Title: A very fast, very strong, topologically meaningful and fun knot invariant.

Abstract. This is a research announcement introducing θ, a 2-variable polynomial knot invariant which is:
(a) Theoritically fast and practically fast: It runs in polynomial time and we computed it on random knots with up to 300 crossings, and evaluated on simple rational numbers on knots with up to 1000 crossings.
(b) Strong: It's separation power is much greater than, say, the HOMFLY-PT polynomial and Khovanov homology on knots with up to 15 crossings (while computing much faster).
(c) Topologically meaningful: It gives a genus bound, and there are reasons to hope that it would do more.
(d) Fun: Scroll to the pictures on page ...

A VERY FAST, VERY STRONG, TOPOLOGICALLY MEANINGFUL AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

ABSTRACT. This is a research announcement introducing $\Theta = (\Delta, \theta)$, a pair of polynomial knot invariants which is:

- Theoretically and practically fast: It runs in polynomial time and we computed it in full on random knots with over 300 crossings, and its evaluation on on simple rational numbers on random knots with over 700 crossings.
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- Fun: Scroll to Figures 1.1 and 1.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh]. Yet our formulas are much simpler and enable its computation even on very large knots.

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1. Fun

The word "fun" rarely appears in the title of a math paper, so let us start with a brief justification.

Θ is a pair of polynomials. The first, Δ, is a one-variable Laurent polynomial in a variable T. For example, Δ([trefoil]) $= T^{-1} - 1 + T$. We turn such a polynomial to a list of coefficients (for [trefoil], it is $(1\ \ -1\ \ 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with brightness proportional to the magnitude of the coefficients). The result is a "bar code", and for the trefoil [trefoil] is it .

Similiarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2. We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array

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Key words and phrases. Alexander polynomial, TBW .

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Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a "hexagonal QR code". For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. In addition, the hexagonal QR codes of some 15 knots with $\geqslant 300$ crossings are in Figure 1.2.

Clearly there are patterns in Figures 1.1 and 1.2. There is a hexaginal symmetry and the QR codes are hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.2 there seem to be large-scale "sand table patterns". We can't prove any of these things, and the last one, we can't even formulate properly. Yet it is clearly there, too clear to be the result of chance alone.

We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

2. The Formulas

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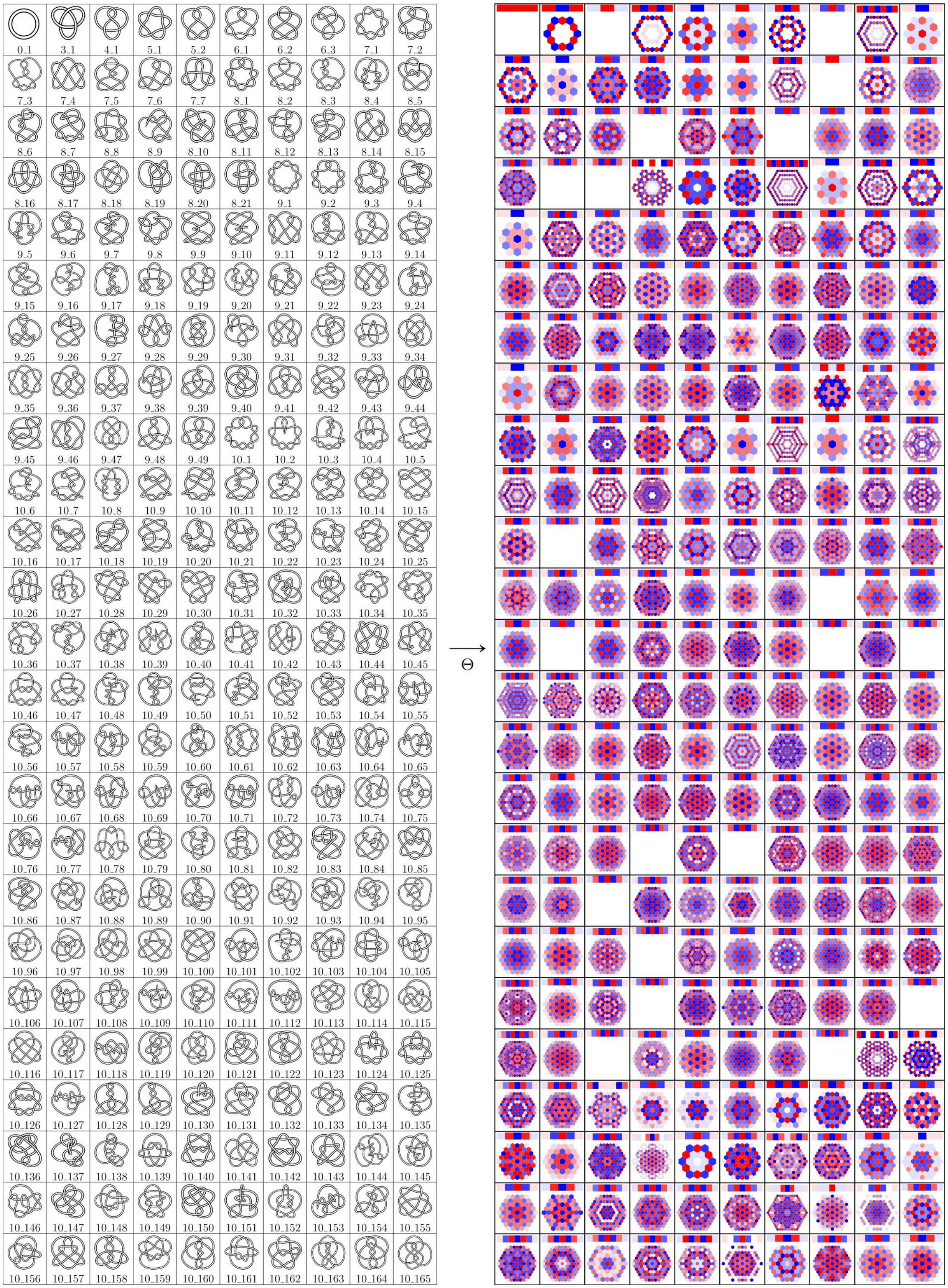

FIGURE 1.1. Θ as a bar code and a hexagonal QR code, for all the knots in the Rolfsen table.

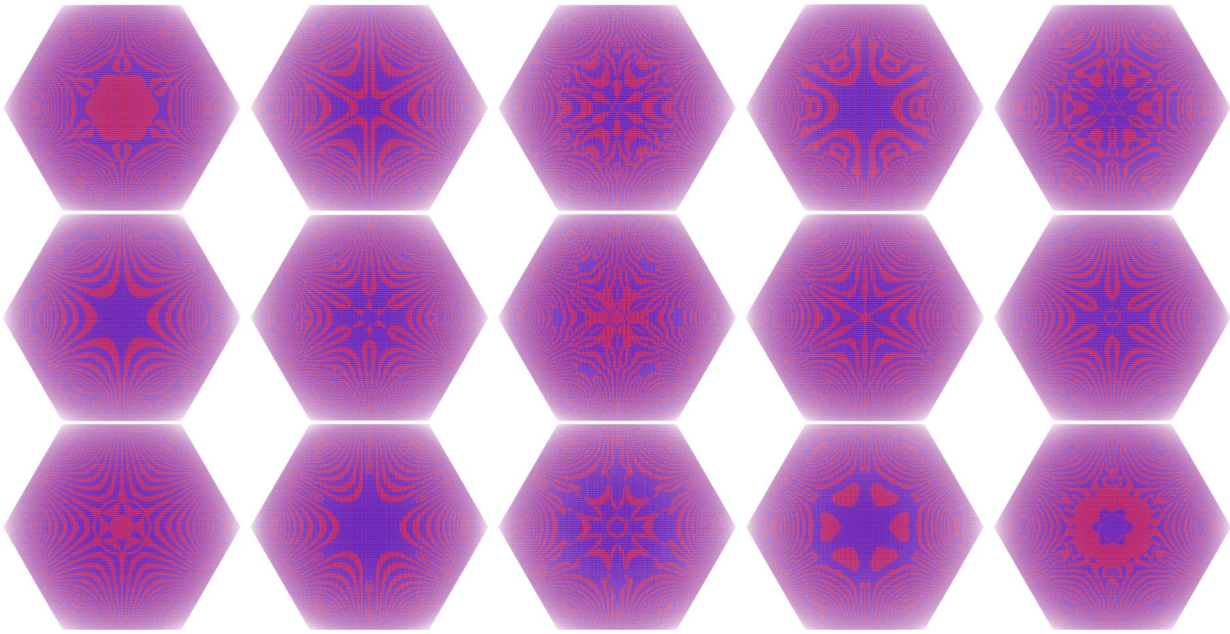

FIGURE 1.2. θ (hexagonal QR code only) of the 15 largest knots that we have computed by September 16, 2024. They are all "generic" in as much as we know, and they all have $\geqslant 300$ crossings. The knots come from . . .

Reverse the sign of Θ!

Global R->V.

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Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh], continuing Rozansky, Garoufalidis, and Kricker [GR, Ro1, Ro2, Ro3, Kr]. Yet our formulas, proofs, and programs are much simpler and enable its computation even on very large knots.

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1. FUN

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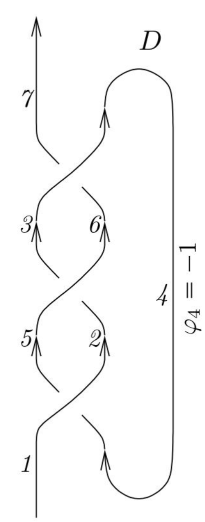
We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

2. Formulas

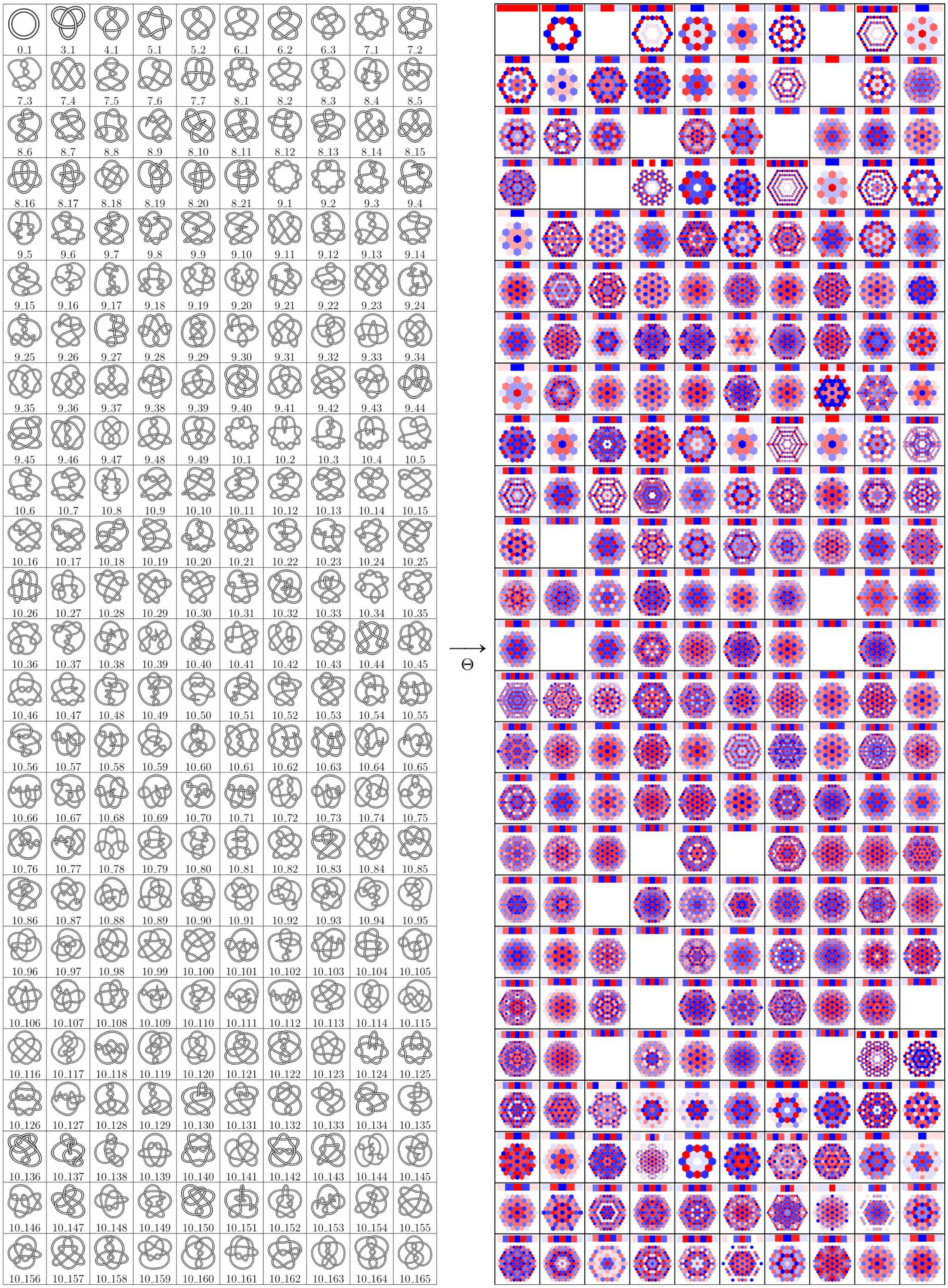
2.1. Old Formulas. The setup leading to the definition of Θ is the same as the setup leading to the definition of the invariant ρ_1 of [BV1], and hence we copy a few relevant paragraphs from [BV1] nearly varbatim, with only a few examples removed.



Given an oriented n-crossing knot K, we draw it in the plane as a long knot diagram D in such a way that the two strands intersecting at each crossing are pointed up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram.* An example of an upright knot diagram is shown on the right.

We then label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

A Technicality. Some Reidemeister moves create or lose an edge and to avoid the need for renumbering it is beneficial to also allow labelling the edges with non-consecutive labels.



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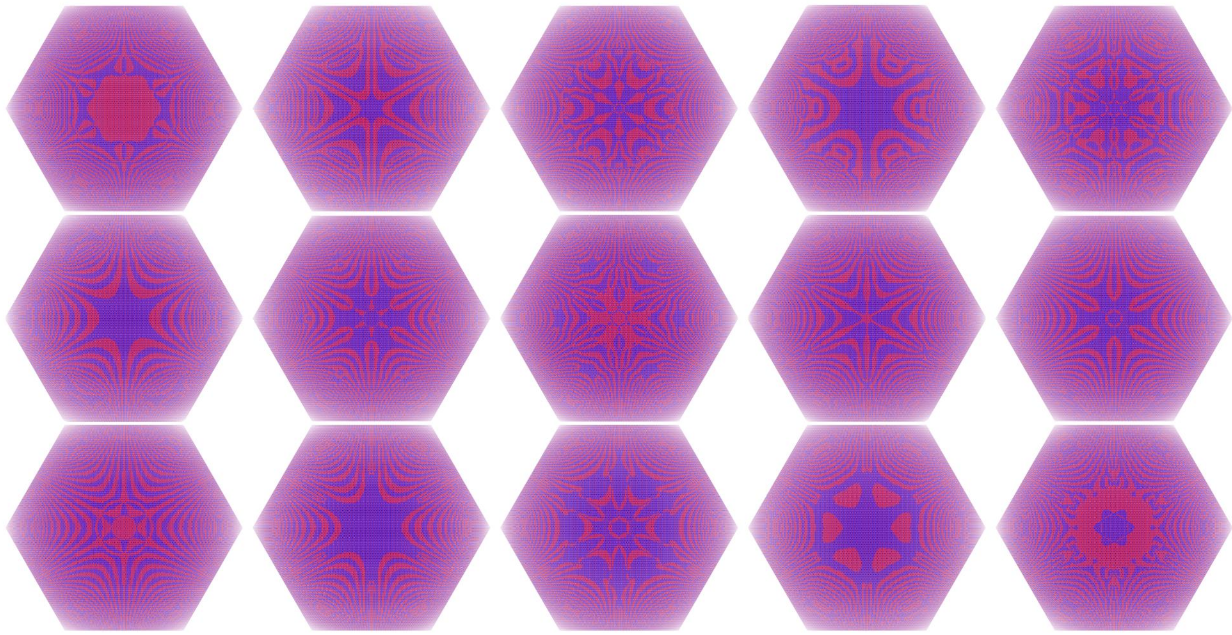


FIGURE 1.2. θ (hexagonal QR code only) of the 15 largest knots that we have computed by September 16, 2024. They are all "generic" in as much as we know, and they all have $\geqslant 300$ crossings. The knots come from [DHOEBL].

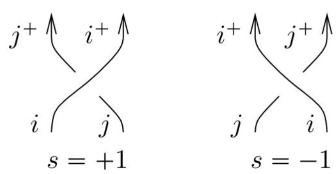
Hence we allow that, and write i^+ for the successor of the label i along the knot, and i^{++} for the successor of i^+ (these are $i+1$ and $i+2$ if the labelling is by consecutive integers). Also, "1" will always refer to the label of the first edge, and "$2n+1$" will always refer to the label of the last.

We let A be the $(2n+1)\times(2n+1)$ matrix of Laurent polynomials in the formal variable T defined by

$$A := I - \sum_c \left(T^s E_{i,i^+} + (1-T^s)E_{i,j^+} + E_{j,j^+}\right),$$

where I is the identity matrix and $E_{\alpha\beta}$ denotes the elementary matrix with 1 in row α and column β and zeros elsewhere. The summation is over the crossings $c=(s,i,j)$ of the diagram D, and once c is chosen, s denotes its sign and i and j denote the labels below the crossing where the label i belongs to the over-strand and j to the under-strand.

Alternatively, $A = I + \sum_c A_c$, where A_c is a matrix of zeros except for the blocks as follows:



$\longrightarrow$

A_c	column i^+	column j^+
row i	$-T^s$	T^s-1
row j	0	-1

(1)

We note (as we did in [BV1]) that the determinant of A is equal up to a unit to the normalized Alexander polynomial Δ of K. In fact, we have that

$$\Delta = T^{(-\varphi(D)-w(D))/2}\det(A), \tag{2}$$

where $\varphi(D) := \sum_k \varphi_k$ is the total rotation number of D and where $w(D) = \sum_c s_c$ is the writhe of D, namely the sum of the signs s_c of all the crossings c in D.

We let $G = (g_{\alpha\beta}) = A^{-1}$ and, thinking of it as a function $g_{\alpha\beta}$ of a pair of edges α and β, we call it the Green function of the diagram D. When inspired by physics (e.g. [BN1]) we sometimes call it "the 2-point function", and when thinking of car traffic (e.g. [BN2]) we sometimes call it "the traffic function".

We note that the computation of G is the bottleneck in the computation of Θ. It requires inverting a $(2n+1) \times (2n+1)$ matrix whose entries that are (degree 1) Laurent polynomials in T. It's a daunting task yet it takes polynomial time, it can be performed in practice even if n is in the hundreds, and everything which follows is easier.

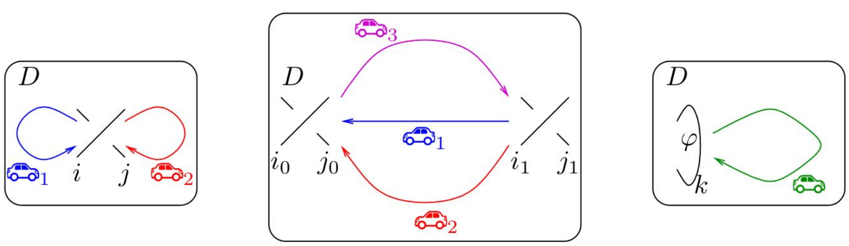
2.2. **New Formulas.** Let T_1 and T_2 be indeteminates and let $T_3 := T_1T_2$. Let $\Delta_\nu := \Delta/(T \to T_\nu)$ and $G_\nu = (g_{\nu\alpha\beta}) := G/(T \to T_\nu)$ be Δ and G affected by the substitution $T \to T_\nu$, where $\nu = 1, 2, 3$ (these are easy to compute once Δ and G have been computed).

The formulas for θ depend on three fixed polynomials $R_{11}(c)$, $R_{12}(c_0, c_1)$ and $\Gamma_1(\varphi, k)$ in the $g_{\nu\alpha\beta}$'s, which we admit, are rather ugly. So we prefer to assert their existance and postpone displaying them to a few paragraphs later.

Theorem 1 (Proof in Section 4)**.** *With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic polynomial $R_{11}(c) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$ in the $g_{\nu\alpha\beta}$'s with coefficients in the ring of rational functions in T_1 and T_2 and with $\alpha, \beta \in \{i, j\}$, and similarly a cubic $R_{12}(c_0, c_1) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $\Gamma_1(\varphi, k)$ such that the following is a knot invariant:*

$$\theta(D) := \Delta_1\Delta_2\Delta_3 \left(\sum_c R_{11}(c) + \sum_{c_0, c_1} R_{12}(c_0, c_1) + \sum_k \Gamma_1(\varphi_k, k) \right). \tag{3}$$

We note without detail that there is an alternative formula for θ in terms of perturbed Gaussian integration [BN1]. In that language, and using also the traffic motifs of [BV1, BN2], the three summands in (3) become Feynman diagrams for processes in which cars governed by parameter $T = T_1, T_2$, or T_3 interact:



In particular, the middle diagram which resembles the greek letter Θ gave the invariant its name.

We note also that computationaly, the worst term in (3) is the middle one, and even it takes merely $\sim n^2$ operations in the ring $\mathbb{Q}(T_1, T_2)$.

The polynomials $R_{11}(c)$, $R_{12}(c_0, c_1)$ and $\Gamma_1(\varphi, k)$ are not unique, and we are not certain that we have the cleanest possible formulas for them. As admitted, they are human-ugly. Yet from a computational perspective, having 18 terms (as is the case for $\Gamma_1(\varphi, k)$) isn't really a problem; computers don't care. Anyway, here are the formulas:

$$\begin{aligned}R_{11}(c) = s\,[1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ + (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}] \\ + \frac{s}{T_2^s - 1}\,[(T_1^s - 1) T_2^s \,(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}) \\ + (T_3^s - 1)\,(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}) \\ - (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}]\end{aligned}$$

$$R_{12}(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1j_1i_0} g_{3j_0i_1}}{T_2^{s_1} - 1}\,(T_2^{s_0} g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0} g_{2j_1i_0} - g_{2i_1j_0})$$

$$\Gamma_1(\varphi, k) = \varphi(-1/2 + g_{3kk})$$

3. Implementation and Examples

A concise yet reasonably efficient implementation is worth a thousand formulas. It completely removes ambiguities, it tests the theories, and it allows for experimentation. Hence our next task is to implement. The section that follows was generated from a Mathematica [Wo] notebook which is available at [BV2, Theta.nb].

We start by loading the package KnotTheory' — it is only needed because it had many specific knots pre-defined:

```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Next we quietly define the commands Rot, used to compute rotation numbers, and PolyPlot, used to plot polynomials as bar codes and as hexagonal QR codes. Neither is a part of the core of the computation of Θ, so neither is shown; yet we do show some usage examples.

```
(* Rot suppressed *)
```

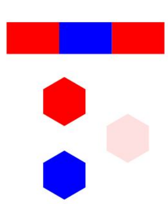
```
Rot[Mirror@Knot[3, 1]]
```

```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0}}
```

We urge the reader to compare the above output with the knot diagram in Section 2.1.

```
(* PolyPlot suppressed *)
```

```
PolyPlot[{T - 1 + T^-1, T1 + 2 T2 - 2 T1^-1 T2^-1}, ImageSize → Tiny]
```



4. Proof of Invariance

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6. Conjectures and Dreams

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Reverse the sign of Θ?

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Ohtsuki:TwoLoop
GaroufalidisRozansky:LoopExpansion, Rozansky:Contribution, Rozansky:Burau, Rozansky:U1RCC, Kricke

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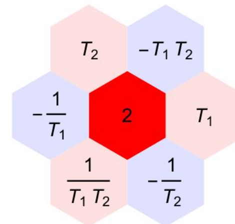


Similiarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2. We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely into a picture! To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a certain shear transformation to the plane before printing. So the colour of a monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $\begin{pmatrix}1 & -1/2\\ 0 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix}$ instead of the more traditional $\begin{pmatrix}n_1\\ n_2\end{pmatrix}$. On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^{-1} + T_1^{-1}T_2^{-1} - T_2^{-1}$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a "hexagonal QR code". For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. In addition, the hexagonal QR codes of some 15 knots with $\geqslant 300$ crossings are in Figure 1.2.

Clearly there are patterns in ~~Figures 1.1 and 1.2~~. There is a hexaginal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.2 there seem to be large-scale "sand table patterns" or "diffraction patterns". We can't prove any of these things, and the last one, we can't even formulate properly. Yet they are clearly there, too clear to be the result of chance alone.

We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

2. Formulas

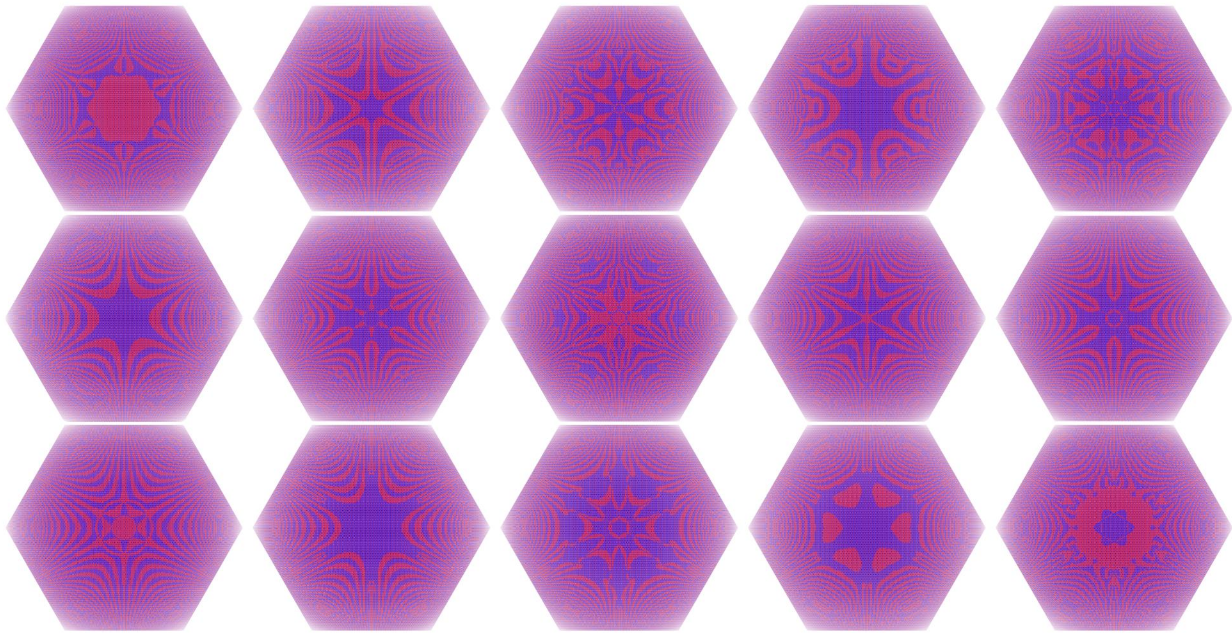
2.1. Old Formulas[1]**.** The setup leading to the definition of Θ is the same as the setup leading to the definition of the invariant ρ_1 of [BV1], and hence we copy a few relevant paragraphs from [BV1] nearly varbatim, with only a few ~~examples removed.~~

[1]"Old" means that these formulas appeared already in [BV1].

a bit larger?

FIGURE 1.1. Θ as a bar code and a hexagonal QR code, for all the knots in the Rolfsen table.

ig:Rolfsen

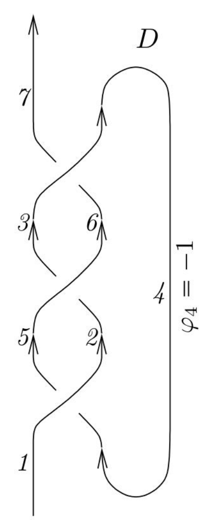


Change to honeycomb

fig:300

FIGURE 1.2. θ (hexagonal QR code only) of the 15 largest knots that we have computed by September 16, 2024. They are all "generic" in as much as we know, and they all have $\geqslant 300$ crossings. The knots come from [DHOEBL].

DHOEBL:Random



Given an oriented n-crossing knot K, we draw it in the plane as a long knot diagram D in such a way that the two strands intersecting at each crossing are pointing up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram*. An example of an upright knot diagram is shown on the right.

We then label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1. Insert A

A Technicality. Some Reidemeister moves create or lose an edge and to avoid the need for renumbering it is beneficial to also allow labelling the edges with non-consecutive labels. Hence we allow that, and write i^+ for the successor of the label i along the knot, and i^{++} for the successor of i^+ (these are $i+1$ and $i+2$ if the labelling is by consecutive integers). Also, "1" will always refer to the label of the first edge, and "$2n+1$" will always refer to the label of the last.

Move to the first place it is used

We let A be the $(2n+1)\times(2n+1)$ matrix of Laurent polynomials in the formal variable T defined by

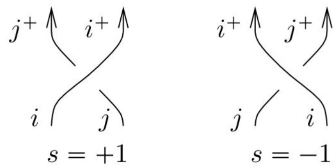
$$A := I - \sum_c \left(T^s E_{i,i^+} + (1-T^s)E_{i,j^+} + E_{j,j^+}\right),$$

Define X to be the set of all crossings in the diagram, where we encode each crossing as a triple (sign, overpass, underpass). In our example knot we have

$$X = \{(1,1,4),(1,5,2),(1,3,6)\}$$

where I is the identity matrix and $E_{\alpha\beta}$ denotes the elementary matrix with 1 in row α and column β and zeros elsewhere. The summation is over the crossings $c = (s, i, j)$ of the diagram D, and once c is chosen, s denotes its sign and i and j denote the labels below the crossing where the label i belongs to the over-strand and j to the under-strand.

Alternatively, $A = I + \sum_c A_c$, where A_c is a matrix of zeros except for the blocks as follows:



$\longrightarrow$

A_c	column i^+	column j^+
row i	$-T^s$	$T^s - 1$
row j	0	-1

(1) eq:A

We note (as we did in APAI [BV1]) that the determinant of A is equal up to a unit to the normalized Alexander polynomial Δ of K. In fact, we have that

$$\Delta = T^{(-\varphi(D) - w(D))/2} \det(A), \qquad (2)$$

eq:Delta

where $\varphi(D) := \sum_k \varphi_k$ is the total rotation number of D and where $w(D) = \sum_c s_c$ is the writhe of D, namely the sum of the signs s_c of all the crossings c in D.

We let $G = (g_{\alpha\beta}) = A^{-1}$ and, thinking of it as a function $g_{\alpha\beta}$ of a pair of edges α and β, we call it the Green function of the diagram D. When inspired by physics (e.g. [BN1] IType) we sometimes call it "the 2-point function", and when thinking of car traffic (e.g. [BN2] Toronto-241030) we sometimes call it "the traffic function".

We note that the computation of G is the bottleneck in the computation of Θ. It requires inverting a $(2n+1) \times (2n+1)$ matrix whose entries that are (degree 1) Laurent polynomials in T. It's a daunting task yet it takes polynomial time, it can be performed in practice even if n is in the hundreds, and everything which then follows is easier.

2.2. **New Formulas.** Let T_1 and T_2 be indeteminates and let $T_3 := T_1 T_2$. Let $\Delta_\nu := \Delta / (T \to T_\nu)$ and $G_\nu = (g_{\nu\alpha\beta}) := G / (T \to T_\nu)$ be Δ and G ~~affected by~~ the substitution $T \to T_\nu$, where $\nu = 1, 2, 3$ (these are easy to compute once Δ and G have been computed).

~~The formulas for θ depend on three fixed polynomials $F_1(c)$, $F_2(c_0, c_1)$ and $F_3(\varphi, k)$ in the $g_{\nu\alpha\beta}$'s, which we admit, are rather ugly. So we prefer to assert their existance and postpone displaying them to a few paragraphs later.~~

Too Mathematician / subject to / Put the formulas here & shut up / Simplify

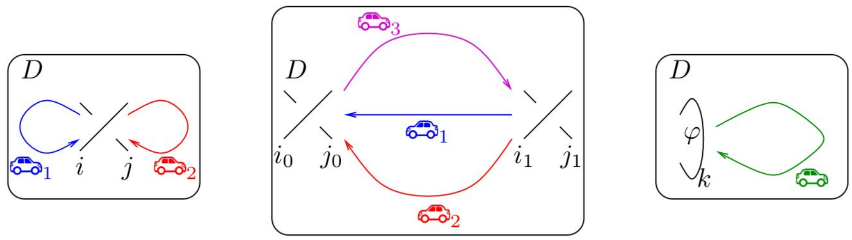
thm:Main

Theorem 1 (Proof in Section 4)**.** *With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic polynomial $F_1(c) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$ in the $g_{\nu\alpha\beta}$'s with coefficients in the ring of rational functions in T_1 and T_2 and with $\alpha, \beta \in \{i, j\}$, and similarly a cubic $F_2(c_0, c_1) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $F_3(\varphi, k)$ such that the following is a knot invariant:*

$$\theta(D) := \Delta_1 \Delta_2 \Delta_3 \left(\sum_c F_1(c) + \sum_{c_0, c_1} F_2(c_0, c_1) + \sum_k F_3(\varphi_k, k) \right). \qquad (3)$$

eq:Main

We note without detail that there is an alternative formula for θ in terms of perturbed Gaussian integration IType [BN1]. In that language, and using also the traffic motifs of APAI, Toronto-241030 [BV1, BN2], the three summands in (3) become Feynman diagrams for processes in which cars governed by parameter $T = T_1, T_2$, or T_3 interact:



In particular, the middle diagram which resembles the greek letter Θ gave the invariant its name.

We note also that computationaly, the worst term in (3) is the middle one, and even it takes merely $\sim n^2$ operations in the ring $\mathbb{Q}(T_1, T_2)$. to compute

The polynomials $F_1(c)$, $F_2(c_0, c_1)$ and $F_3(\varphi, k)$ are not unique, and we are not certain that we have the cleanest possible formulas for them. As admitted, they are human-ugly. Yet from a computational perspective, having 18 terms (as is the case for $F_1(c)$) isn't really a problem; computers don't care. ~~Anyway,~~ Here are the formulas:

$$\begin{aligned}F_1(c) = s\,[&1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\ &+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}] \\ &+ \frac{s}{T_2^s - 1}\,[(T_1^s - 1) T_2^s\,(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}) \\ &+ (T_3^s - 1)\,(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}) \\ &- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}]\end{aligned} \tag{4}$$ eq:F1

$$F_2(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1 j_1 i_0} g_{3 j_0 i_1}}{T_2^{s_1} - 1}\,(T_2^{s_0} g_{2 i_1 i_0} + g_{2 j_1 j_0} - T_2^{s_0} g_{2 j_1 i_0} - g_{2 i_1 j_0}) \tag{5}$$ eq:F2

$$F_3(\varphi, k) = \varphi (g_{3kk} - 1/2) \tag{6}$$ eq:F3

3. Implementation and Examples

A concise yet reasonably efficient implementation is worth a thousand formulas. It completely removes ambiguities, it tests the theories, and it allows for experimentation. Hence our next task is to implement. The section that follows was generated from a Mathematica [Wo] notebook which is available at [BV2, Theta.nb]. Also include a link to https://www.rolandvdv.nl/Theta/

We start by loading the package `KnotTheory`` — it is only needed because it has many specific knots pre-defined:

```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Next we quietly define the commands `Rot`, used to compute rotation numbers, and `PolyPlot`, used to plot polynomials as bar codes and as hexagonal QR codes. Neither is a part of the core of the computation of Θ, so neither is shown; yet we do show ~~some usage examples.~~ one usage example for each.

```
(* Rot suppressed *)
```

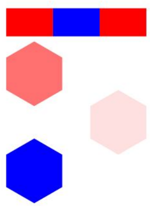
```
Rot[Mirror@Knot[3, 1]]
```

```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0}}
```

We urge the reader to compare the above output with the knot diagram in Section 2.1.

```
(* PolyPlot suppressed *)
```

$$\texttt{PolyPlot}\left[\left\{T - 1 + T^{-1},\ T_1 + 2\,T_2 - 4\,T_1^{-1}\,T_2^{-1}\right\},\ \texttt{ImageSize} \to \texttt{Tiny}\right]$$



We urge the reader to reflect on how the "QR Code" part of the above picture corresponds to the 2-variable polynomial $T_1 + 2T_2 - 4T_1^{-1}T_2^{-1}$.

Next, we decree that $T_3 = T_1T_2$ and define the three "Feynman Diagram" polynomials F_1, F_2, and F_3 (those definitions are printed in a smaller font because they are equal to what was already printed in (4)–(6)):

$$T_3 = T_1\,T_2;$$

$$F_1[\{1, i\_, j\_\}] = \frac{1}{2} + T_2\, g_{1ii}\, g_{2ji} + \frac{(T_1-1)\, T_2^2\, g_{1ji}\, g_{2ji}}{T_2-1} - g_{1ii}\, g_{2jj} - \frac{(T_1-1)\, T_2\, g_{1ji}\, g_{2jj}}{T_2-1} - g_{3ii} + (1-T_2)\, g_{2ji}\, g_{3ii} + 2\, g_{2jj}\, g_{3ii} + \frac{(T_3-1)\, g_{3ji}}{T_2-1} - \frac{T_2\,(T_3-1)\, g_{1ii}\, g_{3ji}}{T_2-1} - \frac{(T_1-1)\,(T_2+1)\,(T_3-1)\, g_{1ji}\, g_{3ji}}{T_2-1} + \frac{(T_3-1)\, g_{2ij}\, g_{3ji}}{T_2-1} + (T_3-1)\, g_{2ji}\, g_{3ji} + \frac{(T_2-2)\,(T_3-1)\, g_{2jj}\, g_{3ji}}{T_2-1} + g_{1ii}\, g_{3jj} + \frac{(T_1-1)\, T_2\, g_{1ji}\, g_{3jj}}{T_2-1} - g_{2ii}\, g_{3jj} - T_2\, g_{2ji}\, g_{3jj};$$

$$F_1[\{-1, i\_, j\_\}] = -\frac{1}{2} - \frac{g_{1ii}\, g_{2ji}}{T_2} - \frac{(T_1-1)\, g_{1ji}\, g_{2ji}}{T_1\,(T_2-1)\,T_2} + g_{1ii}\, g_{2jj} + \frac{(T_1-1)\, g_{1ji}\, g_{2jj}}{T_1\,(T_2-1)} + g_{3ii} - \frac{(T_2-1)\, g_{2ji}\, g_{3ii}}{T_2} - 2\, g_{2jj}\, g_{3ii} - \frac{(T_3-1)\, g_{3ji}}{T_1\,(T_2-1)} + \frac{(T_3-1)\, g_{1ii}\, g_{3ji}}{T_1\,(T_2-1)\,T_2} - \frac{(T_1-1)\,(T_2+1)\,(T_3-1)\, g_{1ji}\, g_{3ji}}{T_1^2\,(T_2-1)\,T_2} - \frac{(T_3-1)\, g_{2ij}\, g_{3ji}}{T_1\,(T_2-1)} + \frac{(T_3-1)\, g_{2ji}\, g_{3ji}}{T_3} + \frac{(2\,T_2-1)\,(T_3-1)\, g_{2jj}\, g_{3ji}}{T_1\,(T_2-1)\,T_2} - g_{1ii}\, g_{3jj} - \frac{(T_1-1)\, g_{1ji}\, g_{3jj}}{T_1\,(T_2-1)} + g_{2ii}\, g_{3jj} + \frac{g_{2ji}\, g_{3jj}}{T_2};$$

$$F_2[\{1, i0\_, j0\_\}, \{1, i1\_, j1\_\}] = \frac{(T_1-1)\, T_2\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1}}{T_2-1} - \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,i1}}{T_2-1} - \frac{(T_1-1)\, T_2\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1}}{T_2-1} + \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,i1}}{T_2-1};$$

$$F_2[\{1, i0\_, j0\_\}, \{-1, i1\_, j1\_\}] = -\frac{(T_1-1)\, T_2\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1}}{T_1\,(T_2-1)} + \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,i1}}{T_1\,(T_2-1)} + \frac{(T_1-1)\, T_2\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1}}{T_1\,(T_2-1)} - \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,i1}}{T_1\,(T_2-1)};$$

$$F_2[\{-1, i0\_, j0\_\}, \{1, i1\_, j1\_\}] = -\frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1}}{T_1\,(T_2-1)\,T_2} + \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,i1}}{T_1\,(T_2-1)} + \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1}}{T_1\,(T_2-1)\,T_2} - \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,i1}}{T_1\,(T_2-1)};$$

$$F_2[\{-1, i0\_, j0\_\}, \{-1, i1\_, j1\_\}] = \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,i1}}{T_1^2\,(T_2-1)\,T_2} - \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,i1}}{T_1^2\,(T_2-1)} - \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,i1}}{T_1^2\,(T_2-1)\,T_2} + \frac{(T_1-1)\,(T_3-1)\, g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,i1}}{T_1^2\,(T_2-1)};$$

$$F_3[\varphi\_, k\_] = \varphi\, g_{3kk} - \varphi / 2;$$

Next comes the main program computing Θ. Fortunately, it matches perfectly with the mathematical description in Section 2. In line 01 we let `Cs` be the list of crossings in an input knot K, and φ the list of its rotation numbers, using the external program `Rot` which we have already mentioned. We also let n be the length of `Cs`, namely, the number of crossings in K. In line 02 we let the starting value of A be the identity matrix, and then in line 03, for each crossing in `Cs` we add to A a 2×2 block, in rows i and j and columns $i+1$ and $j+1$, as explain in Equation (1). In line 04 we compute the normalized Alexander polynomial Δ as in (2). In line 05 we let G be the inverse of A. In line 06 we declare what it means to evaluate, `ev`, a formula $\mathcal{E}$ that may contain symbols of the form $g_{\nu\alpha\beta}$: each such symbol is to be replaced by the entry in position α,β of G, but with T replaced with T_ν. In line 07 we start computing θ by computing the first summand in (3), which in itself, is a sum over the crossings of the knot. In line 08 we add to θ the double sum corresponding to the second term in (3), and in line 09, we add the third summand of (3). Finally, line 10 outputs a pair: Δ, and the re-normalized version of θ.

```
Θ[K_] := Θ[K] = Module[{Cs, φ, n, A, Δ, G, ev, θ},
  (* 01 *) {Cs, φ} = Rot[K]; n = Length[Cs];
  (* 02 *) A = IdentityMatrix[2 n + 1];
  (* 03 *) Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ; 0  -1 ))];
  (* 04 *) Δ = T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det[A];
  (* 05 *) G = Inverse[A];
  (* 06 *) ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :> (G[[α, β]] /. T → T_ν)];
  (* 07 *) θ = ev[Σ_{k=1}^{n} F₁[Cs[[k]]]];
  (* 08 *) θ += ev[Σ_{k1=1}^{n} Σ_{k2=1}^{n} F₂[Cs[[k1]], Cs[[k2]]]];
  (* 09 *) θ += ev[Σ_{k=1}^{2 n} F₃[φ[[k]], k]];
  (* 10 *) Factor@{Δ, (Δ /. T → T₁) (Δ /. T → T₂) (Δ /. T → T₃) θ}
];
```

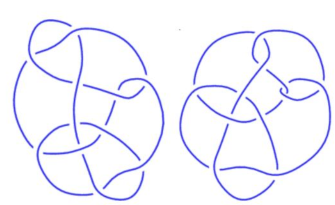
On to examples! Starting with the trefoil knot.

```
Expand[Θ[Knot[3, 1]]]
PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]
```

```
Θ[Knot[3, 1]]
```

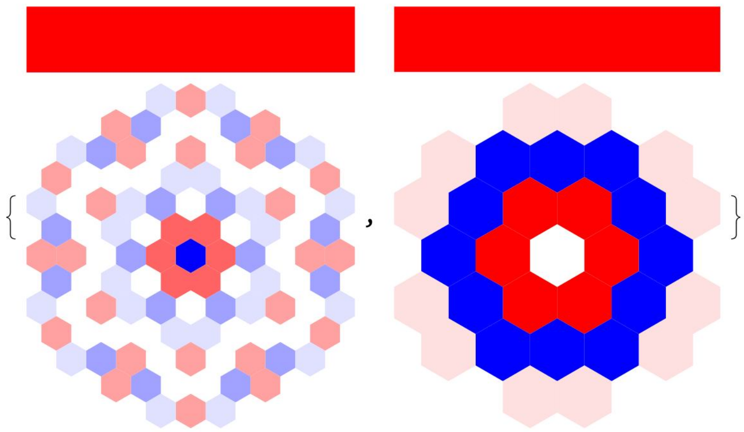
$$\left\{\frac{1-T+T^2}{T},\ -\frac{1-T_1+T_1^2-T_2-T_1^3T_2+T_2^2+T_1^4T_2^2-T_1T_2^3-T_1^4T_2^3+T_1^2T_2^4-T_1^3T_2^4+T_1^4T_2^4}{T_1^2T_2^2}\right\}$$

Next are the Conway knot 11_{n34} and the Kinoshita-Terasaka knot 11_{n42}. The two are mutants and famously hard to separate: they both have $\Delta = 1$ (as evidenced by their one-bar bar codes below), and they have the same HOMFLY-PT polynomial and Khovanov homology. Yet their θ invariants are different. Note that the genus of the Conway knot is 3, while the genus of the Kinoshita-Terasaka knot is 2.



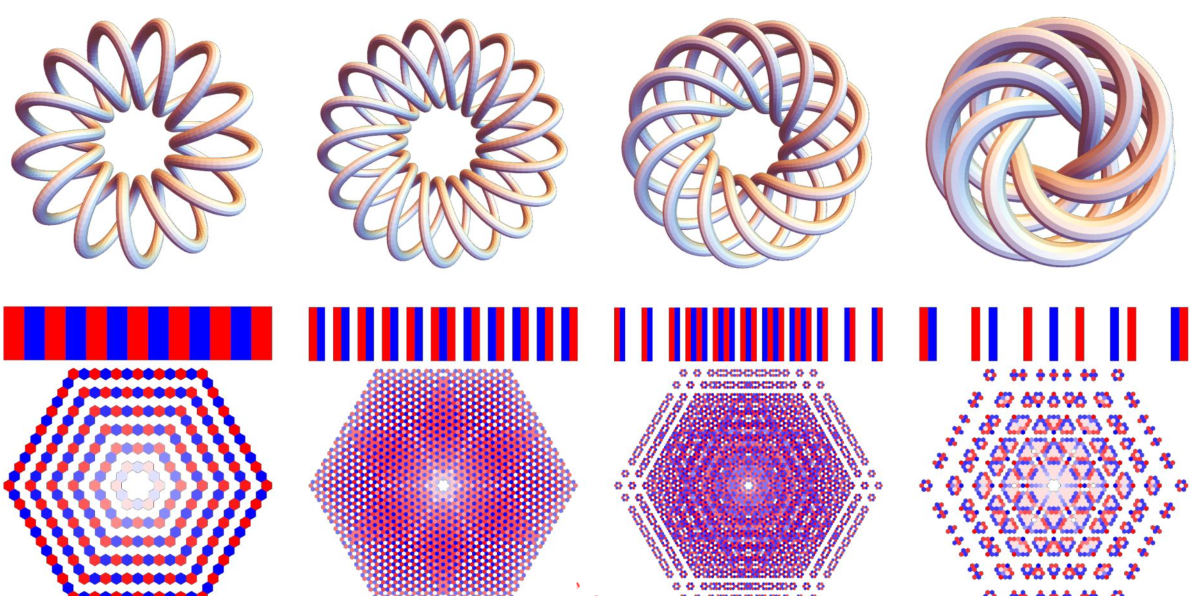
This agrees with the apparent higher complexity of the QR code of the Conway polynomial, and with the observations in Section 5.

```
PolyPlot[Θ[Knot[#]], ImageSize → Small] & /@ {"K11n34", "K11n42"}
```



Torus knots have particularly nice-looking Θ invariants. Here are the torus knots $T_{13/2}$, $T_{17/3}$, $T_{13/5}$, and $T_{7/6}$:

```
GraphicsGrid[{
  TubePlot[TorusKnot @@ #] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}},
  PolyPlot[Θ[TorusKnot @@ #]] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}
 }]
```



The next line shows the computation time is seconds for the 132-crossing torus knot $T_{22/7}$ on a 2024 laptop, without actually showing the output. The output plot is in Figure 3.1.

```
AbsoluteTiming[Θ[TorusKnot[22, 7]];]
```

```
{715.344, Null}
```

```
ImageCompose[PolyPlot[Θ[TorusKnot[22, 7]], ImageSize → 720],
  TubePlot[TorusKnot[22, 7], ImageSize → 360], {Right, Bottom}, {Right, Bottom}]
```

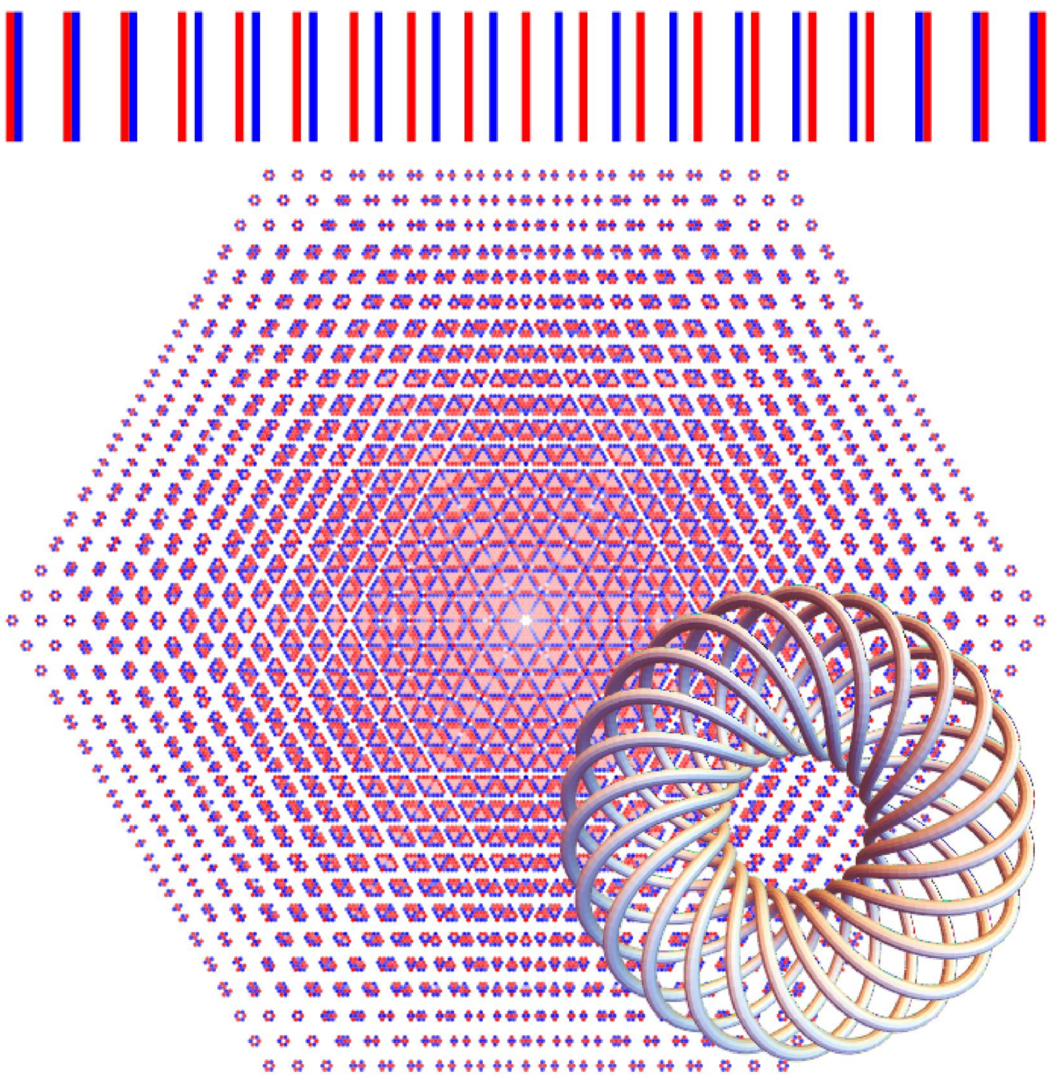


fig:T227

FIGURE 3.1. The 132-crossing torus knot $T_{22/7}$ and a plot of its Θ invariant

sec:Proof

4. Proof of Invariance

Our proof of the invariance of θ (Theorem 1) is very similar, and uses many of the same pieces, as the proofs of invariance of ρ_1 in [BV1]. Thus instead of repeating everything we just summarize those steps that are identical and then provide the needed details for the steps that differ.

Like in [BV1, Lemma 3], we know that for any crossing $c = (s, i, j)$ in a knot diagram D, the Green functions $G_\nu = (g_{\nu\alpha\beta})$ of D satisfy the following "g-rules" for $\nu = 1, 2, 3$ and with δ denoting the Kronecker delta:

$$\begin{aligned} g_{\nu i\beta} &= \delta_{i\beta} + T_\nu^s g_{\nu,i^+,\beta} + (1 - T_\nu^s) g_{\nu,j^+,\beta}, & g_{\nu j\beta} &= \delta_{j\beta} + g_{\nu,j^+,\beta}, \\ g_{\nu,\alpha,i^+} &= T_\nu^s g_{\nu\alpha i} + \delta_{\alpha,i^+}, & g_{\nu,\alpha,j^+} &= g_{\nu\alpha j} + (1 - T_\nu^s) g_{\nu\alpha i} + \delta_{\alpha,j^+}. \end{aligned} \quad (7)$$

eq:gRules

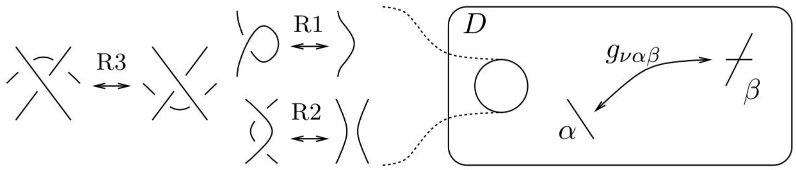


FIGURE 4.1. The Green functions $g_{\nu\alpha\beta}$ are invariant under Reidemeister moves performed away from where they are measured.

Invariance

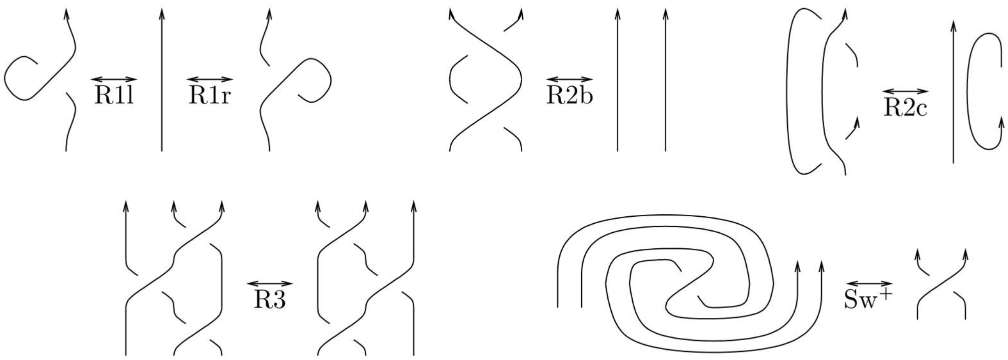


FIGURE 4.2. The upright Reidemeister moves: Reidemeister 1 left and right, Reidemeister 2 braid-like and cyclic, Reidemeister 3, and (the +) Swirl.

ightRMoves

The following theorem, a consequence of the g-rules above and/or of the interpretation of g as a traffic function as in [BV1], was not named in [BV1], yet it was stated there as the first part of the first proof of [BV1, Theorem 1]. It is proven by a simple verification for each of the upright Reidemeister moves, and these verifications appear as the first halves of [BV1, Propositions 6-9] for R3, R2c, R1l (and with R1r, R2b, and Sw$^+$ omitted for brevity).

eInvariant

Theorem 2. *The Green functions $g_{\nu\alpha\beta}$ are "relative invariants", meaning that once edges α and β are fixed within a knot diagram D, the values of $g_{\nu\alpha\beta}$ do not change if Reidemeister moves are performed* away *from the edges α and β. An illustration appears in Figure 4.1.*

We can now move on to the main part of the proof of Theorem 1. As follows from [BV1, Theorem 2], we need to prove the invariance of θ under the "upright Reidemeister" moves of Figure 4.2. We start with the hardest, R3:

prop:R3

Proposition 3. *The quantity θ is invariant under R3.*

Proof. Let D_l and D_r be two knot diagrams that differ only by an R3 move, and label their relevant edges and crossings as in Figure 4.3. Let $g^l_{\nu\alpha\beta}$ and $g^r_{\nu\alpha\beta}$ be their corresponding Green functions. Let $F^l_1(c)$, $F^l_2(c_0, c_1)$ and $F^l_3(\varphi, k)$ be defined from $g^l_{\nu\alpha\beta}$ as in (4)–(6), and similarly make F^r_1, F^r_2 and F^r_3 using $g^r_{\nu\alpha\beta}$.

By the invariance of the Alexander polynomial, the pre-factor $\Delta_1\Delta_2\Delta_3$ is the same for $\theta(D^l)$ and for $\theta(D^r)$ (see Equation (3)). By Theorem 2, $g^l_{\nu\alpha\beta} = g^r_{\nu\alpha\beta}$ so long as $\alpha, \beta \notin \{i^+, j^+, k^+\}$. And so the only terms that may differ in $\theta(D^h)$ between $h = l$ and $h = r$ are

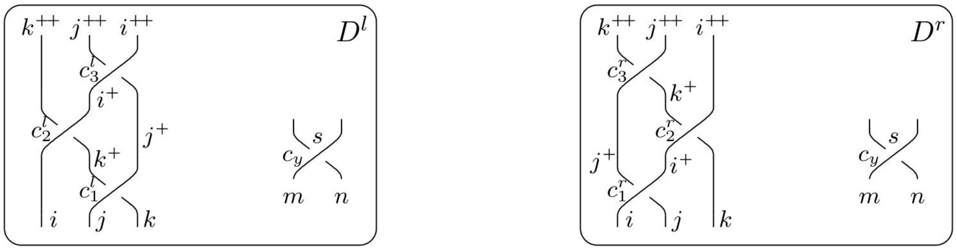


FIGURE 4.3. The two sides D^l and D^r of the R3 move. The left side D^l consists of 3 distinuighed crossings $c_1^l = (1, j, k)$, $c_2^l = (1, i, k^+)$, $c_3^l = (1, i^+, j^+)$ and a collection of further crossings $c_y = (s, m, n) \in Y$, where Y is the set of crossings not participating in the R3 move. The right side D^r consists of $c_1^r = (1, i, j)$, $c_2^r = (1, i^+, k)$, $c_3^r = (1, j^+, k^+)$ and the same set Y of further crossings c_y.

fig:R3

the terms

$$A^h = \sum_{c\in\{c^h_{1,2,3}\}} F_1^h(c) + \sum_{c_0,c_1\in\{c^h_{1,2,3}\}} F_2^h(c_0,c_1), \quad B^h = \sum_{c_0\in\{c^h_{1,2,3}\},\, c_y\in Y} F_2^h(c_0,c_y), \quad \text{and } C^h = \sum_{c_1\in\{c^h_{1,2,3}\},\, c_y\in Y} F_2^h(c_y,c_1). \qquad (8)$$

eq:ABC

We claim that $A^l = A^r$, $B^l = B^r$, and $C^l = C^r$.

To show that $A^l = A^r$, we need to compare polynomials in $g^l_{\nu\alpha\beta}$ with polynomials in $g^r_{\nu\alpha\beta}$ in which α and β may belong to the set $\{i^+, j^+, k^+\}$ on which it may be that $g^l \neq g^r$. Fortunately the g-rules of Equation (7) allow us to rewrite the offending g's, namely the ones with subscripts in $\{i^+, j^+, k^+\}$, in terms of other g's whose subscripts are in $\{i, j, k, i^{++}, j^{++}, k^{++}\}$, where $g^l = g^r$. So it is enough to show that

$$A^l \;/.\; (\text{the } g\text{-rules for } c_1^l,\, c_2^l,\, c_3^l) = A^r \;/.\; (\text{the } g\text{-rules for } c_1^r,\, c_2^r,\, c_3^r) \qquad \text{under } g^l = g^r, \qquad (9)$$

eq:R3A

where the symbol /. means "apply the rules". This is a finite computation that can in-principle be carried out by hand. But each A^h is a sum of $3+9 = 12$ polynomials in the g^h's, these polynomials are rather unpleasant (see (4) and (5)), and applying the relevant g-rules adds a bit further to the complexity. Luckily, we can delegate this pages-long calculation to an entity that doesn't complain.

First, we implement the Kronecker δ-function, the g-rules for a crossing (s, i, j), and the g-rules for a list of crossings X:

```
δ_{i_,j_} := If[i === j, 1, 0];
gRules[{s_Integer, i_, j_}] := {
    g_{ν_ jβ_} :> g_{ν j⁺ β} + δ_{jβ}, g_{ν_ iβ_} :> T_ν^s g_{ν i⁺ β} + (1 - T_ν^s) g_{ν j⁺ β} + δ_{iβ},
    g_{ν_ α_ i⁺} :> T_ν^s g_{ν α i} + δ_{α i⁺}, g_{ν_ α_ j⁺} :> g_{ν α j} + (1 - T_ν^s) g_{ν α i} + δ_{α j⁺}
  };
gRules[{X___List}] := Union @@ Table[gRules[c], {c, {X}}]
```

We then let `Xl` be the three crossings in the left-hand-side of the R3 move, as in Figure 4.3, we let `Al` be the A^l term of (8), and we let `lhs` be the result of applying the g-rules for the crossings in `Xl` to `Al`. We print only a "`Short`" version of `lhs` because the full thing would cover about 2.5 pages:

```
Xl = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}};
Al = Sum[F₁[c], {c, Xl}] + Sum[F₂[c0, c1], {c0, Xl}, {c1, Xl}];
lhs = Simplify[Al //. gRules[Xl]];
Short[lhs, 5]
```

$$-\frac{1}{2\,(1-T_2)}\Big(3-3\,T_2+\ll 128\gg+2\,(1-T_2)\,T_2\,g_{2,(k^+)^+,i}\,\big(1+(1-T_1\,T_2)\,g_{3,(k^+)^+,j}+g_{3,(k^+)^+,k}\big)+ 2\,(1-T_2)\,\big(1+T_2\,\big(T_2\,g_{2,(i^+)^+,i}-(-1+T_2)\,g_{2,(j^+)^+,i}\big)-(-1+T_2)\,g_{2,(k^+)^+,i}\big)\,\big(1+(1-T_1\,T_2)\,g_{3,(k^+)^+,j}+g_{3,(k^+)^+,k}\big)\Big)$$

We do the same for A^r, except this time, without printing at all:

```
Xr = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}};
Ar = Sum[F₁[c], {c, Xr}] + Sum[F₂[c0, c1], {c0, Xr}, {c1, Xr}];
rhs = Simplify[Ar //. gRules[Xr]];
```

We then compare `lhs` with `rhs`. The output, `True`, tells us that we have proven (9):

```
Simplify[lhs == rhs]
```

```
True
```

We show that $B^l = B^r$ by following exactly the same procedure. Note that we ignore the summation over c_y and instead treat it as fixed. If an equality is proven for every fixed c_y, it is of course also proven for the sum over $c_y \in Y$. Note also that we repeat the test twice, for the two choices of the sign of c_y:

```
Table[
 cy = {s, m, n};
 lhs = Sum[F₂[c0, cy], {c0, Xl}] //. gRules[Xl];
 rhs = Sum[F₂[c0, cy], {c0, Xr}] //. gRules[Xr];
 Simplify[lhs == rhs],
 {s, {1, -1}}
]
```

```
{True, True}
```

Similarly we prove that $C^l = C^r$, and this concludes the proof of Proposition 3.

```
Table[
 cy = {s, m, n};
 lhs = Sum[F₂[cy, c1], {c1, Xl}] //. gRules[Xl];
 rhs = Sum[F₂[cy, c1], {c1, Xr}] //. gRules[Xr];
 Simplify[lhs == rhs],
 {s, {1, -1}}
]
```

```
{True, True}
```

□

rem:E **Remark 4.** The computations above were carried out for generic $g_{\nu\alpha\beta}$ and for a generic $c_y = (s, m, n)$; namely, without specifying the knot diagrams in full, and hence without assigning specific values to $g_{\nu\alpha\beta}$, and without specifying m and n. Under these conditions the three parts of (8) cannot mix (namely, terms from, say, A^h cannot cancel terms in B^h or C^h), and so it would have been enough to show that $E^l = E^r$, where E^h combines A^h and B^h and C^h (and a few harmless further terms) by adding c_y to the summation corresponding to A^h:

$$E^h = \sum_{c \in \{c^h_{1,2,3,y}\}} F^h_1(c) + \sum_{c_0, c_1 \in \{c^h_{1,2,3,y}\}} F^h_2(c_0, c_1).$$

But that's a simpler computation:

```
Xl = {{1, j, k}, {1, i, k^+}, {1, i^+, j^+}};
Xr = {{1, i, j}, {1, i^+, k}, {1, j^+, k^+}};
ESum[X_] := (Sum[F_1[c], {c, X}] + Sum[F_2[c0, c1], {c0, X}, {c1, X}]) //. gRules[X];
Table[
 Simplify[ESum[Append[Xl, {s, m, n}]] == ESum[Append[Xr, {s, m, n}]]],
 {s, {1, -1}}
]
```

```
{True, True}
```

5. Strong and Meaningful

sec:SandM

6. Conjectures and Dreams

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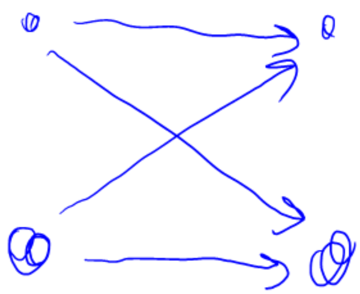

I don't understand partial inversion?

$$\begin{pmatrix} -I & A & \\ & -I & B \end{pmatrix} \sim \begin{pmatrix} -I & 0 & AB \\ 0 & -I & B \end{pmatrix}$$

	k	k^+	k^{++}	rest
k	I	A	B	0
k^+	0	I	C	0
k^{++}	D	0	I	E
rest	F	0	0	G

See http://drorbn.net/AP/Projects/Theta/

Reverse the sign of θZ

A VERY FAST, VERY STRONG, TOPOLOGICALLY MEANINGFUL AND FUN KNOT INVARIANT

DROR BAR-NATAN AND ROLAND VAN DER VEEN

Abstract. In this paper we introduce $\Theta = (\Delta, \theta)$, a pair of polynomial knot invariants which is:

- Theoretically and practically fast: Θ can be computed in polynomial time and we computed it in full on random knots with over 300 crossings, and its evaluation on on simple rational numbers on random knots with over 700 crossings.
- Strong: Its separation power is much greater than, say, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings (while computing much faster).
- Topologically meaningful: It gives a genus bound, and there are reasons to hope that it would do more.
- Fun: Scroll to Figures 1.1, 1.2, and 3.1.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh], continuing Rozansky, Garoufalidis, and Kricker [GR, Ro1, Ro2, Ro3, Kr]. Yet our formulas, proofs, and programs are much simpler and enable its computation even on very large knots.

Ohtsuki:TwoLoop
GaroufalidisRozansky:LoopExpansion, Rozansky:Contribution, Rozansky:Burau, Rozansky:U1RCC, Kricke

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1. Fun

The word "fun" rarely appears in the title of a math paper, so let us start with a brief justification.

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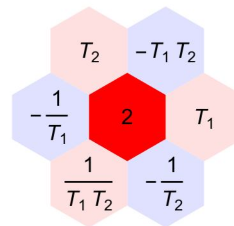
This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC). It is available in electronic form, along with source files and a demo *Mathematica* notebook at http://drorbn.net/Theta and at arXiv:????.?????.

Θ is a pair of polynomials. The first, Δ, is old news, the Alexander polynomial [Al]. It is a one-variable Laurent polynomial in a variable T. For example, $\Delta(☘) = T^{-1} - 1 + T$. We turn such a polynomial to a list of coefficients (for ☘, it is $(1\ \ -1\ \ 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a "bar code", and for the trefoil ☘ is it .



Similiarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2. We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely into a picture! To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a certain shear transformation to the plane before printing. So the colour of a monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $\begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ instead of the more traditional $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^{-1} + T_1^{-1}T_2^{-1} - T_2^{-1}$.

Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a "hexagonal QR code". For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. In addition, the hexagonal QR codes of some 15 knots with $\geqslant 300$ crossings are in Figure 1.2, and Θ of a 132-crossing torus knot is in Figure 3.1.

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.2 there seem to be large-scale "sand table patterns" or "diffraction patterns". We can't prove any of these things, and the last one, we can't even formulate properly. Yet they are clearly there, too clear to be the result of chance alone.

We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

2. Formulas

2.1. **Old Formulas**[1].

The setup leading to the definition of Θ is the same as the setup leading to the definition of the invariant ρ_1 of [BV1], and hence we copy a few relevant paragraphs from [BV1] nearly varbatim, with only a few modifications.

[1] "Old" means that these formulas appeared already in [BV1].

ig:Rolfsen

FIGURE 1.1. Θ as a bar code and a QR code, for all the knots in the Rolfsen table.

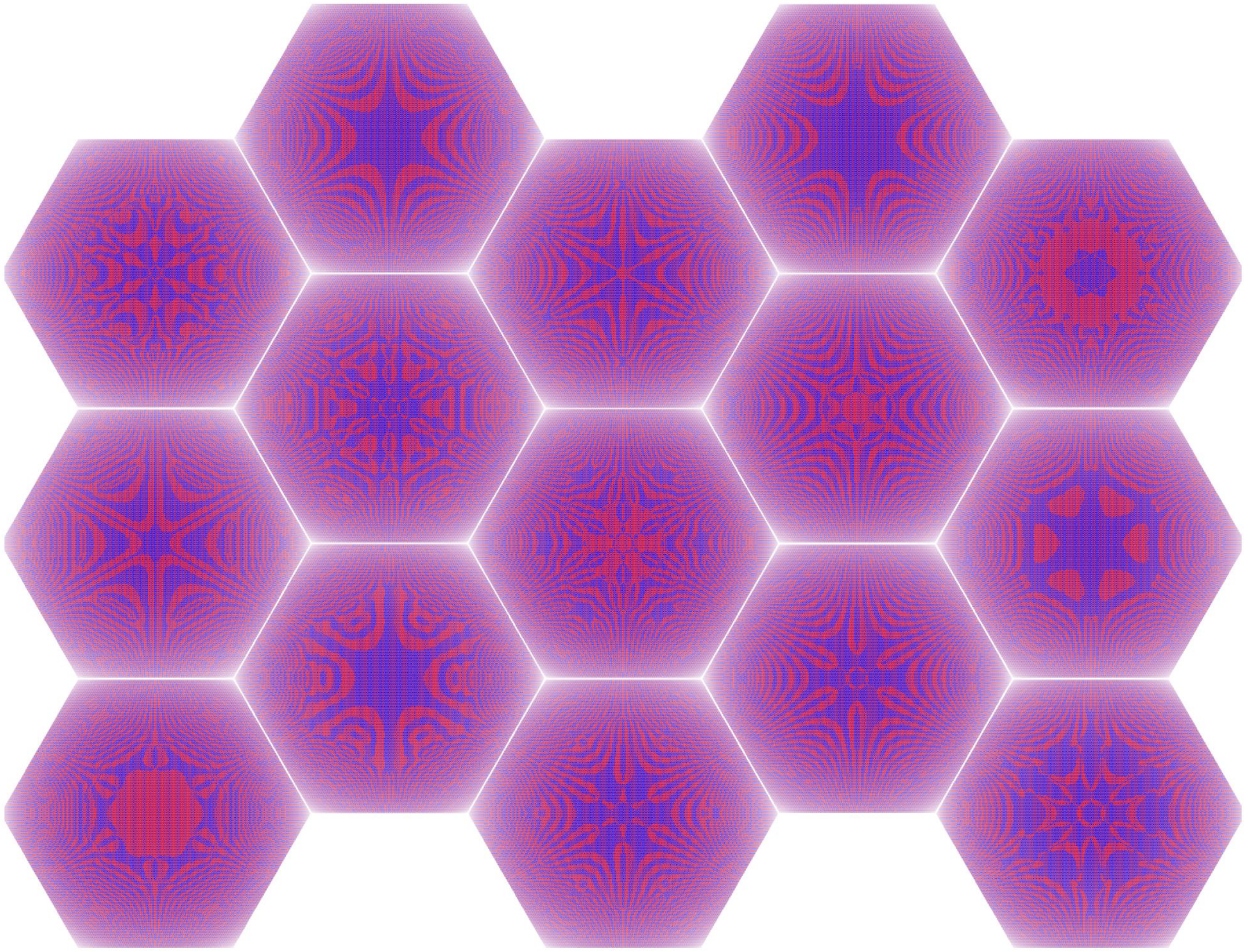
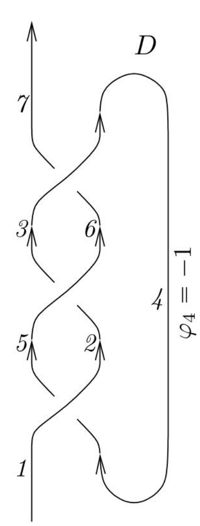


fig:300

FIGURE 1.2. θ (hexagonal QR code only) of the 15 largest knots that we have computed by September 16, 2024. They are all "generic" in as much as we know, and they all have $\geqslant 300$ crossings. The knots come from [DHOEBL].

Given an oriented n-crossing knot K, we draw it in the plane as a long knot diagram D in such a way that the two strands intersecting at each crossing are pointing up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram.* An example of an upright knot diagram is shown on the right.



We then label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n + 1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

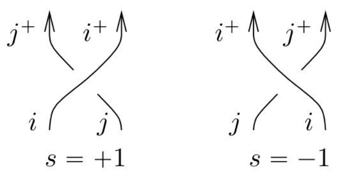
Let X be the set of all crossings in the diagram D, where we encode each crossing as a triple (sign, incoming over edge, incoming under edge). In our example we have $X = \{(1,1,4),(1,5,2),(1,3,6)\}$.

We let A be the $(2n+1)\times(2n+1)$ matrix of Laurent polynomials in a formal variable T, defined by

$$A := I - \sum_c \left(T^s E_{i,i^+} + (1-T^s)E_{i,j^+} + E_{j,j^+}\right),$$

where I is the identity matrix and $E_{\alpha\beta}$ denotes the elementary matrix with 1 in row α and column β and zeros elsewhere. The summation is over the crossings $c = (s,i,j)$ of the diagram D, and once c is chosen, s denotes its sign and i and j denote the labels below the crossing where the label i belongs to the over-strand and j to the under-strand.

Alternatively, $A = I + \sum_c A_c$, where A_c is a matrix of zeros except for the blocks as follows:



$\longrightarrow$

A_c	column i^+	column j^+
row i	$-T^s$	T^s-1
row j	0	-1

(1) eq:A

We note (as we did in [BV1]) that the determinant of A is equal up to a unit to the normalized Alexander polynomial Δ of K. In fact, we have that

$$\Delta = T^{(-\varphi(D)-w(D))/2}\det(A), \tag{2}$$

eq:Delta

where $\varphi(D) := \sum_k \varphi_k$ is the total rotation number of D and where $w(D) = \sum_c s_c$ is the writhe of D, namely the sum of the signs s_c of all the crossings c in D.

We let $G = (g_{\alpha\beta}) = A^{-1}$ and, thinking of it as a function $g_{\alpha\beta}$ of a pair of edges α and β, we call it the Green function of the diagram D. When inspired by physics (e.g. [BN2]) we sometimes call it "the 2-point function", and when thinking of car traffic (e.g. [BN3]) we sometimes call it "the traffic function".

We note that the computation of G is the bottleneck in the computation of Θ. It requires inverting a $(2n+1)\times(2n+1)$ matrix whose entries are (degree 1) Laurent polynomials in T. It's a daunting task yet it takes polynomial time, it can be performed in practice even if n is in the hundreds, and everything which then follows is easier.

2.2. **New Formulas.** Let T_1 and T_2 be indeteminates and let $T_3 := T_1T_2$. Let $\Delta_\nu := \Delta_{T\to T_\nu}$ and $G_\nu = (g_{\nu\alpha\beta}) := G_{T\to T_\nu}$ be Δ and G subject to the substitution $T \to T_\nu$, where $\nu = 1,2,3$ (these are easy to compute once Δ and G have been computed).

Given crossings $c = (s,i,j)$, $c_0 = (s_0,i_0,j_0)$, and $c_1 = (s_1,i_1,j_1)$ in X, let

$$\begin{aligned}
F_1(c) = {} & s\,[1/2 - g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\
& \quad + (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}] \\
& + \frac{s}{T_2^s - 1} \,[(T_1^s - 1) T_2^s \left(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}\right) \\
& \quad + (T_3^s - 1)\left(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}\right) \\
& \quad - (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}]
\end{aligned} \tag{3}$$

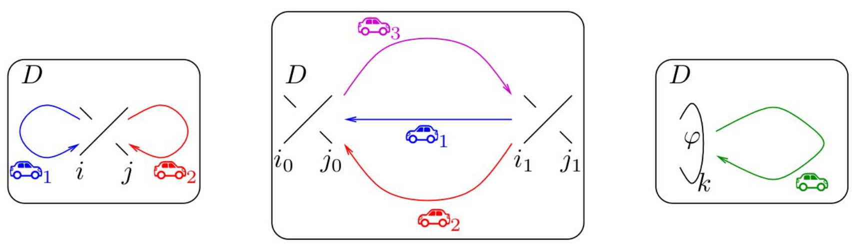
$$F_2(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1 j_1 i_0} g_{3 j_0 i_1}}{T_2^{s_1} - 1} \left(T_2^{s_0} g_{2 i_1 i_0} + g_{2 j_1 j_0} - T_2^{s_0} g_{2 j_1 i_0} - g_{2 i_1 j_0}\right) \tag{4}$$

$$F_3(\varphi_k, k) = \varphi_k (g_{3kk} - 1/2) \tag{5}$$

Theorem 1 (Proof in Section 4)**.** *The following is a knot invariant:*

$$\theta(D) := \Delta_1 \Delta_2 \Delta_3 \left(\sum_c F_1(c) + \sum_{c_0, c_1} F_2(c_0, c_1) + \sum_k F_3(\varphi_k, k) \right). \tag{6}$$

We note without detail that there is an alternative formula for θ in terms of perturbed Gaussian integration [BN2]. In that language, and using also the traffic motifs of [BV1, BN3], the three summands in (6) become Feynman diagrams for processes in which cars governed by parameter $T = T_1, T_2$, or T_3 interact:



In particular, the middle diagram which resembles the greek letter Θ gave the invariant its name.

We note also that computationaly, the worst term in (6) is the middle one, and even it takes merely $\sim n^2$ operations in the ring $\mathbb{Q}(T_1, T_2)$ to complete.

The polynomials $F_1(c)$, $F_2(c_0, c_1)$ and $F_3(\varphi, k)$ are not unique, and we are not certain that we have the cleanest possible formulas for them. They are human-ugly, yet from a computational perspective, having 18 terms (as is the case for $F_1(c)$) isn't really a problem; computers don't care.

3. Implementation and Examples

A concise yet reasonably efficient implementation is worth a thousand formulas. It completely removes ambiguities, it tests the theories, and it allows for experimentation. Hence our next task is to implement. The section that follows was generated from a Mathematica [Wo] notebook which is available at [BV2, Theta.nb]. A second implemntation of Θ, using Python and SageMath (https://www.sagemath.org/) is available at https://www.rolandvdv.nl/Theta/.

We start by loading the package `KnotTheory‘` — it is only needed because it has many specific knots pre-defined:

```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Next we quietly define the commands `Rot`, used to compute rotation numbers, and `PolyPlot`, used to plot polynomials as bar codes and as hexagonal QR codes. Neither is a part of the core of the computation of Θ, so neither is shown; yet we do show one usage example for each.

```
(* Rot suppressed *)
```

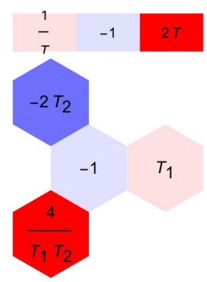
```
Rot[Mirror@Knot[3, 1]]
```

```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0, 0}}
```

We urge the reader to compare the above output with the knot diagram in Section 2.1. ssec:OldFormulas

```
(* PolyPlot suppressed *)
```

```
PolyPlot[{2 T - 1 + T^-1, -1 + T1 - 2 T2 + 4 T1^-1 T2^-1},
  ImageSize → 100, Labeled → True]
```



The definition of `CF` below is a technicality telling the computer how to best store polynomials in the $g_{\nu\alpha\beta}$ such as F_1 and F_2. The programs would run just the same without it, albeit a bit more slowly:

```
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

Next, we decree that $T_3 = T_1T_2$ and define the three "Feynman Diagram" polynomials F_1, F_2, and F_3:

```
T3 = T1 T2;
```

```
F1[{s_, i_, j_}] := CF[
  s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii - (1 - T3^s) g2ji g3ji -
      g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
        (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) / (T2^s - 1))]
```

```
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
  CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
    ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
```

```
F3[φ_, k_] = φ g3kk - φ/2;
```

Next comes the main program computing Θ. Fortunately, it matches perfectly with the mathematical description in Section 2. sec:Formulas In line 01 we let X be the list of crossings in an input knot K, and φ the list of its rotation numbers, using the external program `Rot` which we have already mentioned. We also let n be the length of X, namely, the number of crossings in K. In line 02 we let the starting value of A be the identity matrix, and then in line 03, for each crossing in X we add to A a 2×2 block, in rows i and j and columns $i+1$ and $j+1$, as explain in Equation (1). In line 04 we compute the normalized Alexander polynomial Δ as in (2). In line 05 we let G be the inverse of A. In line 06 we declare what it means to evaluate, `ev`, a formula $\mathcal{E}$ that may contain symbols of the form $g_{\nu\alpha\beta}$: each such symbol is to be replaced by the entry in position α, β of G, but with T replaced with T_ν. In line 07 we start computing θ by computing the first summand in (6), which in itself, is a sum over the crossings of the knot. In line 08 we add to θ the double sum corresponding to the second term in (6), and in line 09, we add the third summand of (6). Finally, line 10 outputs a pair: Δ, and the re-normalized version of θ.

```
Θ[K_] := Θ[K] = Module[{X, φ, n, A, Δ, G, ev, θ},
  (* 01 *) {X, φ} = Rot[K]; n = Length[X];
  (* 02 *) A = IdentityMatrix[2 n + 1];
  (* 03 *) Cases[X, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ; 0  -1 ))];
  (* 04 *) Δ = T^((-Total[φ]-Total[X[[All,1]]])/2) Det[A];
  (* 05 *) G = Inverse[A];
  (* 06 *) ev[ℰ_] := Factor[ℰ /. g_{ν_,α_,β_} :> (G[[α, β]] /. T → T_ν)];
  (* 07 *) θ = ev[∑_{k=1}^{n} F1[X[[k]]]];
  (* 08 *) θ += ev[∑_{k1=1}^{n} ∑_{k2=1}^{n} F2[X[[k1]], X[[k2]]]];
  (* 09 *) θ += ev[∑_{k=1}^{2 n} F3[φ[[k]], k]];
  (* 10 *) Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}
];
```

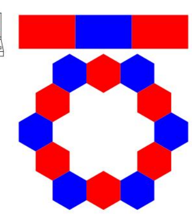
Consider a pro-numeric version

On to examples! Starting with the trefoil knot.

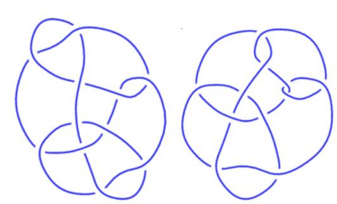
```
Expand[Θ[Knot[3, 1]]]
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2 T_2^2}+\frac{1}{T_1 T_2^2}+\frac{1}{T_1^2 T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2 T_2-T_2^2+T_1 T_2^2-T_1^2 T_2^2\right\}$$

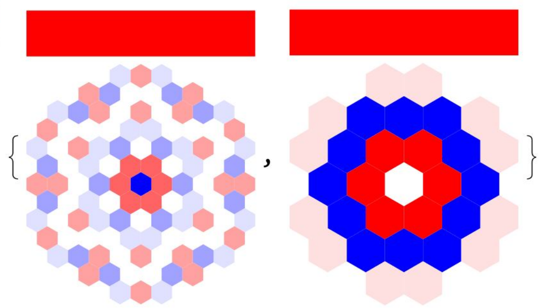
```
PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]
```



Next are the Conway knot 11_{n34} and the Kinoshita-Terasaka knot 11_{n42}. The two are mutants and famously hard to separate: they both have $\Delta = 1$ (as evidenced by their one-bar bar codes below), and they have the same HOMFLY-PT polynomial and Khovanov homology. Yet their θ invariants are different. Note that the genus of the Conway knot is 3, while the genus of the Kinoshita-Terasaka knot is 2. This agrees with the apparent higher complexity of the QR code of the Conway polynomial, and with the observations in Section 5.

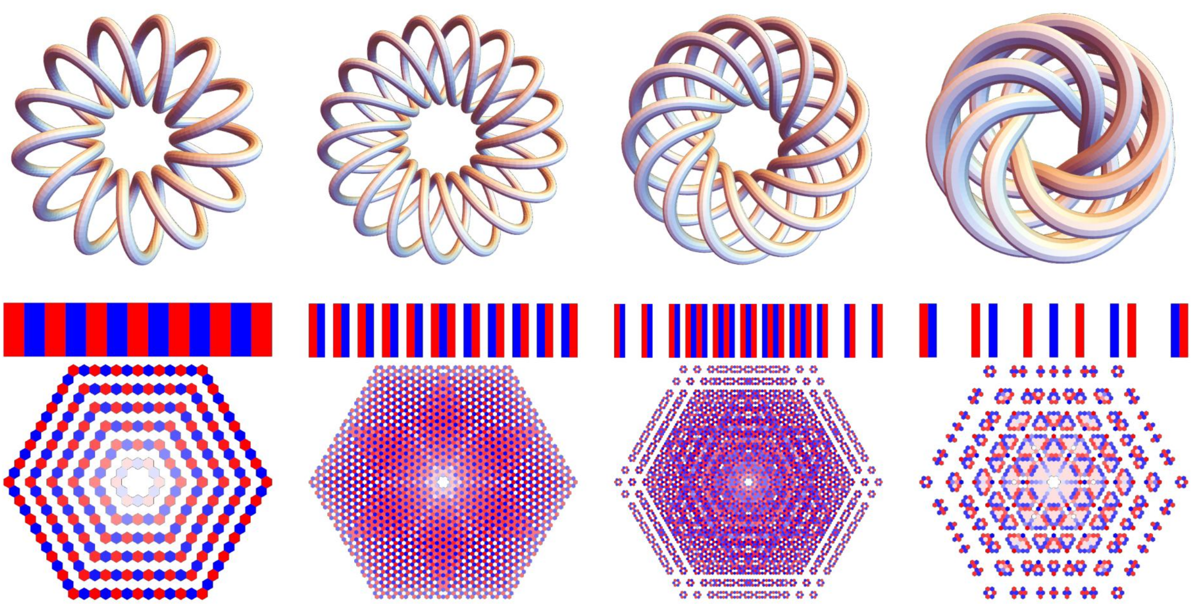


```
PolyPlot[Θ[Knot[#]], ImageSize → 120] & /@
  {"K11n34", "K11n42"}
```



Torus knots have particularly nice-looking Θ invariants. Here are the torus knots $T_{13/2}$, $T_{17/3}$, $T_{13/5}$, and $T_{7/6}$:

```
GraphicsGrid[{
  TubePlot[TorusKnot @@ #] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}},
  PolyPlot[Θ[TorusKnot @@ #]] & /@ {{13, 2}, {17, 3}, {13, 5}, {7, 6}}
 }]
```



The next line shows the computation time in seconds for the 132-crossing torus knot $T_{22/7}$ on a 2024 laptop, without actually showing the output. The output plot is in Figure 3.1.

```
AbsoluteTiming[Θ[TorusKnot[22, 7]];]
```

```
{1020.73, Null}
```

4. Proof of Invariance

Our proof of the invariance of θ (Theorem 1) is very similar, and uses many of the same pieces, as the proof of the invariance of ρ_1 in [BV1]. Thus instead of repeating everything we just summarize those steps that are identical and then provide the needed details for the steps that differ.

Like in [BV1, Lemma 3], we know that for any crossing $c = (s, i, j)$ in a knot diagram D, the Green functions $G_\nu = (g_{\nu\alpha\beta})$ of D satisfy the following "g-rules" for $\nu = 1, 2, 3$ and with

```
ImageCompose[PolyPlot[Θ[TorusKnot[22, 7]], ImageSize → 720],
  TubePlot[TorusKnot[22, 7], ImageSize → 360], {Right, Bottom}, {Right, Bottom}]
```

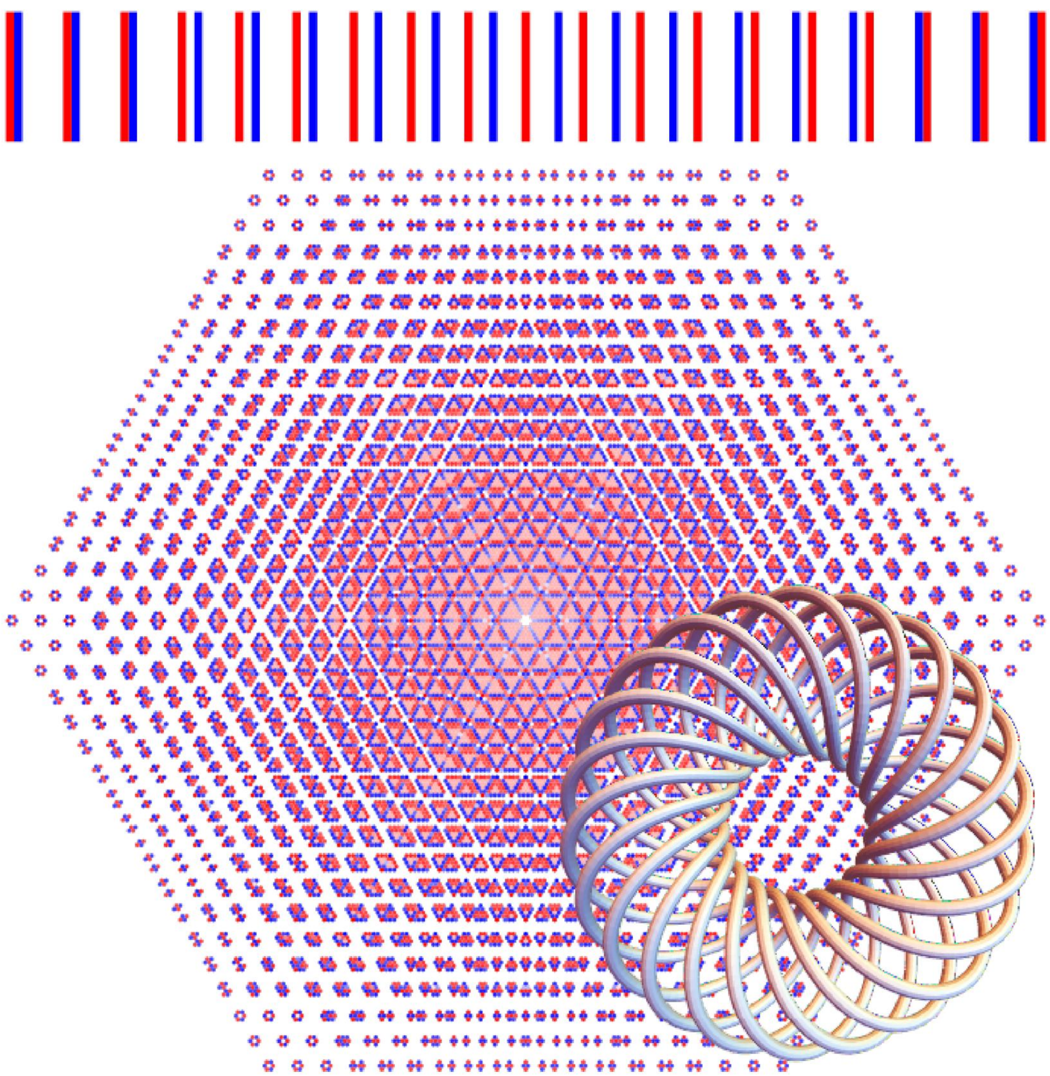


fig:T227

FIGURE 3.1. The 132-crossing torus knot $T_{22/7}$ and a plot of its Θ invariant

δ denoting the Kronecker delta:

$$
\begin{aligned}
g_{\nu i\beta} &= \delta_{i\beta} + T^s_\nu g_{\nu,i^+,\beta} + (1-T^s_\nu) g_{\nu,j^+,\beta}, \qquad & g_{\nu j\beta} &= \delta_{j\beta} + g_{\nu,j^+,\beta}, \\
g_{\nu,\alpha,i^+} &= T^s_\nu g_{\nu\alpha i} + \delta_{\alpha,i^+}, & g_{\nu,\alpha,j^+} &= g_{\nu\alpha j} + (1-T^s_\nu) g_{\nu\alpha i} + \delta_{\alpha,j^+}.
\end{aligned} \tag{7}
$$

eq:gRules

We also need a variant $\bar{g}_{\nu ab}$ of $g_{\nu\alpha\beta}$, defined whenever a and b are two distinct points on the edges of a knot diagram D, away from the crossings. If α is the edge on which a lies and β is the edge on which b lies, $\bar{g}_{\nu ab}$ is defined as follows:

$$
\bar{g}_{\nu ab} = \begin{cases} g_{\nu\alpha\beta} & \text{if } \alpha \neq \beta, \\ g_{\nu\alpha\beta} & \text{if } \alpha = \beta \text{ and } a < b \text{ relative to the orientation of the edge } \alpha = \beta, \\ g_{\nu\alpha\beta} - 1 & \text{if } \alpha = \beta \text{ and } a > b \text{ relative to the orientation of the edge } \alpha = \beta. \end{cases}
$$

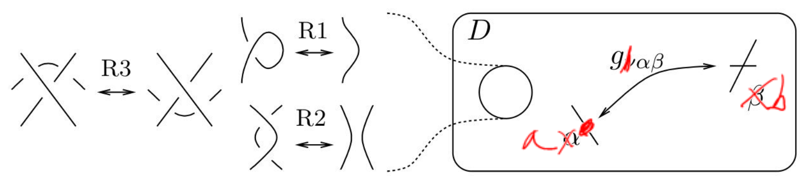


FIGURE 4.1. The Green functions $g_{\nu\alpha\beta}$ are invariant under Reidemeister moves performed away from where they are measured.

It is clear that g and $\bar{g}$ contain the same information and are easily computable from each other. The variant $\bar{g}$ is, strictly speaking, not a matrix and so g is a bit more suitable for computations. Yet $\bar{g}$ is a bit better behaved when we try to track, as below, the behaviour of g / $\bar{g}$ under Reidemeister moves. Indeed Reidemeister moves may changes the indexing α and β of edges even when those edges are far from the move location, while it makes sense to keep points like a and b in place when the moves are away. Furthemore and more importantly, the distinction between $a < b$ and $a > b$ when a and b are on the same edge matters below.

The following theorem, was not named in [BV1], yet it was stated there as the first part of the first proof of [BV1, Theorem 1].

Theorem 2. *The modified Green functions $\bar{g}_{\nu ab}$ are "relative invariants", meaning that once points a and b are fixed within a knot diagram D, the values of $\bar{g}_{\nu ab}$ do not change if Reidemeister moves are performed away from the points a and b. An illustration appears in Figure 4.1.*

The proof of Theorem 2 is perhaps best understood in terms of the function of [BV1, BN1, BN3]: One simply needs to verify that for each of the Reidemeister moves, traffic entering the tangle diagram for the left hand side of the move exits it in the same manner as traffic entering the tangle diagram for the right hand side of the move, and each of these verifications, as explained in [BV1, BN1, BN3], is very easy. Yet this proof is a bit informal, so we opt here to give a fully formal proof along the lines of the first halves of [BV1, Propositions 7-9].

Proof of Theorem 2.

We can now move on to the main part of the proof of Theorem 1. As follows from [BV1, Theorem 2], we need to prove the invariance of θ under the "upright Reidemeister" moves of Figure 4.2. We start with the hardest, R3:

Proposition 3. *The quantity θ is invariant under R3.*

Proof. Let D_l and D_r be two knot diagrams that differ only by an R3 move, and label their relevant edges and crossings as in Figure 4.3. Let $g^l_{\nu\alpha\beta}$ and $g^r_{\nu\alpha\beta}$ be their corresponding Green functions. Let $F^l_1(c)$, $F^l_2(c_0, c_1)$ and $F^l_3(\varphi, k)$ be defined from $g^l_{\nu\alpha\beta}$ as in (3)–(5), and similarly make F^r_1, F^r_2 and F^r_3 using $g^r_{\nu\alpha\beta}$.

By the invariance of the Alexander polynomial, the pre-factor $\Delta_1\Delta_2\Delta_3$ is the same for $\theta(D^l)$ and for $\theta(D^r)$ (see Equation (6)). By Theorem 2, $g^l_{\nu\alpha\beta} = g^r_{\nu\alpha\beta}$ so long as $\alpha, \beta \notin \{i^+, j^+, k^+\}$. And so the only terms that may differ in $\theta(D^h)$ between $h = l$ and $h = r$ are the terms

$$A^h = \sum_{c\in\{c^h_{1,2,3}\}} F^h_1(c) + \sum_{c_0,c_1\in\{c^h_{1,2,3}\}} F^h_2(c_0, c_1), \quad B^h = \sum_{c_0\in\{c^h_{1,2,3}\},\, c_y\in Y} F^h_2(c_0, c_y), \quad \text{and } C^h = \sum_{c_1\in\{c^h_{1,2,3}\},\, c_y\in Y} F^h_2(c_y, c_1). \quad (8)$$

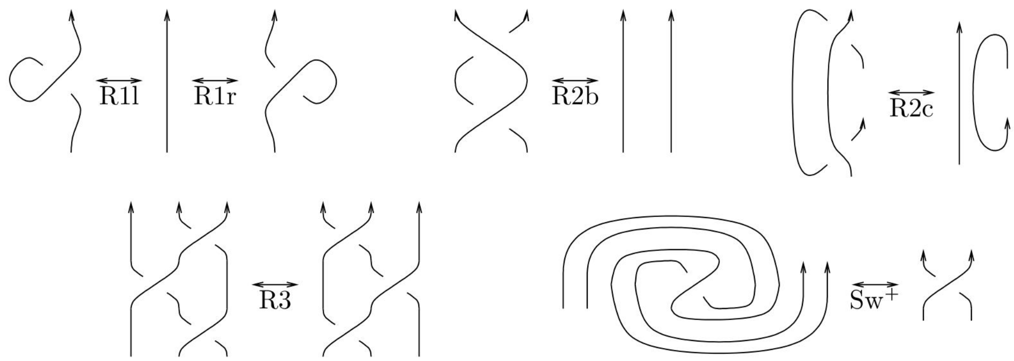


FIGURE 4.2. The upright Reidemeister moves: Reidemeister 1 left and right, Reidemeister 2 braid-like and cyclic, Reidemeister 3, and (the +) Swirl.

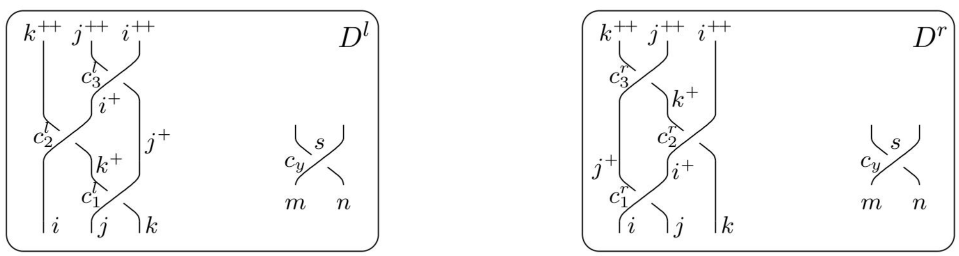


FIGURE 4.3. The two sides D^l and D^r of the R3 move. The left side D^l consists of 3 distinuighed crossings $c_1^l = (1, j, k)$, $c_2^l = (1, i, k^+)$, $c_3^l = (1, i^+, j^+)$ and a collection of further crossings $c_y = (s, m, n) \in Y$, where Y is the set of crossings not participating in the R3 move. The right side D^r consists of $c_1^r = (1, i, j)$, $c_2^r = (1, i^+, k)$, $c_3^r = (1, j^+, k^+)$ and the same set Y of further crossings c_y.

We claim that $A^l = A^r$, $B^l = B^r$, and $C^l = C^r$.

To show that $A^l = A^r$, we need to compare polynomials in $g^l_{\nu\alpha\beta}$ with polynomials in $g^r_{\nu\alpha\beta}$ in which α and β may belong to the set $\{i^+, j^+, k^+\}$ on which it may be that $g^l \neq g^r$. Fortunately the g-rules of Equation (7) allow us to rewrite the offending g's, namely the ones with subscripts in $\{i^+, j^+, k^+\}$, in terms of other g's whose subscripts are in $\{i, j, k, i^{++}, j^{++}, k^{++}\}$, where $g^l = g^r$. So it is enough to show that

$$A^l \; /. \; (\text{the } g\text{-rules for } c_1^l,\, c_2^l,\, c_3^l) = A^r \; /. \; (\text{the } g\text{-rules for } c_1^r,\, c_2^r,\, c_3^r) \qquad \text{under } g^l = g^r, \quad (9)$$

where the symbol /. means "apply the rules". This is a finite computation that can in-principle be carried out by hand. But each A^h is a sum of $3+9 = 12$ polynomials in the g^h's, these polynomials are rather unpleasant (see (3) and (4)), and applying the relevant g-rules adds a bit further to the complexity. Luckily, we can delegate this pages-long calculation to an entity that doesn't complain.

First, we implement the Kronecker δ-function, the g-rules for a crossing (s, i, j), and the g-rules for a list of crossings X:

```
δ_{i_,j_} := If[i === j, 1, 0];
gRules[{s_, i_, j_}] := {
   g_{ν_jβ_} :> g_{νj⁺β} + δ_{jβ}, g_{ν_iβ_} :> T_ν^s g_{νi⁺β} + (1 - T_ν^s) g_{νj⁺β} + δ_{iβ},
   g_{ν_α_i⁺} :> T_ν^s g_{ναi} + δ_{αi⁺}, g_{ν_α_j⁺} :> g_{ναj} + (1 - T_ν^s) g_{ναi} + δ_{αj⁺}
  };
gRules[X___List] := Union @@ Table[gRules[c], {c, {X}}]
```

fig:R3

We then let Xl be the three crossings in the left-hand-side of the R3 move, as in Figure 4.3, we let Al be the A^l term of (8), and we let lhs be the result of applying the g-rules for the crossings in Xl to Al. We print only a "Short" version of lhs because the full thing would cover about 2.5 pages:

```
Xl = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}};
Al = Sum[F₁[c], {c, Xl}] + Sum[F₂[c0, c1], {c0, Xl}, {c1, Xl}];
lhs = Simplify[Al //. gRules @@ Xl];
Short[lhs, 5]
```

$$-\frac{1}{2\,(1-T_2)}\Big(3-3\,T_2+\ll 129\gg+ 2\,(1-T_2)\,\big(1+T_2\,(T_2\,g_{2,\ll 1\gg^+,i}-(-1+T_2)\,g_{2,\ll 1\gg,i})-(-1+T_2)\,g_{2,(k^+)^+,i}\big)\,\big(1+(1-T_1\,T_2)\,g_{3,(k^+)^+,j}+g_{3,(k^+)^+,k}\big)\Big)$$

We do the same for A^r, except this time, without printing at all:

```
Xr = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}};
Ar = Sum[F₁[c], {c, Xr}] + Sum[F₂[c0, c1], {c0, Xr}, {c1, Xr}];
rhs = Simplify[Ar //. gRules @@ Xr];
```

We then compare lhs with rhs. The output, True, tells us that we have proven (9):

```
Simplify[lhs == rhs]
```

True

We show that $B^l = B^r$ by following exactly the same procedure. Note that we ignore the summation over c_y and instead treat it as a fixed crossing $c_y = (s, m, n)$. If an equality is proven for every fixed c_y, it is of course also proven for the sum over $c_y \in Y$.

```
lhs = Sum[F₂[c0, {s, m, n}], {c0, Xl}] //. gRules @@ Xl;
rhs = Sum[F₂[c0, {s, m, n}], {c0, Xr}] //. gRules @@ Xr;
Simplify[lhs == rhs]
```

True

Similarly we prove that $C^l = C^r$, and this concludes the proof of Proposition 3.

```
lhs = Sum[F₂[{s, m, n}, c1], {c1, Xl}] //. gRules @@ Xl;
rhs = Sum[F₂[{s, m, n}, c1], {c1, Xr}] //. gRules @@ Xr;
Simplify[lhs == rhs]
```

True

□

rem:E **Remark 4.** The computations above were carried out for generic $g_{\nu\alpha\beta}$ and for a generic $c_y = (s, m, n)$; namely, without specifying the knot diagrams in full, and hence without assigning specific values to $g_{\nu\alpha\beta}$, and without specifying m and n. Under these conditions the three parts of (8) cannot mix (namely, terms from, say, A^h cannot cancel terms in B^h or

C^h), and so it would have been enough to show that $E^l = E^r$, where E^h combines A^h and B^h and C^h (and a few harmless further terms) by adding c_y to the summation corresponding to A^h:

$$E^h = \sum_{c\in\{c^h_{1,2,3,y}\}} F^h_1(c) + \sum_{c_0,c_1\in\{c^h_{1,2,3,y}\}} F^h_2(c_0,c_1).$$

But that's a simpler computation:

```
ESum[X_] := (Sum[F₁[c], {c, X}] + Sum[F₂[c0, c1], {c0, X}, {c1, X}]) //. gRules @@ X;
```

```
Xl = {{1, j, k}, {1, i, k⁺}, {1, i⁺, j⁺}};
Xr = {{1, i, j}, {1, i⁺, k}, {1, j⁺, k⁺}};
Simplify[ESum[Append[Xl, {s, m, n}]] == ESum[Append[Xr, {s, m, n}]]]
```

```
True
```

5. Strong and Meaningful

sec:SandM

6. Conjectures and Dreams

References

Invariants [Al] J. W. Alexander, *Topological invariants of knots and link,* Trans. Amer. Math. Soc. **30** (1928) 275–306. See pp. 2.

axaca-2210 [BN1] D. Bar-Natan, *Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants,* talk given in Oaxaca, October 2022. Video and handout at http://drorbn.net/oa22. See pp. 11.

IType [BN2] Dror Bar-Natan, *Knot Invariants from Finite Dimensional Integration,* talks in Beijing (July 2024, http://drorbn.net/icbs24) and in Geneva (August 2024, http://drorbn.net/ge24). See pp. 5, 6.

nto-241030 [BN3] Dror Bar-Natan, *The Strongest Genuinely Computable Knot Invariant in 2024,* talk given in Toronto (October 2024, http://drorbn.net/to24). See pp. 5, 6, 11.

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