A VERY FAST, VERY STRONG, TOPOLOGICALLY MEANINGFUL AND FUN KNOT INVARIANT

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ABSTRACT. In this paper we introduce $\Theta = (\Delta, \theta)$, a pair of polynomial knot invariants which is:

- Theoretically and practically fast: Θ can be computed in polynomial time and we computed it in full on random knots with over 300 crossings, and its evaluation on on simple rational numbers on random knots with over 700 crossings.
- Strong: Its separation power is much greater than, say, the HOMFLY-PT polynomial and Khovanov homology (taken together) on knots with up to 15 crossings (while computing much faster).
- Topologically meaningful: It gives a genus bound, and there are reasons to hope that it would do more.
- Fun: Scroll to Figures 1.1 and 1.2.

Δ is merely the Alexander polynomial. θ is almost certainly equal to an invariant that was studied extensively by Ohtsuki [Oh], continuing Rozansky, Garoufalidis, and Kricker [GR, Ro1, Ro2, Ro3, Kr]. Yet our formulas, proofs, and programs are much simpler and enable its computation even on very large knots.

CONTENTS

1. FUN

The word "fun" rarely appears in the title of a math paper, so let us start with a brief justification.

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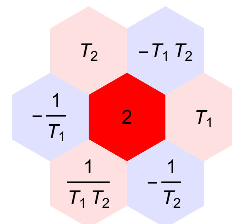
Θ is a pair of polynomials. The first, Δ, is old news, the Alexander polynomial [Al]. It is a one-variable Laurent polynomial in a variable T. For example, $\Delta( ) = T^{-1} - 1 + T$. We turn such a polynomial to a list of coefficients (for , it is $(1\ \ -1\ 1)$), and then to a chain of bars of varying colours: white for the zero coefficients, and red and blue for the positive and negative coefficients (with intensity proportional to the magnitude of the coefficients). The result is a "bar code", and for the trefoil is it .

Similiarly, θ is a 2-variable Laurent polynomial, in variables T_1 and T_2. We can turn such a polynomial into a 2D array of coefficients and then using the same rules, into a 2D array of colours, namely into a picture! To highlight a certain conjectured hexagonal symmetry of the resulting pictures, we apply a certain shear transformation to the plane before printing. So the colour of a monomial $cT_1^{n_1}T_2^{n_2}$ gets printed at position $\begin{pmatrix}1 & -1/2\\ 0 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix}$ instead of the more traditional $\begin{pmatrix}n_1\\ n_2\end{pmatrix}$. On the right is the 2D picture corresponding to the polynomial $2 + T_1 - T_1T_2 + T_2 - T_1^{-1} + T_1^{-1}T_2^{-1} - T_2^{-1}$.



Thus Θ becomes a pair of pictures: a bar code, and a 2D picture that we call a "hexagonal QR code". For the knots in the Rolfsen table (with the unknot prepended at the start), they are in Figure 1.1. In addition, the hexagonal QR codes of some 15 knots with $\geqslant 300$ crossings are in Figure 1.2.

Clearly there are patterns in Figures 1.1 and 1.2. There is a hexaginal symmetry and the QR codes are nearly always hexagons (these are independent properties). Much more can be seen in Figure 1.1. In Figure 1.2 there seem to be large-scale "sand table patterns" or "diffraction patterns". We can't prove any of these things, and the last one, we can't even formulate properly. Yet they are clearly there, too clear to be the result of chance alone.

We plan to have fun over the next few years observing and proving these patterns. We hope that others will join us too.

2. Formulas

2.1. **Old Formulas[1].** The setup leading to the definition of Θ is the same as the setup leading to the definition of the invariant ρ_1 of [BV1], and hence we copy a few relevant paragraphs from [BV1] nearly varbatim, with only a few examples removed.

[1] "Old" means that these formulas appeared already in [BV1].

FIGURE 1.1. Θ as a bar code and a hexagonal QR code, for all the knots in the Rolfsen table.

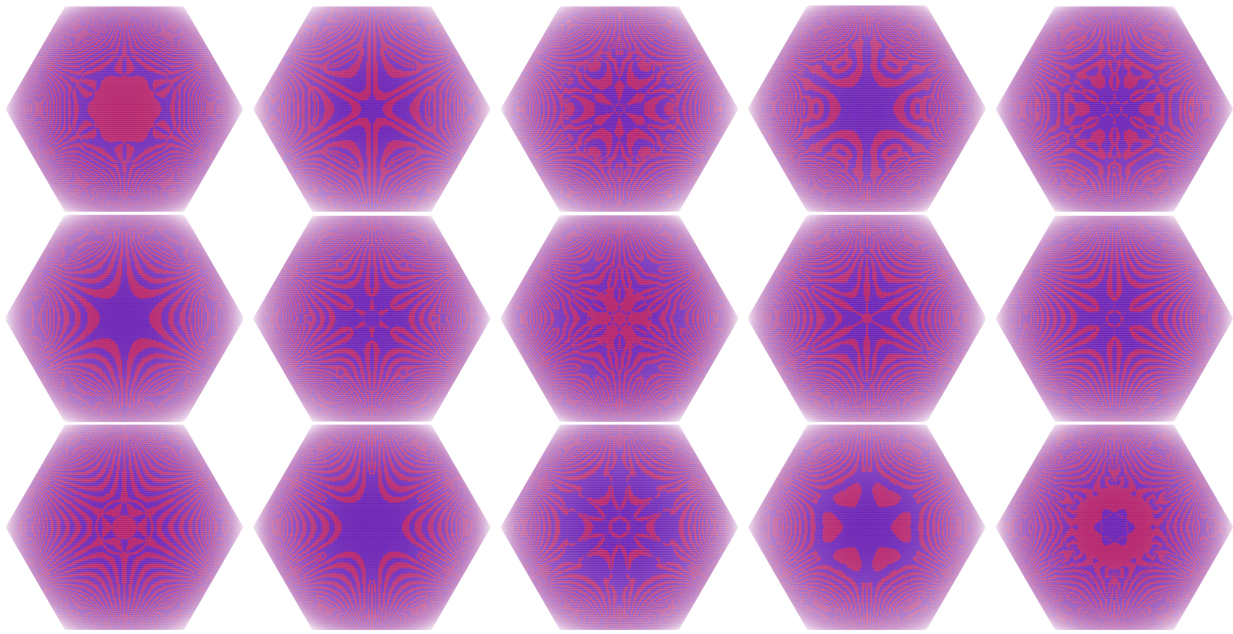
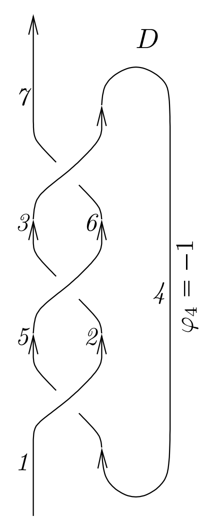


FIGURE 1.2. θ (hexagonal QR code only) of the 15 largest knots that we have computed by September 16, 2024. They are all "generic" in as much as we know, and they all have $\geqslant 300$ crossings. The knots come from [DHOEBL].



Given an oriented n-crossing knot K, we draw it in the plane as a long knot diagram D in such a way that the two strands intersecting at each crossing are pointing up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram.* An example of an upright knot diagram is shown on the right.

We then label each edge of the diagram with two integer labels: a running index k which runs from 1 to $2n+1$, and a "rotation number" φ_k, the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with $+1$ signs and caps with -1; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for φ_4, which is -1.

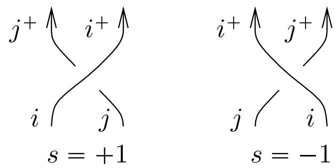
A Technicality. Some Reidemeister moves create or lose an edge and to avoid the need for renumbering it is beneficial to also allow labelling the edges with non-consecutive labels. Hence we allow that, and write i^+ for the successor of the label i along the knot, and i^{++} for the successor of i^+ (these are $i+1$ and $i+2$ if the labelling is by consecutive integers). Also, "1" will always refer to the label of the first edge, and "$2n+1$" will always refer to the label of the last.

We let A be the $(2n+1)\times(2n+1)$ matrix of Laurent polynomials in the formal variable T defined by

$$A := I - \sum_c \left(T^s E_{i,i^+} + (1-T^s)E_{i,j^+} + E_{j,j^+}\right),$$

where I is the identity matrix and $E_{\alpha\beta}$ denotes the elementary matrix with 1 in row α and column β and zeros elsewhere. The summation is over the crossings $c = (s, i, j)$ of the diagram D, and once c is chosen, s denotes its sign and i and j denote the labels below the crossing where the label i belongs to the over-strand and j to the under-strand.

Alternatively, $A = I + \sum_c A_c$, where A_c is a matrix of zeros except for the blocks as follows:



$$\longrightarrow \qquad \begin{array}{c|cc} A_c & \text{column } i^+ & \text{column } j^+ \\ \hline \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{array} \tag{1}$$

We note (as we did in [BV1]) that the determinant of A is equal up to a unit to the normalized Alexander polynomial Δ of K. In fact, we have that

$$\Delta = T^{(-\varphi(D) - w(D))/2} \det(A), \tag{2}$$

where $\varphi(D) := \sum_k \varphi_k$ is the total rotation number of D and where $w(D) = \sum_c s_c$ is the writhe of D, namely the sum of the signs s_c of all the crossings c in D.

We let $G = (g_{\alpha\beta}) = A^{-1}$ and, thinking of it as a function $g_{\alpha\beta}$ of a pair of edges α and β, we call it the Green function of the diagram D. When inspired by physics (e.g. [BN1]) we sometimes call it "the 2-point function", and when thinking of car traffic (e.g. [BN2]) we sometimes call it "the traffic function".

We note that the computation of G is the bottleneck in the computation of Θ. It requires inverting a $(2n+1) \times (2n+1)$ matrix whose entries that are (degree 1) Laurent polynomials in T. It's a daunting task yet it takes polynomial time, it can be performed in practice even if n is in the hundreds, and everything which then follows is easier.

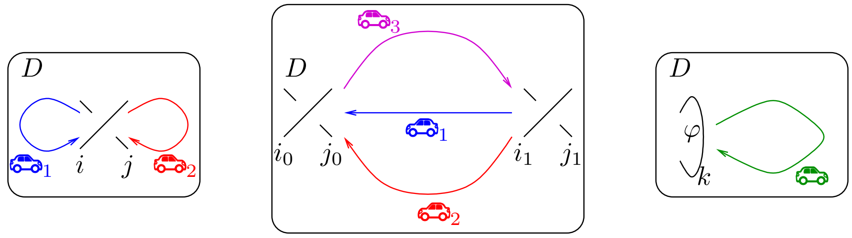
2.2. **New Formulas.** Let T_1 and T_2 be indeteminates and let $T_3 := T_1T_2$. Let $\Delta_\nu := \Delta / (T \to T_\nu)$ and $G_\nu = (g_{\nu\alpha\beta}) := G / (T \to T_\nu)$ be Δ and G affected by the substitution $T \to T_\nu$, where $\nu = 1, 2, 3$ (these are easy to compute once Δ and G have been computed).

The formulas for θ depend on three fixed polynomials $R_{11}(c)$, $R_{12}(c_0, c_1)$ and $\Gamma_1(\varphi, k)$ in the $g_{\nu\alpha\beta}$'s, which we admit, are rather ugly. So we prefer to assert their existance and postpone displaying them to a few paragraphs later.

Theorem 1 (Proof in Section 4)**.** *With $c = (s, i, j)$, $c_0 = (s_0, i_0, j_0)$, and $c_1 = (s_1, i_1, j_1)$ denoting crossings, there is a quadratic polynomial $R_{11}(c) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i, j\}]$ in the $g_{\nu\alpha\beta}$'s with coefficients in the ring of rational functions in T_1 and T_2 and with $\alpha, \beta \in \{i, j\}$, and similarly a cubic $R_{12}(c_0, c_1) \in \mathbb{Q}(T_1, T_2)[g_{\nu\alpha\beta} : \alpha, \beta \in \{i_0, j_0, i_1, j_1\}]$, and a linear $\Gamma_1(\varphi, k)$ such that the following is a knot invariant:*

$$\theta(D) := \Delta_1\Delta_2\Delta_3 \left(\sum_c R_{11}(c) + \sum_{c_0, c_1} R_{12}(c_0, c_1) + \sum_k \Gamma_1(\varphi_k, k) \right). \tag{3}$$

We note without detail that there is an alternative formula for θ in terms of perturbed Gaussian integration [BN1]. In that language, and using also the traffic motifs of [BV1, BN2], the three summands in (3) become Feynman diagrams for processes in which cars governed by parameter $T = T_1, T_2$, or T_3 interact:



In particular, the middle diagram which resembles the greek letter Θ gave the invariant its name.

We note also that computationaly, the worst term in (3) is the middle one, and even it takes merely $\sim n^2$ operations in the ring $\mathbb{Q}(T_1, T_2)$.

The polynomials $R_{11}(c)$, $R_{12}(c_0, c_1)$ and $\Gamma_1(\varphi, k)$ are not unique, and we are not certain that we have the cleanest possible formulas for them. As admitted, they are human-ugly. Yet from a computational perspective, having 18 terms (as is the case for $R_{11}(c)$) isn't really a problem; computers don't care. Anyway, here are the formulas:

$$\begin{aligned}
R_{11}(c) = s\,[1/2 &- g_{3ii} + T_2^s g_{1ii} g_{2ji} - T_2^s g_{3jj} g_{2ji} - (T_2^s - 1) g_{3ii} g_{2ji} \\
&+ (T_3^s - 1) g_{2ji} g_{3ji} - g_{1ii} g_{2jj} + 2 g_{3ii} g_{2jj} + g_{1ii} g_{3jj} - g_{2ii} g_{3jj}] \\
&+ \frac{s}{T_2^s - 1}\,[(T_1^s - 1) T_2^s \left(g_{3jj} g_{1ji} - g_{2jj} g_{1ji} + T_2^s g_{1ji} g_{2ji}\right) \\
&+ (T_3^s - 1)\left(g_{3ji} - T_2^s g_{1ii} g_{3ji} + g_{2ij} g_{3ji} + (T_2^s - 2) g_{2jj} g_{3ji}\right) \\
&- (T_1^s - 1)(T_2^s + 1)(T_3^s - 1) g_{1ji} g_{3ji}]
\end{aligned}$$

$$R_{12}(c_0, c_1) = \frac{s_1 (T_1^{s_0} - 1)(T_3^{s_1} - 1) g_{1j_1i_0} g_{3j_0i_1}}{T_2^{s_1} - 1} \left(T_2^{s_0} g_{2i_1i_0} + g_{2j_1j_0} - T_2^{s_0} g_{2j_1i_0} - g_{2i_1j_0}\right)$$

$$\Gamma_1(\varphi, k) = \varphi(-1/2 + g_{3kk})$$

3. Implementation and Examples

A concise yet reasonably efficient implementation is worth a thousand formulas. It completely removes ambiguities, it tests the theories, and it allows for experimentation. Hence our next task is to implement. The section that follows was generated from a Mathematica [Wo] notebook which is available at [BV2, Theta.nb].

We start by loading the package `KnotTheory‘` — it is only needed because it has many specific knots pre-defined:

```
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Next we quietly define the commands `Rot`, used to compute rotation numbers, and `PolyPlot`, used to plot polynomials as bar codes and as hexagonal QR codes. Neither is a part of the core of the computation of Θ, so neither is shown; yet we do show some usage examples.

```
(* Rot suppressed *)
```

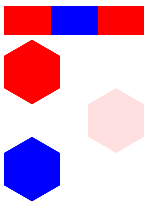
```
Rot[Mirror@Knot[3, 1]]
```

```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0}}
```

We urge the reader to compare the above output with the knot diagram in Section 2.1.

```
(* PolyPlot suppressed *)
```

```
PolyPlot[{T - 1 + T^-1, T_1 + 2 T_2 - 2 T_1^-1 T_2^-1}, ImageSize → Tiny]
```



```
T_3 = T_1 T_2;
```

```
R_11[{s_, i_, j_}] =
  s (1 / 2 - g_3ii + T_2^s g_1ii g_2ji - g_1ii g_2jj - (T_2^s - 1) g_2ji g_3ii + 2 g_2jj g_3ii - (1 - T_3^s) g_2ji g_3ji -
      g_2ii g_3jj - T_2^s g_2ji g_3jj + g_1ii g_3jj +
     ((T_1^s - 1) g_1ji (T_2^(2 s) g_2ji - T_2^s g_2jj + T_2^s g_3jj) +
        (T_3^s - 1) g_3ji (1 - T_2^s g_1ii - (T_1^s - 1) (T_2^s + 1) g_1ji + (T_2^s - 2) g_2jj + g_2ij)) / (T_2^s - 1));
```

```
R_12[{s0_, i0_, j0_}, {s1_, i1_, j1_}] =
  s1 (T_1^s0 - 1) (T_2^s1 - 1)^-1 (T_3^s1 - 1) g_1,j1,i0 g_3,j0,i1
   ((T_2^s0 g_2,i1,i0 - g_2,i1,j0) - (T_2^s0 g_2,j1,i0 - g_2,j1,j0));
```

```
Γ_1[φ_, k_] = -φ / 2 + φ g_3kk;
```

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                                0    -1     ))];
   Δ = T^((-Total[φ] - Total[Cs[[All,1]]]) / 2) Det[A];
   G = Inverse[A];
   ev[ℰ_] := Factor[ℰ /. g_ν_,α_,β_ :> (G[[α, β]] /. T → T_ν)];
   θ = ev[Σ_{k1=1}^n Σ_{k2=1}^n R_12[Cs[[k1]], Cs[[k2]]]];
   θ += ev[Σ_{k=1}^n R_11[Cs[[k]]]];
   θ += ev[Σ_{k=1}^{2 n} Γ_1[φ[[k]], k]];
   Factor@{Δ, (Δ /. T → T_1) (Δ /. T → T_2) (Δ /. T → T_3) θ}
  ];
```

4. Proof of Invariance

5. Strong and Meaningful

6. Conjectures and Dreams

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