```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
        Once[<< Theta.m]
        SetOptions[PolyPlot, ImageSize → Tiny];
        Clear[Θ]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= RandomVK[n_] := {
          Prepend[#, 2 RandomInteger[1] - 1] & /@
           Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
          Table[RandomInteger[{-1, 1}], 2 n + 1]
         };
```

```
In[•]:= RandomVK[5]
```

```
Out[•]=
{{{-1, 10, 6}, {-1, 4, 8}, {1, 2, 3}, {-1, 1, 9}, {-1, 5, 7}},
 {-1, -1, 1, -1, 0, 1, 1, -1, 0, 0, 0}}
```

```
In[•]:= CF[ℰ_] := Expand@Collect[ℰ, g__ | χ__, F] /. F → Factor@*PowerExpand;
```

```
In[•]:= Short[Options[Θ] = {F1 → (F1i = F1[{s0, i0, j0}]),
          F2 → (F2i = F2[{s0, i0, j0}, {s1, i1, j1}]), F3 → (F3i = F3[φ, k])}]
```

```
Out[•]//Short=
{F1 → s0/2 + s0 T2^s0 g<<1>> g2,j0,i0 + <<24>>, F2 → <<1>>, F3 → -φ/2 + φ g3,k,k}
```

```
In[•]:= Θ[K_, opts___Rule] := Module[{X, φφ, n, A, Δ, G, ev, Θ, kk, k0, k1, f1, f2, f3},
          f1 = F1 /. {opts} /. Options[Θ];
          f2 = F2 /. {opts} /. Options[Θ];
          f3 = F3 /. {opts} /. Options[Θ];
          {X, φφ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
          Cases[X, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
          Δ = T^((-Total[φφ] - Total[X[[All, 1]]])/2) Det[A];
          G = Inverse[A];
          ev[ℰ_] := Factor[ℰ /. {k_^+ :> k + 1, $ → 2 n + 1} /.
              {g_α_,β_ :> G[[α, β]], g_ν_,α_,β_ :> (G[[α, β]] /. T → T_ν)}];
          Θ = ev@Sum[f1 /. Thread[{s0, i0, j0} → X[[kk]]], {kk, n}];
          Θ += ev@Sum[f2 /. Thread[{s0, i0, j0} → X[[k0]]] //.
               Thread[{s1, i1, j1} → X[[k1]]], {k0, n}, {k1, n}];
          Θ += ev@Sum[f3 /. {φ → φφ[[kk]], k → kk}, {kk, Length@φφ}];
          Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) Θ}
         ];
```

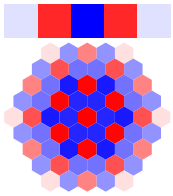
In[•]:=
```
K1 = Knot[7, 6];
K1 // Rot
K1 // Θ
K1 // Θ // PolyPlot
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
$$\{\{\{-1, 4, 1\}, \{-1, 8, 3\}, \{-1, 12, 5\}, \{1, 14, 9\}, \{1, 10, 13\}, \{-1, 6, 11\}, \{-1, 2, 7\}\},\ \{0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, -1, -1, 1, 0\}\}$$

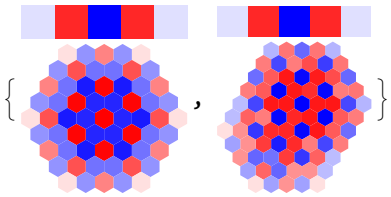
Out[•]=
$$\left\{-\frac{1 - 5\,T + 7\,T^2 - 5\,T^3 + T^4}{T^2},\right.$$
$$\frac{1}{T_1^4\,T_2^4}\,\left(1 - 5\,T_1 + 7\,T_1^2 - 5\,T_1^3 + T_1^4 - 5\,T_2 + 20\,T_1\,T_2 - 10\,T_1^2\,T_2 - 10\,T_1^3\,T_2 + 20\,T_1^4\,T_2 - 5\,T_1^5\,T_2 + 7\,T_2^2 - 10\,T_1\,T_2^2 - 64\,T_1^2\,T_2^2 + 98\,T_1^3\,T_2^2 - 64\,T_1^4\,T_2^2 - 10\,T_1^5\,T_2^2 + 7\,T_1^6\,T_2^2 - 5\,T_2^3 - 10\,T_1\,T_2^3 + 98\,T_1^2\,T_2^3 - 50\,T_1^3\,T_2^3 - 50\,T_1^4\,T_2^3 + 98\,T_1^5\,T_2^3 - 10\,T_1^6\,T_2^3 - 5\,T_1^7\,T_2^3 + T_2^4 + 20\,T_1\,T_2^4 - 64\,T_1^2\,T_2^4 - 50\,T_1^3\,T_2^4 + 108\,T_1^4\,T_2^4 - 50\,T_1^5\,T_2^4 - 64\,T_1^6\,T_2^4 + 20\,T_1^7\,T_2^4 + T_1^8\,T_2^4 - 5\,T_1\,T_2^5 - 10\,T_1^2\,T_2^5 + 98\,T_1^3\,T_2^5 - 50\,T_1^4\,T_2^5 - 50\,T_1^5\,T_2^5 + 98\,T_1^6\,T_2^5 - 10\,T_1^7\,T_2^5 - 5\,T_1^8\,T_2^5 + 7\,T_1^2\,T_2^6 - 10\,T_1^3\,T_2^6 - 64\,T_1^4\,T_2^6 + 98\,T_1^5\,T_2^6 - 64\,T_1^6\,T_2^6 - 10\,T_1^7\,T_2^6 + 7\,T_1^8\,T_2^6 - 5\,T_1^3\,T_2^7 + 20\,T_1^4\,T_2^7 - 10\,T_1^5\,T_2^7 - 10\,T_1^6\,T_2^7 + 20\,T_1^7\,T_2^7 - 5\,T_1^8\,T_2^7 + T_1^4\,T_2^8 - 5\,T_1^5\,T_2^8 + 7\,T_1^6\,T_2^8 - 5\,T_1^7\,T_2^8 + T_1^8\,T_2^8\right)\Big\}$$

Out[•]=



In[•]:=
```
PolyPlot /@ {Θ[K1], Θ[K1, F3 → 0]}
```

Out[•]=



In[•]:=
```
χ_True = 1; χ_False = 0;
χ_{α_≤α_} = 1; χ_{ph_}^{p_} /; p > 1 ^:= χ_ph; χ_{α_==β_} /; OrderedQ[{β, α}] := χ_{β==α};
χ_{α_⁺==β_⁺} := χ_{α==β};
δ_{i_,j_} := χ_{i==j};
```

```
In[•]:= gRules[{s_, i_, j_}] := {g_{ν_,jβ_} :> g_{νj⁺β} + δ_{jβ}, g_{ν_,iβ_} :> T^s_ν g_{νi⁺β} + (1 - T^s_ν) g_{νj⁺β} + δ_{iβ},
          g_{ν_,α_,i⁺} :> T^s_ν g_{ναi} + δ_{αi⁺}, g_{ν_,α_,j⁺} :> g_{ναj} + (1 - T^s_ν) g_{ναi} + δ_{αj⁺},
          g_{jβ_} :> g_{j⁺β} + δ_{jβ}, g_{iβ_} :> T^s g_{i⁺β} + (1 - T^s) g_{j⁺β} + δ_{iβ},
          g_{α_,i⁺} :> T^s g_{αi} + δ_{αi⁺}, g_{α_,j⁺} :> g_{αj} + (1 - T^s) g_{αi} + δ_{αj⁺}
        };
        bRules[{s_, i_, j_}] := { (* b for "push indices backwards" *)
          g_{j⁺,β_} :> g_{j,β} - δ_{j,β}, g_{i⁺,β_} :> T^-s g_{i,β} + (1 - T^-s) g_{j,β} - T^-s δ_{i,β} - (1 - T^-s) δ_{j,β},
          g_{α_,i⁺} :> T^s g_{α,i} + δ_{α,i⁺}, g_{α_,j⁺} :> g_{α,j} + (1 - T^s) g_{α,i} + δ_{α,j⁺},
          χ_{i⁺≤β_} :> χ_{i≤β} - χ_{i==β}, χ_{j⁺≤β_} :> χ_{j≤β} - χ_{j==β}
        };
```

```
In[•]:= {g_{i,β}, g_{j,β}} /. gRules[{s, i, j}]
```

```
Out[•]= {χ_{i==β} + T^s g_{i⁺,β} + (1 - T^s) g_{j⁺,β}, χ_{j==β} + g_{j⁺,β}}
```

```
In[•]:= Expand[{g_{i,β}, g_{j,β}} /. gRules[{s, i, j}] /. bRules[{s, i, j}]]
```

```
Out[•]= {g_{i,β}, g_{j,β}}
```

```
In[•]:= R1 = CF[Residue[F_1[{s0, i0, j0}], {T_2, 1}] //.
          {T_1 → T, g_{2,α_,β_} :> χ_{α≤β}, g_{1|3,α_,β_} :> g_{α,β}, χ_{j0≤i0} → 1 - χ_{i0≤j0}}]
```

```
Out[•]= (1 - T^s0) g_{i0,i0} g_{j0,i0} - 2 (-1 + T^s0)^2 g^2_{j0,i0} + (-1 + T^s0) g_{j0,i0} g_{j0,j0}
```

```
In[•]:= R2 = CF[Residue[F_2[{s0, i0, j0}, {s1, i1, j1}], {T_2, 1}] //.
          {T_1 → T, g_{2,α_,β_} :> χ_{α≤β}, g_{1|3,α_,β_} :> g_{α,β}}]
```

```
Out[•]= (-1 + T^s0) (-1 + T^s1) χ_{i1≤i0} g_{j0,i1} g_{j1,i0} - (-1 + T^s0) (-1 + T^s1) χ_{i1≤j0} g_{j0,i1} g_{j1,i0} -
         (-1 + T^s0) (-1 + T^s1) χ_{j1≤i0} g_{j0,i1} g_{j1,i0} + (-1 + T^s0) (-1 + T^s1) χ_{j1≤j0} g_{j0,i1} g_{j1,i0}
```

```
In[•]:= Table[Θ[K, F1 → R1, F2 → R2, F3 → 0][[2]], {K, AllKnots[{3, 8}]}]
```

```
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
         0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= Table[Θ[RandomVK[10], F1 → R1, F2 → R2, F3 → 0][[2]], 30]
```

```
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= D_{{s_,i_,j_}}[ℰ_] :=
          CF[((ℰ /. ⌗ → i⁺) + (ℰ /. ⌗ → j⁺) - (ℰ /. ⌗ → i) - (ℰ /. ⌗ → j)) //. bRules[{s, i, j}]];
        B[ℰ_] := CF[((ℰ /. ⌗ → $) - (ℰ /. ⌗ → 1))];
```

```
In[•]:= bas = List @@ Expand[(χ#≤i0 + χ#≤j0) (gi0⁺,# + gj0⁺,#) (g#,i0 + g#,j0)]
```

```
Out[•]=
{χ#1≤i0 g#1,i0 gi0⁺,#1, χ#1≤j0 g#1,i0 gi0⁺,#1, χ#1≤i0 g#1,j0 gi0⁺,#1, χ#1≤j0 g#1,j0 gi0⁺,#1,
 χ#1≤i0 g#1,i0 gj0⁺,#1, χ#1≤j0 g#1,i0 gj0⁺,#1, χ#1≤i0 g#1,j0 gj0⁺,#1, χ#1≤j0 g#1,j0 gj0⁺,#1}
```

```
In[•]:= (B /@ bas) /. {g(i0|j0)⁺,1 → 0, g$,i0|j0 → 0}
```

```
Out[•]=
{0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= D{s1,i1,j1}[bas] // Short
```

```
Out[•]//Short=
{T^-s1 χi0==i1 + χi0==j1 + <<34>>, <<1>>, <<4>>, T^-s1 χ<<1>> χ<<1>> + <<35>>, T^-s1 χi1==j0 + <<35>>}
```

```
In[•]:= sRules[ℰ_] := FixedPoint[CF[# /. bRules[{s0, i0, j0}] ⋃ bRules[{s1, i1, j1}] ⋃ {
        χi0==j1 → 0, χi1==j0 → 0, χi0==j0 → 0,
        χj0==j1 → χi0==i1,
        χi1⁺≤i0 → χi1≤i0 − χi0==i1, χi0⁺==i1⁺ → χi0==i1,
        χi0==i1 χi1≤i0 → χi0==i1, χi0==i1 χj1≤j0 → χi0==i1,
        χj0≤i0 → 1 − χi0≤j0, χi1≤i0 → 1 − χi0≤i1 + χi1==i0,
        χj1≤i0 → 1 − χi0≤j1 + χj1==i0, χi1≤i0 → 1 − χi0≤i1 + χi1==i0, χj1≤j0 → 1 − χj0≤j1 + χj1==j0,
        χi0==i1 γ_ /; ! FreeQ[γ, s1 | i1 | j1] :↦ χi0==i1 (γ /. {s1 → s0, i1 → i0, j1 → j0}),
        χα_==β_ gβ_,γ_ :↦ χα==β gα,γ, χα_==β_ gγ_,β_ :↦ χα==β gγ,α
       }] &,
  ℰ]
```

In[•]:=

```
nulls = sRules[D{s1,i1,j1}[bas]]
Table[Θ[K, F1 → 0, F2 → #, F3 → 0]⟦2⟧ & /@ nulls, {K, AllKnots[{3, 8}]}]
```

Out[•]=

$$\{T^{-s0}\chi_{i0==i1} g_{i0,i0} - T^{-s0}\chi_{i0==i1} g^2_{i0,i0} + T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{j0,i0} -$$
$$T^{-s0}(-1+T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{i0,i0} g_{j0,i0} -$$
$$T^{-s0}(-1+T^{s1})\chi_{i0\le i1} g_{i0,i1} g_{j1,i0} + T^{-s0}(-1+T^{s1})\chi_{i0\le j1} g_{i0,i1} g_{j1,i0} -$$
$$T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i0\le i1} g_{j0,i1} g_{j1,i0} + T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i0\le j1} g_{j0,i1} g_{j1,i0},$$
$$T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{j0,i0} - T^{-s0}(-1+T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - T^{-s0}\chi_{i0==i1} g_{i0,j0} g_{j0,i0} -$$
$$T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{j0,i0} g_{j0,j0} - T^{-s0}(-1+T^{s1}) g_{i0,i1} g_{j1,i0} + T^{-s0}(-1+T^{s1})\chi_{i1\le j0} g_{i0,i1} g_{j1,i0} +$$
$$T^{-s0}(-1+T^{s1})\chi_{j0\le j1} g_{i0,i1} g_{j1,i0} - T^{-s0}(-1+T^{s0})(-1+T^{s1}) g_{j0,i1} g_{j1,i0} +$$
$$T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i1\le j0} g_{j0,i1} g_{j1,i0} + T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{j0\le j1} g_{j0,i1} g_{j1,i0},$$
$$T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{i0,i0} - T^{-s0}(-1+T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{i0,i0} +$$
$$T^{-s0}\chi_{i0==i1}\chi_{i0\le j0} g_{i0,j0} - T^{-s0}\chi_{i0==i1} g_{i0,i0} g_{i0,j0} + T^{-s0}(-1+T^{s0})^2\chi_{i0==i1} g_{j0,i0} -$$
$$T^{-s0}(-1+T^{s0})^2\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{i0,j0} g_{j0,i0} -$$
$$T^{-s0}(-1+T^{s1})\chi_{i0\le i1} g_{i0,i1} g_{j1,j0} + T^{-s0}(-1+T^{s1})\chi_{i0\le j1} g_{i0,i1} g_{j1,j0} -$$
$$T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i0\le i1} g_{j0,i1} g_{j1,j0} + T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i0\le j1} g_{j0,i1} g_{j1,j0},$$
$$-T^{-s0}(-1+T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{i0,i0} + T^{-s0}\chi_{i0==i1}\chi_{i0\le j0} g_{i0,j0} - T^{-s0}(-1+T^{s0})^2\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} +$$
$$T^{-s0}(-1+T^{s0})\chi_{i0==i1} g_{j0,j0} - T^{-s0}\chi_{i0==i1} g_{i0,j0} g_{j0,j0} - T^{-s0}(-1+T^{s0})\chi_{i0==i1} g^2_{j0,j0} -$$
$$T^{-s0}(-1+T^{s1}) g_{i0,i1} g_{j1,j0} + T^{-s0}(-1+T^{s1})\chi_{i1\le j0} g_{i0,i1} g_{j1,j0} +$$
$$T^{-s0}(-1+T^{s1})\chi_{j0\le j1} g_{i0,i1} g_{j1,j0} - T^{-s0}(-1+T^{s0})(-1+T^{s1}) g_{j0,i1} g_{j1,j0} +$$
$$T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{i1\le j0} g_{j0,i1} g_{j1,j0} + T^{-s0}(-1+T^{s0})(-1+T^{s1})\chi_{j0\le j1} g_{j0,i1} g_{j1,j0},$$
$$\chi_{i0==i1} g_{j0,i0} - \chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - \chi_{i0==i1} g_{i0,i0} g_{j0,i0} + (1-T^{s1})\chi_{i0\le i1} g_{j0,i1} g_{j1,i0} +$$
$$(-1+T^{s1})\chi_{i0\le j1} g_{j0,i1} g_{j1,i0},\ \chi_{i0==i1} g_{j0,i0} - \chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - \chi_{i0==i1} g_{j0,i0} g_{j0,j0} +$$
$$(1-T^{s1}) g_{j0,i1} g_{j1,i0} + (-1+T^{s1})\chi_{i1\le j0} g_{j0,i1} g_{j1,i0} + (-1+T^{s1})\chi_{j0\le j1} g_{j0,i1} g_{j1,i0},$$
$$(-1+T^{s0})\chi_{i0==i1} g_{j0,i0} + (1-T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} - \chi_{i0==i1} g_{i0,j0} g_{j0,i0} +$$
$$(1-T^{s1})\chi_{i0\le i1} g_{j0,i1} g_{j1,j0} + (-1+T^{s1})\chi_{i0\le j1} g_{j0,i1} g_{j1,j0},$$
$$(1-T^{s0})\chi_{i0==i1}\chi_{i0\le j0} g_{j0,i0} + \chi_{i0==i1} g_{j0,j0} - \chi_{i0==i1} g^2_{j0,j0} + (1-T^{s1}) g_{j0,i1} g_{j1,j0} +$$
$$(-1+T^{s1})\chi_{i1\le j0} g_{j0,i1} g_{j1,j0} + (-1+T^{s1})\chi_{j0\le j1} g_{j0,i1} g_{j1,j0}\}$$

Out[•]=

```
{{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}}
```

In[•]:= **sRules[R2]**

Out[•]=

$$2\left(-1+T^{s0}\right)^2\chi_{i0=i1}\,g_{j0,i0}^2+\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)g_{j0,i1}\,g_{j1,i0}-\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,i0}+\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,i0}-\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,i0}-\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,i0}$$

In[•]:= **CF[nulls /. $\chi_{i0=i1}\to 0$]**

Out[•]=

$$\Big\{-T^{-s0}\left(-1+T^{s1}\right)\chi_{i0\le i1}\,g_{i0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{i0,i1}\,g_{j1,i0}-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,i0},$$
$$-T^{-s0}\left(-1+T^{s1}\right)g_{i0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{i0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{i0,i1}\,g_{j1,i0}-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)g_{j0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,i0},$$
$$-T^{-s0}\left(-1+T^{s1}\right)\chi_{i0\le i1}\,g_{i0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{i0,i1}\,g_{j1,j0}-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,j0},$$
$$-T^{-s0}\left(-1+T^{s1}\right)g_{i0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{i0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{i0,i1}\,g_{j1,j0}-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)g_{j0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,j0},$$
$$\left(1-T^{s1}\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,i0}+\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,i0},$$
$$\left(1-T^{s1}\right)g_{j0,i1}\,g_{j1,i0}+\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,i0}+\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,i0},$$
$$\left(1-T^{s1}\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,j0}+\left(-1+T^{s1}\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,j0},$$
$$\left(1-T^{s1}\right)g_{j0,i1}\,g_{j1,j0}+\left(-1+T^{s1}\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,j0}+\left(-1+T^{s1}\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,j0}\Big\}$$

In[•]:= **Coefficient[CF[nulls /. $\chi_{i0=i1}\to 0$], $g_{j0,i1}\,g_{j1,i0}$]**

Out[•]=

$$\Big\{-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le i1}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i0\le j1},$$
$$-T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{i1\le j0}+T^{-s0}\left(-1+T^{s0}\right)\left(-1+T^{s1}\right)\chi_{j0\le j1},$$
$$0,\ 0,\ \left(1-T^{s1}\right)\chi_{i0\le i1}+\left(-1+T^{s1}\right)\chi_{i0\le j1},\ 1-T^{s1}+\left(-1+T^{s1}\right)\chi_{i1\le j0}+\left(-1+T^{s1}\right)\chi_{j0\le j1},\ 0,\ 0\Big\}$$

In[•]:= **CF[Coefficient[nulls, $\chi_{i0=i1}$]]**

Out[•]=

$$\Big\{T^{-s0}\,g_{i0,i0}-T^{-s0}\,g_{i0,i0}^2+T^{-s0}\left(-1+T^{s0}\right)g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)\chi_{i0\le j0}\,g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)g_{i0,i0}\,g_{j0,i0},$$
$$T^{-s0}\left(-1+T^{s0}\right)g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)\chi_{i0\le j0}\,g_{j0,i0}-T^{-s0}\,g_{i0,j0}\,g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)g_{j0,i0}\,g_{j0,j0},$$
$$T^{-s0}\left(-1+T^{s0}\right)g_{i0,i0}-T^{-s0}\left(-1+T^{s0}\right)\chi_{i0\le j0}\,g_{i0,i0}+T^{-s0}\,\chi_{i0\le j0}\,g_{i0,j0}-T^{-s0}\,g_{i0,i0}\,g_{i0,j0}+T^{-s0}\left(-1+T^{s0}\right)^2g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)^2\chi_{i0\le j0}\,g_{j0,i0}-T^{-s0}\left(-1+T^{s0}\right)g_{i0,j0}\,g_{j0,i0},$$
$$-T^{-s0}\left(-1+T^{s0}\right)\chi_{i0\le j0}\,g_{i0,i0}+T^{-s0}\,\chi_{i0\le j0}\,g_{i0,j0}-T^{-s0}\left(-1+T^{s0}\right)^2\chi_{i0\le j0}\,g_{j0,i0}+T^{-s0}\left(-1+T^{s0}\right)g_{j0,j0}-T^{-s0}\,g_{i0,j0}\,g_{j0,j0}-T^{-s0}\left(-1+T^{s0}\right)g_{j0,j0}^2,$$
$$g_{j0,i0}-\chi_{i0\le j0}\,g_{j0,i0}-g_{i0,i0}\,g_{j0,i0},\ g_{j0,i0}-\chi_{i0\le j0}\,g_{j0,i0}-g_{j0,i0}\,g_{j0,j0},$$
$$\left(-1+T^{s0}\right)g_{j0,i0}+\left(1-T^{s0}\right)\chi_{i0\le j0}\,g_{j0,i0}-g_{i0,j0}\,g_{j0,i0},\ \left(1-T^{s0}\right)\chi_{i0\le j0}\,g_{j0,i0}+g_{j0,j0}-g_{j0,j0}^2\Big\}$$

In[•]:=
```
CF[sRules[R2] + Table[a_i, {i, 8}].nulls /. χ_{i0=!=i1} → 0]
```

Out[•]=

$$-T^{-s0}\left(-1+T^{s1}\right)a_2\,g_{i0,i1}\,g_{j1,i0}-T^{-s0}\left(-1+T^{s1}\right)a_1\,\chi_{i0\le i1}\,g_{i0,i1}\,g_{j1,i0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)a_1\,\chi_{i0\le j1}\,g_{i0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s1}\right)a_2\,\chi_{i1\le j0}\,g_{i0,i1}\,g_{j1,i0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)a_2\,\chi_{j0\le j1}\,g_{i0,i1}\,g_{j1,i0}+T^{-s0}\left(-1+T^{s1}\right)\left(-T^{s0}+T^{2\,s0}+a_2-T^{s0}\,a_2-T^{s0}\,a_6\right)g_{j0,i1}\,g_{j1,i0}-$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-T^{s0}+T^{2\,s0}-a_1+T^{s0}\,a_1+T^{s0}\,a_5\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,i0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-T^{s0}+T^{2\,s0}-a_1+T^{s0}\,a_1+T^{s0}\,a_5\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,i0}-$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-T^{s0}+T^{2\,s0}+a_2-T^{s0}\,a_2-T^{s0}\,a_6\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,i0}-$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-T^{s0}+T^{2\,s0}+a_2-T^{s0}\,a_2-T^{s0}\,a_6\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,i0}-$$
$$T^{-s0}\left(-1+T^{s1}\right)a_4\,g_{i0,i1}\,g_{j1,j0}-T^{-s0}\left(-1+T^{s1}\right)a_3\,\chi_{i0\le i1}\,g_{i0,i1}\,g_{j1,j0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)a_3\,\chi_{i0\le j1}\,g_{i0,i1}\,g_{j1,j0}+T^{-s0}\left(-1+T^{s1}\right)a_4\,\chi_{i1\le j0}\,g_{i0,i1}\,g_{j1,j0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)a_4\,\chi_{j0\le j1}\,g_{i0,i1}\,g_{j1,j0}-T^{-s0}\left(-1+T^{s1}\right)\left(-a_4+T^{s0}\,a_4+T^{s0}\,a_8\right)g_{j0,i1}\,g_{j1,j0}-$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-a_3+T^{s0}\,a_3+T^{s0}\,a_7\right)\chi_{i0\le i1}\,g_{j0,i1}\,g_{j1,j0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-a_3+T^{s0}\,a_3+T^{s0}\,a_7\right)\chi_{i0\le j1}\,g_{j0,i1}\,g_{j1,j0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-a_4+T^{s0}\,a_4+T^{s0}\,a_8\right)\chi_{i1\le j0}\,g_{j0,i1}\,g_{j1,j0}+$$
$$T^{-s0}\left(-1+T^{s1}\right)\left(-a_4+T^{s0}\,a_4+T^{s0}\,a_8\right)\chi_{j0\le j1}\,g_{j0,i1}\,g_{j1,j0}$$

In[•]:=
```
CF[sRules[R2] + Table[a_i, {i, 8}].nulls /. χ_{i0=!=i1} → 0 //.
  {a_{1|2|3|4|7|8} → 0, a_6 → T^s0 - 1, a_5 → 1 - T^s0}]
```

Out[•]=

$$0$$

In[•]:=
```
tw = Table[a_i, {i, 8}].bas //. {a_{1|2|3|4|7|8} → 0, a_6 → T^s0 - 1, a_5 → 1 - T^s0}
```

Out[•]=

$$\left(1-T^{s0}\right)\chi_{\#1\le i0}\,g_{\#1,i0}\,g_{j0^+,\#1}+\left(-1+T^{s0}\right)\chi_{\#1\le j0}\,g_{\#1,i0}\,g_{j0^+,\#1}$$

In[•]:=
```
CF@sRules[R2 + D_{s1,i1,j1}[tw]]
```

Out[•]=

$$\left(-1+T^{s0}\right)\chi_{i0=!=i1}\,g_{i0,i0}\,g_{j0,i0}+2\left(-1+T^{s0}\right)^2\chi_{i0=!=i1}\,g_{j0,i0}^2+\left(1-T^{s0}\right)\chi_{i0=!=i1}\,g_{j0,i0}\,g_{j0,j0}$$

In[•]:=
```
R1
```

Out[•]=

$$\left(1-T^{s0}\right)g_{i0,i0}\,g_{j0,i0}-2\left(-1+T^{s0}\right)^2g_{j0,i0}^2+\left(-1+T^{s0}\right)g_{j0,i0}\,g_{j0,j0}$$

In[•]:=
```
CF@sRules[χ_{i0=!=i1} R1 + R2 + D_{s1,i1,j1}[tw]]
```

Out[•]=

$$0$$