A FAST, STRONG, TOPOLOGICALLY MEANINGFUL, AND FUN KNOT INVARIANT

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Abstract. In this paper we discuss a pair of polynomial knot invariants $\Theta = (\Delta, \theta)$ which is:

- Theoretically and practically fast: Θ can be computed in polynomial time. We can compute it in full on random knots with over 300 crossings, and its evaluation at simple rational numbers on random knots with over 600 crossings.
- Strong: Its separation power is much greater than the hyperbolic volume, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings (while being computable on much larger knots).
- Topologically meaningful: It gives a genus bound, and there are reasons to hope that it would do more.
- Fun: Scroll to Figures 1.1–1.4, 3.1, and 6.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh2], continuing Rozansky, Kricker, and Garoufalidis [Roz1, Roz2, Roz3, Kr, GR]. Yet our formulas, proofs, and programs are much simpler and enable its computation even on very large knots.

Contents

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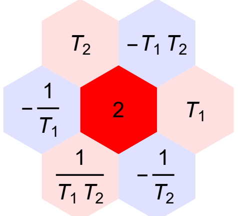
This paper is available in electronic form, along with source files and a demo *Mathematica* notebook at http://drorbn.net/Theta and at arXiv:2509.18456.

1. Fun

The word "fun" rarely appears in the title of a math paper, so let us start with a brief justification (more can be found in a Quanta Magazine article, [Kl]).

Θ is a pair of polynomials. The first, Δ, is old news, the Alexander polynomial [Al]. It is a one-variable Laurent polynomial in a variable T. For example, $\Delta(\text{trefoil}) = T^{-1} - 1 + T$. We turn such a polynomial into a list of coefficients (for the trefoil, it is $(1, -1, 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a "bar code", and for the trefoil it is a bar code.

Similarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2. We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely, into a picture. To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a shear transformation to the plane before printing. So a monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $(n_1 - n_2/2, \sqrt{3}n_2/2)$ instead of the more straightforward (n_1, n_2). On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^{-1} + T_1^{-1}T_2^{-1} - T_2^{-1}$.



Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a "hexagonal QR code". For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. For some alternating square weave knots, they are in Figure 1.2, and for a random square weave, in Figure 1.3. In addition, the hexagonal QR codes of 15 knots with $\geqslant 300$ crossings are in Figure 1.4, and Θ of a 132-crossing torus knot is in Figure 3.1. Some further computations and figures, also highlighting the parity of coefficients rather than just their signs, are at [Lal].

Clearly there are patterns in these figures. There is a hexagonal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.4 there seem to be large-scale patterns perhaps reminiscent of the "Chladni figures" formed by powders atop vibrating plates (on right). We can't prove any of these things, and the last one, we can't even formulate properly. Yet they are clearly there, too clear to be the result of chance alone.

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We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

Figure 1.1. Θ as a bar code and a QR code, for all the knots in the Rolfsen table.