In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
Once[<< Theta.m]
SetOptions[PolyPlot, ImageSize → Tiny];
Clear[Θ]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
RandomVK[n_] := {
   Prepend[#, 2 RandomInteger[1] - 1] & /@
    Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
   Table[RandomInteger[{-1, 1}], 2 n + 1]
  };
```

In[∘]:=
```
RandomVK[5]
```

Out[∘]=
```
{{{1, 7, 3}, {-1, 6, 2}, {1, 4, 9}, {-1, 10, 1}, {1, 5, 8}},
 {-1, -1, 0, 1, 0, -1, 1, 1, -1, 1, 0}}
```

In[∘]:=
```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor@*PowerExpand;
```

In[∘]:=
```
Short[Options[Θ] = {F1 → (F1i = F₁[{s0, i0, j0}]),
   F2 → (F2i = F₂[{s0, i0, j0}, {s1, i1, j1}]), F3 → (F3i = F₃[φ, k])}]
```

Out[∘]//Short=
```
{ ≪1≫ }
```

In[∘]:=
```
Θ[K_, opts___Rule] := Module[{X, φφ, n, A, Δ, G, ev, θ, kk, k0, k1, f1, f2, f3},
   f1 = F1 /. {opts} /. Options[Θ];
   f2 = F2 /. {opts} /. Options[Θ];
   f3 = F3 /. {opts} /. Options[Θ];
   {X, φφ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
   Cases[X, {s_, i_, j_} ⧴ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                              0    -1     ))];
   Δ = T^((-Total[φφ]-Total[X[[All,1]]])/2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[
     ℰ /. {k_^+ ⧴ k + 1, $ → 2 n + 1} /. {g_{ν_,α_,β_} ⧴ (G[[α, β]] /. T → T_ν), χ_True → 1, χ_False → 0}];
   θ = ev@Sum[f1 /. Thread[{s0, i0, j0} → X[[kk]]], {kk, n}];
   θ += ev@Sum[f2 /. Thread[{s0, i0, j0} → X[[k0]]] //.
       Thread[{s1, i1, j1} → X[[k1]]], {k0, n}, {k1, n}];
   θ += ev@Sum[f3 /. {φ → φφ[[kk]], k → kk}, {kk, Length@φφ}];
   Factor@{Δ, (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) θ}
  ];
```

In[•]:=
```
Θ[Knot[7, 3]]
```

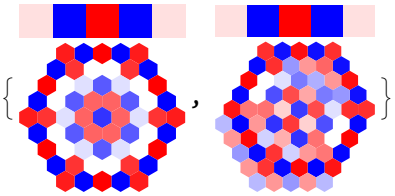
KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\right.$$
$$\frac{1}{T_1^4\,T_2^4}\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-\right.$$
$$T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+$$
$$17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-$$
$$\left.\left.7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-\right.\right.$$
$$\left.\left.25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\}$$

In[•]:=
```
PolyPlot /@ {Θ[Knot[7, 3]], Θ[Knot[7, 3], F3 → 0]}
```

Out[•]=



In[•]:=
```
δ_,j_ := χ_{i==j}; χ^p_ph_ /; p > 1 ^:= χ_ph;
bRules[{s_, i_, j_}] := { (* b for "push indices backwards" *)
    g_ν_,j^+,β_ :> g_ν,j,β - δ_j,β, g_ν_,i^+,β_ :> T_ν^-s g_ν,i,β + (1 - T_ν^-s) g_ν,j,β - T_ν^-s δ_i,β - (1 - T_ν^-s) δ_j,β,
    g_ν_,α_,i^+ :> T_ν^s g_ν,α,i + δ_α,i^+, g_ν_,α_,j^+ :> g_ν,α,j + (1 - T_ν^s) g_ν,α,i + δ_α,j^+
  };
bRules[X___List] := Union @@ Table[bRules[c], {c, {X}}]
```

In[•]:=
```
{g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}]
```

Out[•]=

$$\{\chi_{i==\beta}+T_\nu^s\,g_{\nu,i^+,\beta}+(1-T_\nu^s)\,g_{\nu,j^+,\beta},\ \chi_{j==\beta}+g_{\nu,j^+,\beta}\}$$

In[•]:=
```
Expand[{g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}] /. bRules[{s, i, j}]]
```

Out[•]=

$$\{g_{\nu,i,\beta},\ g_{\nu,j,\beta}\}$$

```
In[•]:= D_{s_,i_,j_}[ℰ_] := CF[Expand[Plus[
              ℰ /. {g_{ν_,#,β_} :> g_{ν,i⁺,β}, g_{ν_,α_,#} :> g_{ν,α,i⁺}},
              ℰ /. {g_{ν_,#,β_} :> g_{ν,j⁺,β}, g_{ν_,α_,#} :> g_{ν,α,j⁺}},
              -ℰ /. {g_{ν_,#,β_} :> g_{ν,i,β}, g_{ν_,α_,#} :> g_{ν,α,i}},
              -ℰ /. {g_{ν_,#,β_} :> g_{ν,j,β}, g_{ν_,α_,#} :> g_{ν,α,j}}
            ] /. bRules[{s, i, j}] /. bRules[{s0, i0, j0}]] /. χ_{eq_Equal} :> χ_{Sort@eq} //. {
       χ_{j0==j1} → χ_{i0==i1}, χ_{j0⁺==j1⁺} → χ_{j0==j1},
       χ_{i0==j1} → 0, χ_{i1==j0} → 0, χ_{i0⁺==j1⁺} → 0, χ_{i0⁺==i1⁺} → χ_{i0==i1}, χ_{i1⁺==j0⁺} → 0, χ_{i0⁺==j0⁺} → 0,
       χ_{i0==i1} A_ :> χ_{i0==i1} (A /. {s1 → s0, i1 → i0, j1 → j0})
      } /. {χ_{i0==j0} → 0, χ_{i0⁺==j0⁺} → 0, χ_{True} → 1}
  ];
B[ℰ_] := CF[
    (ℰ /. {g_{ν_,#,β_} :> g_{ν,$,β}, g_{ν_,α_,#} :> g_{ν,α,$}}) - (ℰ /. {g_{ν_,#,β_} :> g_{ν,1,β}, g_{ν_,α_,#} :> g_{ν,α,1}}) /. {
       g_{_,$,β_} → χ_{$==β}, g_{_,_,$} → 1, g_{_,α_,1} → χ_{α==1}} /. {χ_{$==i0} → 0, χ_{i0⁺==1} → 0, χ_{$==j0} → 0, χ_{j0⁺==1} → 0}
  ];
```

```
In[•]:= D_{s1,i1,j1}[g_{1,#,i0}]
```

Out[•]=

$$-T_1^{-s0}\,\chi_{i0==i1}-T_1^{-s1}\left(-1+T_1^{s1}\right)g_{1,i1,i0}+T_1^{-s1}\left(-1+T_1^{s1}\right)g_{1,j1,i0}$$

```
In[•]:= D_{s1,i1,j1}[g_{3,j0,#}]
```

Out[•]=

$$\chi_{j0==i1^+}+\chi_{j0==j1^+}$$

```
In[•]:= B[g_{3,j0,#}]
```

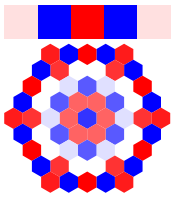
Out[•]=

$$1-\chi_{j0==1}$$

```
In[•]:= tw = g_{1,#,i0};
        ⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $$\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$$
$$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-\right.$$
$$25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-$$
$$T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-$$
$$\left.\left.T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\}$$

Out[•]=



In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
```

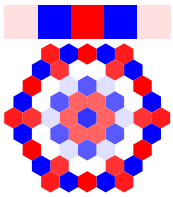
» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ 0\right\}$

Out[•]=

In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$
$\left(17 - 25\,T_1 + 25\,T_1^2 - 25\,T_1^3 + 17\,T_1^4 - 25\,T_2 + 12\,T_1\,T_2 + 12\,T_1^4\,T_2 - 25\,T_1^5\,T_2 + 25\,T_2^2 - T_1^2\,T_2^2 - 7\,T_1^3\,T_2^2 - T_1^4\,T_2^2 + 25\,T_1^6\,T_2^2 - 25\,T_2^3 - 7\,T_1^2\,T_2^3 + 6\,T_1^3\,T_2^3 + 6\,T_1^4\,T_2^3 - 7\,T_1^5\,T_2^3 - 25\,T_1^7\,T_2^3 + 17\,T_2^4 + 12\,T_1\,T_2^4 - T_1^2\,T_2^4 + 6\,T_1^3\,T_2^4 - 12\,T_1^4\,T_2^4 + 6\,T_1^5\,T_2^4 - T_1^6\,T_2^4 + 12\,T_1^7\,T_2^4 + 17\,T_1^8\,T_2^4 - 25\,T_1\,T_2^5 - 7\,T_1^3\,T_2^5 + 6\,T_1^4\,T_2^5 + 6\,T_1^5\,T_2^5 - 7\,T_1^6\,T_2^5 - 25\,T_1^8\,T_2^5 + 25\,T_1^2\,T_2^6 - T_1^4\,T_2^6 - 7\,T_1^5\,T_2^6 - T_1^6\,T_2^6 + 25\,T_1^8\,T_2^6 - 25\,T_1^3\,T_2^7 + 12\,T_1^4\,T_2^7 + 12\,T_1^7\,T_2^7 - 25\,T_1^8\,T_2^7 + 17\,T_1^4\,T_2^8 - 25\,T_1^5\,T_2^8 + 25\,T_1^6\,T_2^8 - 25\,T_1^7\,T_2^8 + 17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



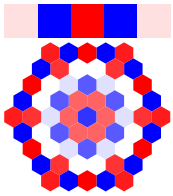
In[•]:=
```
tw = g1,#1,j0 g2,#1,j0 g3,i0,#1;
Θ[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ 0\right\}$

Out[•]=

» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$
$\left(17 - 25\,T_1 + 25\,T_1^2 - 25\,T_1^3 + 17\,T_1^4 - 25\,T_2 + 12\,T_1\,T_2 + 12\,T_1^4\,T_2 - 25\,T_1^5\,T_2 + 25\,T_2^2 - T_1^2\,T_2^2 - 7\,T_1^3\,T_2^2 - T_1^4\,T_2^2 + 25\,T_1^6\,T_2^2 - 25\,T_2^3 - 7\,T_1^2\,T_2^3 + 6\,T_1^3\,T_2^3 + 6\,T_1^4\,T_2^3 - 7\,T_1^5\,T_2^3 - 25\,T_1^7\,T_2^3 + 17\,T_2^4 + 12\,T_1\,T_2^4 - T_1^2\,T_2^4 + 6\,T_1^3\,T_2^4 - 12\,T_1^4\,T_2^4 + 6\,T_1^5\,T_2^4 - T_1^6\,T_2^4 + 12\,T_1^7\,T_2^4 + 17\,T_1^8\,T_2^4 - 25\,T_1\,T_2^5 - 7\,T_1^3\,T_2^5 + 6\,T_1^4\,T_2^5 + 6\,T_1^5\,T_2^5 - 7\,T_1^6\,T_2^5 - 25\,T_1^8\,T_2^5 + 25\,T_1^2\,T_2^6 - T_1^4\,T_2^6 - 7\,T_1^5\,T_2^6 - T_1^6\,T_2^6 + 25\,T_1^8\,T_2^6 - 25\,T_1^3\,T_2^7 + 12\,T_1^4\,T_2^7 + 12\,T_1^7\,T_2^7 - 25\,T_1^8\,T_2^7 + 17\,T_1^4\,T_2^8 - 25\,T_1^5\,T_2^8 + 25\,T_1^6\,T_2^8 - 25\,T_1^7\,T_2^8 + 17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



In[•]:=
```
bas = List @@ Expand[(g1,#,i0 + g1,#,j0) (g2,#,i0 + g2,#,j0) (g3,i0⁺,# + g3,j0⁺,#)]
```

Out[•]=

$\{g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\}$

In[•]:=
```
Column[(# → Θ[Knot[7, 3], F1 → -B[#], F2 → D{s1,i1,j1}[#], F3 → 0][[2]]) & /@ bas]
Column[(# → {B[#], D{s1,i1,j1}[#]}) & /@ bas]
```

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to$
$\{0,\ -T_1^{-s0}\,T_2^{-s0}\left(-2 + T_1^{s1} + T_2^{s1}\right) g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1} + \chi_{i0=i1}\left(T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} + T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{3,j0,i0} + g_{2,i0,i0}\left(-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{3,j0,i0}\right) + g_{1,i0,i0}\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,i0} - T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{3,j0,i0}\right) + g_{1,j0,i0}\left(T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{2,j0,i0} - T_1^{-s0}\left(-1 + T_1^{s0}\right) T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\left(-1 + T_1^{s0}\right) T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{3,j0,i0}\right) + g_{2,j0,i0}\left(-T_1^{-s0}\,T_2^{-s0}\left(-1 + T_2^{s0}\right) g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_2^{s0}\right)\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{3,j0,i0}\right)\right) -$
$T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)\left(-2 + T_1^{s1} + T_2^{s1}\right) g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} +$
$g_{2,i1,i0}\left(T_1^{-s0}\left(-1 + T_1^{s1}\right) T_2^{-s0}\,g_{1,j1,i0}\,g_{3,i0,i1} + T_1^{-s0}\left(-1 + T_1^{s1}\right) T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right) g_{1,j1,i0}\,g_{3,j0,i1}\right) +$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$g_{2,i1,i0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,j0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,i0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}+T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+\right.$$
$$g_{1,i0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{2,i0,i0}\,\left(-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{1,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\right.$$
$$\left.\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{2,j0,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$g_{2,i1,j0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,i0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+\right.$$
$$g_{2,i0,j0}\,\left(-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{2,j0,j0}\,\left(-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{1,i0,i0}$$
$$\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{1,j0,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,j0}+T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$g_{2,i1,j0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,j0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}+\right.$$
$$T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+$$
$$g_{2,i0,j0}\,\left(-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{1,i0,j0}$$
$$\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}+g_{2,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)+$$
$$g_{1,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,j0}+T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to\Big\{0,\;\chi_{i0=i1}$$
$$\left(\left(1-g_{1,i0,i0}-g_{2,i0,i0}\right)\,g_{3,j0,i0}+\left(1-T_2^{s0}\right)\,g_{2,j0,i0}\,g_{3,j0,i0}+g_{1,j0,i0}\,\left(g_{2,j0,i0}+\left(1-T_1^{s0}\right)\,g_{3,j0,i0}\right)\right)+$$
$$\left(-1+T_1^{s1}\right)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)\,g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$\left(2-T_1^{s1}-T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to\Big\{0,\;\left(-1+T_1^{s1}\right)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)\,g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$\left(2-T_1^{s1}-T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\chi_{i0=i1}\,\left(\left(-1+T_1^{s0}-g_{1,i0,j0}+\left(1-T_1^{s0}\right)\,g_{2,i0,i0}\right)\,g_{3,j0,i0}+\right.$$
$$\left.g_{1,j0,j0}\,\left(g_{2,j0,i0}+\left(1-T_1^{s0}\right)\,g_{3,j0,i0}\right)+g_{2,j0,i0}\,\left(\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,j0,i0}-g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to \left\{0,\ \left(-1+T_1^{s1}\right) g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1} + \left(-1+T_2^{s1}\right) g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \left(2-T_1^{s1}-T_2^{s1}\right) g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \chi_{i0==i1}\left(\left(-1+T_2^{s0}+\left(1-T_2^{s0}\right) g_{1,i0,i0} - g_{2,i0,j0}\right) g_{3,j0,i0} + \left(1-T_2^{s0}\right) g_{2,j0,j0}\,g_{3,j0,i0} + g_{1,j0,i0}\left(g_{2,j0,j0} + \left(-2+T_1^{s0}+T_2^{s0}\right) g_{3,j0,i0} - g_{3,j0,j0}\right)\right)\right\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to \left\{0,\ \left(-1+T_1^{s1}\right) g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1} + \left(-1+T_2^{s1}\right) g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \left(2-T_1^{s1}-T_2^{s1}\right) g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \chi_{i0==i1}\left(\left(2-T_1^{s0}-T_2^{s0}+\left(1-T_2^{s0}\right) g_{1,i0,j0} + \left(1-T_1^{s0}\right) g_{2,i0,j0}\right) g_{3,j0,i0} + g_{2,j0,j0}\left(\left(-2+T_1^{s0}+T_2^{s0}\right) g_{3,j0,i0} - g_{3,j0,j0}\right) + g_{1,j0,j0}\left(g_{2,j0,j0} + \left(-2+T_1^{s0}+T_2^{s0}\right) g_{3,j0,i0} - g_{3,j0,j0}\right) + g_{3,j0,j0}\right)\right\}$$

In[•]:=
```
tw = {0, 0, 0, 0, 1, -1, 0, 0}.
   List @@ Expand[(g1,#,i0 + g1,#,j0) (g2,#,i0 + g2,#,j0) (g3,i0^+,# + g3,j0^+,#)] / (-1 + T2)
D{s1,i1,j1}[tw]
⊖[Knot[7, 6], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
⊖[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

Out[•]=
$$\frac{g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} - g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}}{-1+T_2}$$

Out[•]=
$$\frac{\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}}{-1+T_2} - \frac{\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}}{-1+T_2} - \frac{\left(-2+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}}{-1+T_2} - \frac{\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}}{-1+T_2} + \frac{\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}}{-1+T_2} - \frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}}{-1+T_2} + \frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}}{-1+T_2} + \frac{\left(-2+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}}{-1+T_2} - \frac{\left(-3+T_1^{s0}+2\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}}{-1+T_2} + \frac{\left(-1+T_1^{s1}\right) g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2} - \frac{\left(-1+T_1^{s1}\right) g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2} + \frac{\left(-1+T_2^{s1}\right) g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2} - \frac{\left(-1+T_2^{s1}\right) g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2} - \frac{\left(-2+T_1^{s1}+T_2^{s1}\right) g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2} + \frac{\left(-2+T_1^{s1}+T_2^{s1}\right) g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2} + \frac{\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}}{-1+T_2}$$

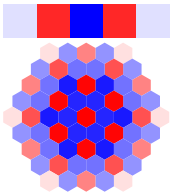
» $$\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\ 0\right\}$$

Out[•]=

» $$\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\right.$$

$$\frac{1}{T_1^4\,T_2^4}\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-\right.$$
$$10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-$$
$$50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-$$
$$64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-$$
$$10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+7\,T_1^2\,T_2^6-10\,T_1^3\,T_2^6-64\,T_1^4\,T_2^6+98\,T_1^5\,T_2^6-64\,T_1^6\,T_2^6-10\,T_1^7\,T_2^6+7\,T_1^8\,T_2^6-5\,T_1^3\,T_2^7+$$
$$\left.\left.20\,T_1^4\,T_2^7-10\,T_1^5\,T_2^7-10\,T_1^6\,T_2^7+20\,T_1^7\,T_2^7-5\,T_1^8\,T_2^7+T_1^4\,T_2^8-5\,T_1^5\,T_2^8+7\,T_1^6\,T_2^8-5\,T_1^7\,T_2^8+T_1^8\,T_2^8\right)\right\}$$

Out[•]=



In[•]:= `CF[F2i + D_{s1,i1,j1}[tw]]`

Out[•]=

$$\frac{\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}}{-1+T_2}-\frac{\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}}{-1+T_2}-\frac{\left(-2+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}}{-1+T_2}-\frac{\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}}{-1+T_2}+$$
$$\frac{\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}}{-1+T_2}+$$
$$\frac{\left(-2+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-3+T_1^{s0}+2\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}}{-1+T_2}+$$
$$\frac{1}{\left(-1+T_2\right)\left(-1+T_2^{s1}\right)}\left(1-T_1^{s1}-s1\,T_2^{s0}+s1\,T_1^{s0}\,T_2^{s0}+s1\,T_2^{1+s0}-s1\,T_1^{s0}\,T_2^{1+s0}-T_2^{s1}+T_1^{s1}\,T_2^{s1}+\right.$$
$$\left.s1\,T_1^{s1}\,T_2^{s0+s1}-s1\,T_1^{s0+s1}\,T_2^{s0+s1}-s1\,T_1^{s1}\,T_2^{1+s0+s1}+s1\,T_1^{s0+s1}\,T_2^{1+s0+s1}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}-$$
$$\frac{\left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2}-\frac{s1\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\,T_2^{s1}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}}{-1+T_2^{s1}}+$$
$$\frac{\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-\frac{\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-$$
$$\frac{1}{\left(-1+T_2\right)\left(-1+T_2^{s1}\right)}\left(2-T_1^{s1}-s1\,T_2^{s0}+s1\,T_1^{s0}\,T_2^{s0}+s1\,T_2^{1+s0}-s1\,T_1^{s0}\,T_2^{1+s0}-3\,T_2^{s1}+T_1^{s1}\,T_2^{s1}+T_2^{2\,s1}+\right.$$
$$\left.s1\,T_1^{s1}\,T_2^{s0+s1}-s1\,T_1^{s0+s1}\,T_2^{s0+s1}-s1\,T_1^{s1}\,T_2^{1+s0+s1}+s1\,T_1^{s0+s1}\,T_2^{1+s0+s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$\frac{\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}+\frac{s1\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\,T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}}{-1+T_2^{s1}}+$$
$$\frac{\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}}{-1+T_2}$$