\,g_{1,i0,i0}\,g_{3,j0,i0}}{T_1\,T_2}-\frac{(-1+T_1)\,(1-T_1+T_1\,T_2)\,g_{1,j0,i0}\,g_{3,j0,i0}}{T_1^2\,T_2}-\frac{(-1+T_1)\,g_{2,i0,i0}\,g_{3,j0,i0}}{T_1\,(-1+T_2)}-g_{2,i0,j0}\,g_{3,j0,i0}+\frac{(-2+T_1+T_1\,T_2)\,g_{2,j0,i0}\,g_{3,j0,i0}}{T_1\,T_2}+\frac{\left(T_1-T_2-2\,T_1\,T_2+2\,T_1\,T_2^2\right)\,g_{2,j0,j0}\,g_{3,j0,i0}}{T_1\,(-1+T_2)\,T_2}-g_{1,i0,i0}\,g_{3,j0,j0}+g_{2,i0,i0}\,g_{3,j0,j0}+\frac{g_{2,j0,i0}\,g_{3,j0,j0}}{T_2}$$

In[•]:=
```
Factor[-T2 (T1 - 1) + (T1 - T2 - 2 T1 T2 + 2 T1 T2^2)]
```

Out[•]=
$$T_1\,(-1+T_2)\,(-1+2\,T_2)$$