

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
Once[<< KnotTheory`]
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[*]:= XFalse = 0; XTrue = 1;
```

```
In[*]:= CF[ $\mathcal{E}_-$ ] := Expand@Collect[ $\mathcal{E}_-$ , g $_{--}$ , F] /. F  $\rightarrow$  Factor;
```

```
In[*]:= T3 = T1 T2;
```

```
In[*]:= F1[{s $_{--}$ , i $_{--}$ , j $_{--}$ }] := CF[
  s (1/2 - g3ii + T2s g1ii g2ji - g1ii g2jj - (T2s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3s) g2ji g3ji -
  g2ii g3jj - T2s g2ji g3jj + g1ii g3jj + ((T1s - 1) g1ji (T22s g2ji - T2s g2jj + T2s g3jj) +
  (T3s - 1) g3ji (1 - T2s g1ii + g2ij + (T2s - 2) g2jj - (T1s - 1) (T2s + 1) g1ji)) / (T2s - 1)]
```

```
In[*]:= F2[{s0 $_{--}$ , i0 $_{--}$ , j0 $_{--}$ }, {s1 $_{--}$ , i1 $_{--}$ , j1 $_{--}$ }] := CF[
  s1 (T1s0 - 1) (T2s1 - 1)-1 (T3s1 - 1) g1,j1,i0 g3,j0,i1 ( (T2s0 g2,i1,i0 - g2,i1,j0) - (T2s0 g2,j1,i0 - g2,j1,j0)) )]
```

```
In[*]:= b0 = Simplify[Together[s (-1 + T2s) F1[{s, i, j}]]] /.
  {T2  $\rightarrow$  1, s2  $\rightarrow$  1, g2, $\alpha$ , $\beta$   $\rightarrow$  X $\alpha \leq \beta$ , g3, $\alpha$ , $\beta$   $\rightarrow$  g1, $\alpha$ , $\beta$ } /. {Xj $\leq$ j  $\rightarrow$  1, Xj $\leq$ i  $\rightarrow$  1 - Xi $\leq$ j}
```

```
Out[*]= -(((-1 + T1s) g1,j,i (g1,i,i + 2(-1 + T1s) g1,j,i - g1,j,j))
```

```
In[*]:= Simplify[b0 /. {(-1 + T1s) g1,j,i  $\rightarrow$  g1,j,j - g1,j,j+}]
```

```
Out[*]= -( (g1,i,i + g1,j,j - 2 g1,j,j+) (g1,j,j - g1,j,j+) )
```

```
In[*]:= b1 = Simplify[Together[s1 (-1 + T2s1) F2[{s0, i0, j0}, {s1, i1, j1}]]] /.
  {T2  $\rightarrow$  1, s12  $\rightarrow$  1, g2, $\alpha$ , $\beta$   $\rightarrow$  X $\alpha \leq \beta$ , g3, $\alpha$ , $\beta$   $\rightarrow$  g1, $\alpha$ , $\beta$ }
```

```
Out[*]= (-1 + T1s0) (-1 + T1s1) (Xi1 $\leq$ i0 - Xi1 $\leq$ j0 - Xj1 $\leq$ i0 + Xj1 $\leq$ j0) g1,j0,i1 g1,j1,i0
```

```
In[*]:= Simplify[b1 /. {(-1 + T1s0) g1,j1,i0  $\rightarrow$  g1,j1,j0 - g1,j1,j0+, (-1 + T1s1) g1,j0,i1  $\rightarrow$  g1,j0,j1 - g1,j0,j1+}]
```

```
Out[*]= (Xi1 $\leq$ i0 - Xi1 $\leq$ j0 - Xj1 $\leq$ i0 + Xj1 $\leq$ j0) (g1,j0,j1 - g1,j0,j1+) (g1,j1,j0 - g1,j1,j0+)
```

```
In[*]:= b2 = Simplify[Together[s0 (-1 + T2s0) F2[{s1, i1, j1}, {s0, i0, j0}]]] /.
  {T2  $\rightarrow$  1, s02  $\rightarrow$  1, g2, $\alpha$ , $\beta$   $\rightarrow$  X $\alpha \leq \beta$ , g3, $\alpha$ , $\beta$   $\rightarrow$  g1, $\alpha$ , $\beta$ }
```

```
Out[*]= (-1 + T1s0) (-1 + T1s1) (Xi0 $\leq$ i1 - Xi0 $\leq$ j1 - Xj0 $\leq$ i1 + Xj0 $\leq$ j1) g1,j0,i1 g1,j1,i0
```

```
In[*]:= Simplify[(b1 + b2) / 2]
Out[*]=

$$\frac{1}{2} \left( -1 + T_1^{s_0} \right) \left( -1 + T_1^{s_1} \right) \left( \chi_{i_0 \leq i_1} - \chi_{i_0 \leq j_1} + \chi_{i_1 \leq i_0} - \chi_{i_1 \leq j_0} - \chi_{j_0 \leq i_1} + \chi_{j_0 \leq j_1} - \chi_{j_1 \leq i_0} + \chi_{j_1 \leq j_0} \right) g_{1,j_0,i_1} g_{1,j_1,i_0}$$

In[*]:= Simplify[Factor[(b1 + b2) / 2] /.
  {(-1 + T_1^{s_0}) g_{1,j_1,i_0} \to g_{1,j_1,j_0} - g_{1,j_1,j_0^*}, (-1 + T_1^{s_1}) g_{1,j_0,i_1} \to g_{1,j_0,j_1} - g_{1,j_0,j_1^*}}]
Out[*]=

$$\frac{1}{2} \left( \chi_{i_0 \leq i_1} - \chi_{i_0 \leq j_1} + \chi_{i_1 \leq i_0} - \chi_{i_1 \leq j_0} - \chi_{j_0 \leq i_1} + \chi_{j_0 \leq j_1} - \chi_{j_1 \leq i_0} + \chi_{j_1 \leq j_0} \right) (g_{1,j_0,j_1} - g_{1,j_0,j_1^*}) (g_{1,j_1,j_0} - g_{1,j_1,j_0^*})$$

```

Testing

```
In[*]:= Rot[X_List] := X;
Rot[K_] := Cases[PD[K], x_X \to { {1, x[[4]], x[[1]]} PositiveQ@x,
  { -1, x[[2]], x[[1]]} True }];

In[*]:= Res[K_] := Module[{X, n, A, G, ev, r, k, k1, k2},
  (* 1 *) X = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  (* 2 *) Cases[X, {s_, i_, j_} \to (A[[{i, j}, {i + 1, j + 1}]] += (
    -T^s T^s - 1))];
  (* 3 *) G = Inverse[A];
  (* 4 *) ev[s_] := Factor[s /. i_ \to i + 1 /. g_{\alpha,\beta} \to G[[\alpha, \beta]]];
  (* 5 *) r = ev[Sum[F1[X[[k]]], {k, n}]];
  (* 6 *) r += ev[Sum[F2[X[[k1]], X[[k2]]], {k1, n}, {k2, n}]];
  (* 7 *) Factor[r]
];

In[*]:= F1[{s_, i_, j_}] := -((g_{i,i} + g_{j,j} - 2 g_{j,j^*}) (g_{j,j} - g_{j,j^*}));
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  -(X_{i1 \leq i0} - X_{i1 \leq j0} - X_{j1 \leq i0} + X_{j1 \leq j0}) (-X_{j0 = j1^*} + X_{j0 = i1^*} + g_{j0,i1} - g_{j0,i1^*}) (g_{j1,j0} - g_{j1,j0^*});
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  (X_{i1 \leq i0} - X_{i1 \leq j0} - X_{j1 \leq i0} + X_{j1 \leq j0}) (g_{j0,j1} - g_{j0,j1^*}) (g_{j1,j0} - g_{j1,j0^*});
F1[{s, i, j}] // TeXForm

In[*]:= Rot[Knot[3, 1]]
Res[Knot[3, 1]]
Out[*]=
{{-1, 4, 1}, {-1, 6, 3}, {-1, 2, 5}}
Out[*]=
0
```



```

In[*]:= nRules[{s0, i0, j0}, {s1, i1, j1}]
Out[*]=

$$\left\{ \begin{aligned} &\chi_{i0 \leq \beta} \mapsto \chi_{i0^+ \leq \beta}, \chi_{i1 \leq \beta} \mapsto \chi_{i1^+ \leq \beta}, \chi_{j0 \leq \beta} \mapsto \chi_{j0^+ \leq \beta}, \chi_{j1 \leq \beta} \mapsto \chi_{j1^+ \leq \beta}, \chi_{\alpha \leq i0} \mapsto \chi_{\alpha \leq i0^+}, \\ &\chi_{\alpha \leq i1} \mapsto \chi_{\alpha \leq i1^+}, \chi_{\alpha \leq j0} \mapsto \chi_{\alpha \leq j0^+}, \chi_{\alpha \leq j1} \mapsto \chi_{\alpha \leq j1^+}, g_{i0, \beta} \mapsto T^{s0} g_{i0^+, \beta} + (1 - T^{s0}) g_{j0^+, \beta}, \\ &g_{i1, \beta} \mapsto T^{s1} g_{i1^+, \beta} + (1 - T^{s1}) g_{j1^+, \beta}, g_{j0, \beta} \mapsto g_{j0^+, \beta}, g_{j1, \beta} \mapsto g_{j1^+, \beta}, g_{\alpha, i0} \mapsto T^{s0} g_{\alpha, i0^+}, \\ &g_{\alpha, i1} \mapsto T^{s1} g_{\alpha, i1^+}, g_{\alpha, j0} \mapsto g_{\alpha, j0^+} + (1 - T^{s0}) g_{\alpha, i0^+}, g_{\alpha, j1} \mapsto g_{\alpha, j1^+} + (1 - T^{s1}) g_{\alpha, i1^+} \end{aligned} \right\}$$


In[*]:= a1 = Simplify[Together[ $\frac{s1}{(-1 + T^{s0})(-1 + T^{s1})} (-1 + T_2^{s1}) F_2[\{s0, i0, j0\}, \{s1, i1, j1\}]$ ] /.
 $\{T_2 \rightarrow 1, s1^2 \rightarrow 1, g_{2, \alpha, \beta} \mapsto \chi_{\alpha \leq \beta}, g_{1|3, \alpha, \beta} \mapsto g_{\alpha, \beta}, T_1 \rightarrow T\}$  /.
nRules[{s0, i0, j0}, {s1, i1, j1}]
]
Out[*]=
 $(\chi_{i1^+ \leq i0} - \chi_{i1^+ \leq j0} - \chi_{j1^+ \leq i0} + \chi_{j1^+ \leq j0}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= a2 = Simplify[Together[ $\frac{s0}{(-1 + T^{s0})(-1 + T^{s1})} (-1 + T_2^{s0}) F_2[\{s1, i1, j1\}, \{s0, i0, j0\}]$ ] /.
 $\{T_2 \rightarrow 1, s0^2 \rightarrow 1, g_{2, \alpha, \beta} \mapsto \chi_{\alpha \leq \beta}, g_{1|3, \alpha, \beta} \mapsto g_{\alpha, \beta}, T_1 \rightarrow T\}$  /.
nRules[{s0, i0, j0}, {s1, i1, j1}]
]
Out[*]=
 $(\chi_{i0^+ \leq i1} - \chi_{i0^+ \leq j1} - \chi_{j0^+ \leq i1} + \chi_{j0^+ \leq j1}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= Simplify[a1 + a2]
Out[*]=
 $(\chi_{i0^+ \leq i1} - \chi_{i0^+ \leq j1} + \chi_{i1^+ \leq i0} - \chi_{i1^+ \leq j0} - \chi_{j0^+ \leq i1} + \chi_{j0^+ \leq j1} - \chi_{j1^+ \leq i0} + \chi_{j1^+ \leq j0}) g_{j0^+, i1} g_{j1^+, i0}$ 

In[*]:= D[{s_, i_, j_}][ $\mathcal{E}_-$ ] :=
CF[( $\mathcal{E} /. \# \rightarrow i^+$ ) + ( $\mathcal{E} /. \# \rightarrow j^+$ ) - ( $\mathcal{E} /. \# \rightarrow i$ ) - ( $\mathcal{E} /. \# \rightarrow j$ )] /. nRules[{s, i, j}]

In[*]:= D[{s, i, j}][{g $_{\alpha, \#}$ , g $_{\alpha, \#}^2$ }]
Out[*]=
 $\{0, 2 T^s (-1 + T^s) g_{\alpha, i}^2 - 2 (-1 + T^s) g_{\alpha, i} g_{\alpha, j}\}$ 

In[*]:= List@@Expand[( $\chi_{\# \leq i0} + \chi_{\# \leq j0}$ )(g $_{i0^+, \#} + g_{j0^+, \#}$ )(g $_{\#, i0} + g_{\#, j0}$ )]
Out[*]=
 $\{\chi_{\#1 \leq i0} g_{\#1, i0} g_{i0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, i0} g_{i0^+, \#1}, \chi_{\#1 \leq i0} g_{\#1, j0} g_{i0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, j0} g_{i0^+, \#1},$ 
 $\chi_{\#1 \leq i0} g_{\#1, i0} g_{j0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, i0} g_{j0^+, \#1}, \chi_{\#1 \leq i0} g_{\#1, j0} g_{j0^+, \#1}, \chi_{\#1 \leq j0} g_{\#1, j0} g_{j0^+, \#1}\}$ 

```

```

In[*]:= {X_{#≤i0} G_{#,i0} G_{i0^*,#}, X_{#≤j0} G_{#,i0} G_{i0^*,#}, X_{#≤i0} G_{#,j0} G_{i0^*,#}, X_{#≤j0} G_{#,j0} G_{i0^*,#},
          X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}, X_{#≤i0} G_{#,j0} G_{j0^*,#}, X_{#≤j0} G_{#,j0} G_{j0^*,#}} // D_{s1,i1,j1}

Out[*]=
{ (-1 + T^{s1}) (X_{i1^*≤i0} - X_{j1^*≤i0}) G_{i0^*,i1} G_{j1^*,i0}, (-1 + T^{s1}) (X_{i1^*≤j0} - X_{j1^*≤j0}) G_{i0^*,i1} G_{j1^*,i0},
  (-1 + T^{s1}) (X_{i1^*≤i0} - X_{j1^*≤i0}) G_{i0^*,i1} G_{j1^*,j0}, (-1 + T^{s1}) (X_{i1^*≤j0} - X_{j1^*≤j0}) G_{i0^*,i1} G_{j1^*,j0},
  (-1 + T^{s1}) (X_{i1^*≤i0} - X_{j1^*≤i0}) G_{j0^*,i1} G_{j1^*,i0}, (-1 + T^{s1}) (X_{i1^*≤j0} - X_{j1^*≤j0}) G_{j0^*,i1} G_{j1^*,i0},
  (-1 + T^{s1}) (X_{i1^*≤i0} - X_{j1^*≤i0}) G_{j0^*,i1} G_{j1^*,j0}, (-1 + T^{s1}) (X_{i1^*≤j0} - X_{j1^*≤j0}) G_{j0^*,i1} G_{j1^*,j0}}

In[*]:= {0, 0, 0, 0, 1, -1, 0, 0} . {X_{#≤i0} G_{#,i0} G_{i0^*,#}, X_{#≤j0} G_{#,i0} G_{i0^*,#}, X_{#≤i0} G_{#,j0} G_{i0^*,#}, X_{#≤j0} G_{#,j0} G_{i0^*,#},
          X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}, X_{#≤i0} G_{#,j0} G_{j0^*,#}, X_{#≤j0} G_{#,j0} G_{j0^*,#}} // D_{s1,i1,j1}

Out[*]=
(-1 + T^{s1}) (X_{i1^*≤i0} - X_{i1^*≤j0} - X_{j1^*≤i0} + X_{j1^*≤j0}) G_{j0^*,i1} G_{j1^*,i0}

In[*]:= ({1, -1} . {X_{#≤i0} G_{#,i0} G_{j0^*,#}, X_{#≤j0} G_{#,i0} G_{j0^*,#}} // D_{s1,i1,j1}) == (-1 + T^{s1}) a1

Out[*]=
True

In[*]:= List @@ Expand[(X_{i1^*≤#} + X_{j1^*≤#}) (G_{#,i1} + G_{#,j1}) (G_{i1^*,#} + G_{j1^*,#})]

Out[*]=
{X_{i1^*≤#1} G_{#1,i1} G_{i1^*,#1}, X_{j1^*≤#1} G_{#1,i1} G_{i1^*,#1}, X_{i1^*≤#1} G_{#1,j1} G_{i1^*,#1}, X_{j1^*≤#1} G_{#1,j1} G_{i1^*,#1},
  X_{i1^*≤#1} G_{#1,i1} G_{j1^*,#1}, X_{j1^*≤#1} G_{#1,i1} G_{j1^*,#1}, X_{i1^*≤#1} G_{#1,j1} G_{j1^*,#1}, X_{j1^*≤#1} G_{#1,j1} G_{j1^*,#1}}

In[*]:= {X_{i1^*≤#1} G_{#1,i1} G_{i1^*,#1}, X_{j1^*≤#1} G_{#1,i1} G_{i1^*,#1}, X_{i1^*≤#1} G_{#1,j1} G_{i1^*,#1}, X_{j1^*≤#1} G_{#1,j1} G_{i1^*,#1},
          X_{i1^*≤#1} G_{#1,i1} G_{j1^*,#1}, X_{j1^*≤#1} G_{#1,i1} G_{j1^*,#1}, X_{i1^*≤#1} G_{#1,j1} G_{j1^*,#1}, X_{j1^*≤#1} G_{#1,j1} G_{j1^*,#1}} // D_{s0,i0,j0}

Out[*]=
{ (-1 + T^{s0}) (X_{i1^*≤i0} - X_{i1^*≤j0}) G_{i1^*,i0} G_{j0^*,i1}, (-1 + T^{s0}) (X_{j1^*≤i0} - X_{j1^*≤j0}) G_{i1^*,i0} G_{j0^*,i1},
  (-1 + T^{s0}) (X_{i1^*≤i0} - X_{i1^*≤j0}) G_{i1^*,i0} G_{j0^*,j1}, (-1 + T^{s0}) (X_{j1^*≤i0} - X_{j1^*≤j0}) G_{i1^*,i0} G_{j0^*,j1},
  (-1 + T^{s0}) (X_{i1^*≤i0} - X_{i1^*≤j0}) G_{j0^*,i1} G_{j1^*,i0}, (-1 + T^{s0}) (X_{j1^*≤i0} - X_{j1^*≤j0}) G_{j0^*,i1} G_{j1^*,i0},
  (-1 + T^{s0}) (X_{i1^*≤i0} - X_{i1^*≤j0}) G_{j0^*,j1} G_{j1^*,i0}, (-1 + T^{s0}) (X_{j1^*≤i0} - X_{j1^*≤j0}) G_{j0^*,j1} G_{j1^*,i0}}

```