```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
        Once[<< Theta.m]
        SetOptions[PolyPlot, ImageSize → Tiny];
        Clear[Θ]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= eq = T2 - 1 == s0 T2^((1-s0)/2) (T2^s0 - 1); Simplify[{eq /. s0 → 1, eq /. s0 → -1}]
```

```
Out[•]=
{True, True}
```

```
In[•]:= RandomVK[n_] := {
          Prepend[#, 2 RandomInteger[1] - 1] & /@
           Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
          Table[RandomInteger[{-1, 1}], 2 n + 1]
         };
```

```
In[•]:= RandomVK[5]
```

```
Out[•]=
{{{1, 7, 3}, {-1, 6, 2}, {1, 4, 9}, {-1, 10, 1}, {1, 5, 8}},
 {-1, -1, 0, 1, 0, -1, 1, 1, -1, 1, 0}}
```

```
In[•]:= CF[ℰ_] := Expand@Collect[ℰ, g__ | χ__, F] /. F → Factor@*PowerExpand;
```

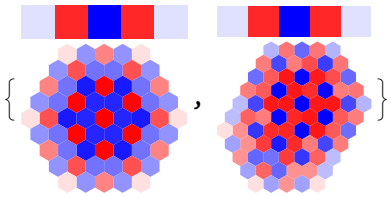
```
In[•]:= Short[Options[Θ] = {F1 → (F1i = F1[{s0, i0, j0}]),
          F2 → (F2i = F2[{s0, i0, j0}, {s1, i1, j1}]), F3 → (F3i = F3[φ, k])}]
```

```
Out[•]//Short=
{F1 → s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + <<19>> + <<1>>/<<1>> - s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0,
 F2 → <<1>>, F3 → -φ/2 + φ g<<1>>}
```

In[•]:=
```
Θ[K_, opts___Rule] := Module[{X, φφ, n, A, Δ, G, ev, θ, kk, k0, k1, f1, f2, f3},
  f1 = F1 /. {opts} /. Options[Θ];
  f2 = F2 /. {opts} /. Options[Θ];
  f3 = F3 /. {opts} /. Options[Θ];
  {X, φφ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  Cases[X, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  Δ = T^((-Total[φφ] - Total[X[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[
    ℰ /. {k_^+ :> k + 1, $ -> 2 n + 1} /. {g_ν_,α_,β_ :> (G[[α, β]] /. T -> T_ν), χ_True -> 1, χ_False -> 0}];
  θ = ev@Sum[f1 /. Thread[{s0, i0, j0} -> X[[kk]]] ⋃ Thread[{s1, i1, j1} -> X[[kk]]], {kk, n}];
  θ += ev@Sum[f2 /.
      Thread[{s0, i0, j0} -> X[[k0]]] ⋃ Thread[{s1, i1, j1} -> X[[k1]]], {k0, n}, {k1, n}];
  θ += ev@Sum[f3 /. {φ -> φφ[[kk]], k -> kk}, {kk, Length@φφ}];
  Factor@{Δ, (Δ /. T -> T_1) (Δ /. T -> T_2) (Δ /. T -> T_3) θ}
 ];
```

In[•]:=
```
PolyPlot /@ {Θ[Knot[7, 6]], Θ[Knot[7, 6], F3 -> 0]}
```

Out[•]=



In[•]:=
```
δ_i_,j_ := χ_i==j;
χ_True = 1; χ_False = 0;
χ_α_==β_ /; OrderedQ[{β, α}] := χ_β==α;
χ^p_ph_ /; p > 1 ^:= χ_ph;
χ_i0==$ = χ_j0==$ = χ_i1==$ = χ_j1==$ = 0;
χ_α_^+==1 = χ_1==α_^+ = 0;
χ_α_^+==β_^+ := χ_α==β;
χ_α_≤α_ = 1;
χ_α_^+≤α_ = 0;
χ_α_^+≤β_ := χ_α≤β - χ_α==β;
```

In[•]:=
```
bRules[{s_, i_, j_}] := { (* b for "push indices backwards" *)
   g_ν_,j^+,β_ :> g_ν,j,β - δ_j,β, g_ν_,i^+,β_ :> T_ν^-s g_ν,i,β + (1 - T_ν^-s) g_ν,j,β - T_ν^-s δ_i,β - (1 - T_ν^-s) δ_j,β,
   g_ν_,α_,i^+ :> T_ν^s g_ν,α,i + δ_α,i^+, g_ν_,α_,j^+ :> g_ν,α,j + (1 - T_ν^s) g_ν,α,i + δ_α,j^+
  };
bRules[X___List] := Union @@ Table[bRules[c], {c, {X}}]
```

In[•]:=
```
Expand@{{g_{ν,i,β}, g_{ν,j,β}} /. gRules[{s, i, j}],
  {g_{ν,i,β}, g_{ν,j,β}} /. gRules[{s, i, j}] /. bRules[{s, i, j}]}
```

Out[•]=
$$\{\{\chi_{i==\beta} + T^{s}_{\nu}\, g_{\nu,i^{+},\beta} + g_{\nu,j^{+},\beta} - T^{s}_{\nu}\, g_{\nu,j^{+},\beta},\ \chi_{j==\beta} + g_{\nu,j^{+},\beta}\},\ \{g_{\nu,i,\beta},\ g_{\nu,j,\beta}\}\}$$

In[•]:=
```
sRules[ℰ_] := FixedPoint[CF[# /. bRules[{s0, i0, j0}]] ⋃ bRules[{s1, i1, j1}] ⋃ {
        χ_{i0==j1} → 0, χ_{i1==j0} → 0, χ_{i0==j0} → 0,
        χ_{j0==j1} → χ_{i0==i1},
        χ_{i1^+≤i0} → χ_{i1≤i0} - χ_{i0==i1}, χ_{i0^+==i1^+} → χ_{i0==i1},
        χ_{i0==i1} χ_{i1≤i0} → χ_{i0==i1}, χ_{i0==i1} χ_{j1≤j0} → χ_{i0==i1},
        χ_{j0≤i0} → 1 - χ_{i0≤j0}, χ_{i1≤i0} → 1 - χ_{i0≤i1} + χ_{i1==i0},
        χ_{j1≤i0} → 1 - χ_{i0≤j1} + χ_{j1==i0}, χ_{i1≤i0} → 1 - χ_{i0≤i1} + χ_{i1==i0}, χ_{j1≤j0} → 1 - χ_{j0≤j1} + χ_{j1==j0},
        χ_{i0==i1} γ_ /; ! FreeQ[γ, s1 | i1 | j1] :> χ_{i0==i1} (γ /. {s1 → s0, i1 → i0, j1 → j0})
       }] &,
  ℰ]
```

In[•]:=
```
D_{s_,i_,j_}[ℰ_] :=
  CF[((ℰ /. # → i^+) + (ℰ /. # → j^+) - (ℰ /. # → i) - (ℰ /. # → j)) //. bRules[{s, i, j}]];
B[ℰ_] := CF[((ℰ /. # → $) - (ℰ /. # → 1)) /. {g_,$,β_ → χ_{$==β}, g_,_,$ → 1, g_,α_,1 → χ_{α==1}}];
```

In[•]:=
```
D_{s1,i1,j1}[g_{1,#,i0}]
```

Out[•]=
$$-T_1^{-s1}\,\chi_{i0==i1} - T_1^{-s1}\left(-1 + 2\,T_1^{s1}\right)\chi_{i0==j1} - T_1^{-s1}\left(-1 + T_1^{s1}\right) g_{1,i1,i0} + T_1^{-s1}\left(-1 + T_1^{s1}\right) g_{1,j1,i0}$$

In[•]:=
```
D_{s1,i1,j1}[g_{3,j0,#}]
```

Out[•]=
$$\chi_{j0==i1^{+}} + \chi_{j0==j1^{+}}$$

In[•]:=
```
B[g_{3,j0,#}]
```

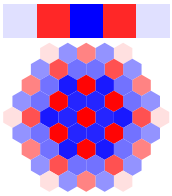
Out[•]=
$$-g_{3,j0,1} + g_{3,j0,\$}$$

In[•]:=
```
tw = g_{1,#,i0};
⊖[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\right.$

$\frac{1}{T_1^4\,T_2^4}\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-\right.$
$10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-$
$50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-$
$64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-$
$10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+7\,T_1^2\,T_2^6-10\,T_1^3\,T_2^6-64\,T_1^4\,T_2^6+98\,T_1^5\,T_2^6-64\,T_1^6\,T_2^6-10\,T_1^7\,T_2^6+7\,T_1^8\,T_2^6-5\,T_1^3\,T_2^7+$
$\left.\left.20\,T_1^4\,T_2^7-10\,T_1^5\,T_2^7-10\,T_1^6\,T_2^7+20\,T_1^7\,T_2^7-5\,T_1^8\,T_2^7+T_1^4\,T_2^8-5\,T_1^5\,T_2^8+7\,T_1^6\,T_2^8-5\,T_1^7\,T_2^8+T_1^8\,T_2^8\right)\right\}$

Out[•]=



In[•]:=
```
tw = g_{1,j0,#};
⊖[Knot[7, 6], F1 → -B[tw], F2 → D_{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
⊖[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot
```

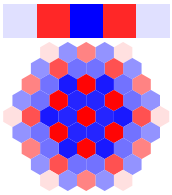
» $\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\,0\right\}$

Out[•]=

» $\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\right.$

$\frac{1}{T_1^4\,T_2^4}\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-\right.$
$10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-$
$50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-$
$64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-$
$10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+7\,T_1^2\,T_2^6-10\,T_1^3\,T_2^6-64\,T_1^4\,T_2^6+98\,T_1^5\,T_2^6-64\,T_1^6\,T_2^6-10\,T_1^7\,T_2^6+7\,T_1^8\,T_2^6-5\,T_1^3\,T_2^7+$
$\left.\left.20\,T_1^4\,T_2^7-10\,T_1^5\,T_2^7-10\,T_1^6\,T_2^7+20\,T_1^7\,T_2^7-5\,T_1^8\,T_2^7+T_1^4\,T_2^8-5\,T_1^5\,T_2^8+7\,T_1^6\,T_2^8-5\,T_1^7\,T_2^8+T_1^8\,T_2^8\right)\right\}$

Out[•]=



```
In[•]:= bas0 = List @@ Expand[(g1,i1⁺,# + g1,j1⁺,#) (g2,i1⁺,# + g2,j1⁺,#) (g3,#,i1 + g3,#,j1)]
```

```
Out[•]=
{g1,i1⁺,#1 g2,i1⁺,#1 g3,#1,i1, g1,j1⁺,#1 g2,i1⁺,#1 g3,#1,i1, g1,i1⁺,#1 g2,j1⁺,#1 g3,#1,i1, g1,j1⁺,#1 g2,j1⁺,#1 g3,#1,i1,
 g1,i1⁺,#1 g2,i1⁺,#1 g3,#1,j1, g1,j1⁺,#1 g2,i1⁺,#1 g3,#1,j1, g1,i1⁺,#1 g2,j1⁺,#1 g3,#1,j1, g1,j1⁺,#1 g2,j1⁺,#1 g3,#1,j1}
```

```
In[•]:= B /@ bas0
```

```
Out[•]=
{0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= Column[(# → ⊖[Knot[7, 6], F1 → -B[#], F2 → D{s0,i0,j0}[#], F3 → 0][[2]]) & /@ bas0]
```

```
Out[•]=
g1,i1⁺,#1 g2,i1⁺,#1 g3,#1,i1 → 0
g1,j1⁺,#1 g2,i1⁺,#1 g3,#1,i1 → 0
g1,i1⁺,#1 g2,j1⁺,#1 g3,#1,i1 → 0
g1,j1⁺,#1 g2,j1⁺,#1 g3,#1,i1 → 0
g1,i1⁺,#1 g2,i1⁺,#1 g3,#1,j1 → 0
g1,j1⁺,#1 g2,i1⁺,#1 g3,#1,j1 → 0
g1,i1⁺,#1 g2,j1⁺,#1 g3,#1,j1 → 0
g1,j1⁺,#1 g2,j1⁺,#1 g3,#1,j1 → 0
```

```
In[•]:= bas1 = List @@ Expand[(g1,#,i0 + g1,#,j0) (g2,#,i0 + g2,#,j0) (g3,i0⁺,# + g3,j0⁺,#)]
```

```
Out[•]=
{g1,#1,i0 g2,#1,i0 g3,i0⁺,#1, g1,#1,j0 g2,#1,i0 g3,i0⁺,#1, g1,#1,i0 g2,#1,j0 g3,i0⁺,#1, g1,#1,j0 g2,#1,j0 g3,i0⁺,#1,
 g1,#1,i0 g2,#1,i0 g3,j0⁺,#1, g1,#1,j0 g2,#1,i0 g3,j0⁺,#1, g1,#1,i0 g2,#1,j0 g3,j0⁺,#1, g1,#1,j0 g2,#1,j0 g3,j0⁺,#1}
```

```
In[•]:= B /@ bas1
```

```
Out[•]=
{0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= Column[(# → ⊖[Knot[7, 6], F1 → -B[#], F2 → D{s1,i1,j1}[#], F3 → 0][[2]]) & /@ bas1]
```

```
Out[•]=
g1,#1,i0 g2,#1,i0 g3,i0⁺,#1 → 0
g1,#1,j0 g2,#1,i0 g3,i0⁺,#1 → 0
g1,#1,i0 g2,#1,j0 g3,i0⁺,#1 → 0
g1,#1,j0 g2,#1,j0 g3,i0⁺,#1 → 0
g1,#1,i0 g2,#1,i0 g3,j0⁺,#1 → 0
g1,#1,j0 g2,#1,i0 g3,j0⁺,#1 → 0
g1,#1,i0 g2,#1,j0 g3,j0⁺,#1 → 0
g1,#1,j0 g2,#1,j0 g3,j0⁺,#1 → 0
```

```
In[•]:= Column[(# → sRules@D{s0,i0,j0}[#]) & /@ bas0]
```

```
Out[•]=
g1,i1⁺,#1 g2,i1⁺,#1 g3,#1,i1 → -T1^-s0 T2^-s0 χ[i0==i1] g1,i0,i0 g2,i0,i0 - T1^-s0 (-1 + T1^s0) T2^-s0 χ[i0==i1] g1,j0,i0 g2,i0,i0 -
  T1^-s0 T2^-s0 (-1 + T2^s0) χ[i0==i1] g1,i0,i0 g2,j0,i0 - T1^-s0 (-1 + T1^s0) T2^-s0 (-1 + T2^s0) χ[i0==i1] g1,j0,i0 g2,j0,i0 -
  T1^-s0 T2^-s0 χ[i0==i1] g3,i0,i0 + T1^-s0 T2^-s0 χ[i0==i1] g1,i0,i0 g3,i0,i0 +
  T1^-s0 (-1 + T1^s0) T2^-s0 χ[i0==i1] g1,j0,i0 g3,i0,i0 + T1^-s0 T2^-s0 χ[i0==i1] g2,i0,i0 g3,i0,i0 +
  T1^-s0 T2^-s0 (-1 + T2^s0) χ[i0==i1] g2,j0,i0 g3,i0,i0 - T1^-s0 (-1 + T1^s0) T2^-s0 (-1 + T2^s0) χ[i0==i1] g3,j0,i0 +
  T1^-s0 T2^-s0 (-1 + T2^s0) χ[i0==i1] g1,i0,j0 g3,j0,i0 + T1^-s0 (-1 + T1^s0) T2^-s0 (-1 + T2^s0) χ[i0==i1] g1,j0,j0 g3,j0,i0 +
  T1^-s0 (-1 + T1^s0) T2^-s0 χ[i0==i1] g2,i0,j0 g3,j0,i0 + T1^-s0 (-1 + T1^s0) T2^-s0 (-1 + T2^s0) χ[i0==i1] g2,j0,j0 g3,j0,i0 +
  T1^-s1 T2^-s1 (-T1^s0 - T2^s0 + 2 T1^s0 T2^s0) g1,i1,i0 g2,i1,i0 g3,j0,i1 - T1^-s1 T2^-s1 (-1 + T2^s0) g1,i1,j0 g2,i1,i0 g3,j0,i1 +
  T1^-s1 (-1 + T1^s1) T2^-s1 (-T1^s0 - T2^s0 + 2 T1^s0 T2^s0) g1,j1,i0 g2,i1,i0 g3,j0,i1 -
  T1^-s1 (-1 + T1^s1) T2^-s1 (-1 + T2^s0) g1,j1,j0 g2,i1,i0 g3,j0,i1 -
```

$$g_{1,j1^+,\#1}\,g_{2,i1^+,\#1}\,g_{3,\#1,i1} \to$$
$$-T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,i0,i0} - T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0} + T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,i0} -$$
$$T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} + T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} + T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0} +$$
$$T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} + T_2^{-s1}\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1} -$$
$$T_2^{-s1}\left(-1+T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1} - \left(-1+T_1^{s0}\right)T_2^{-s1}\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1} +$$
$$T_2^{-s1}\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} -$$
$$T_2^{-s1}\left(-1+T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} - \left(-1+T_1^{s0}\right)T_2^{-s1}\left(-1+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}$$

$$g_{1,i1^+,\#1}\,g_{2,j1^+,\#1}\,g_{3,\#1,i1} \to$$
$$-T_1^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,j0,i0} - T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0} + T_1^{-s0}\,\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,i0} -$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} + T_1^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0} + T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} + T_1^{-s1}\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} -$$
$$T_1^{-s1}\left(-1+T_2^{s0}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} + T_1^{-s1}\left(-1+T_1^{s1}\right)\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} -$$
$$T_1^{-s1}\left(-1+T_1^{s1}\right)\left(-1+T_2^{s0}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} -$$
$$T_1^{-s1}\left(-1+T_1^{s0}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} - T_1^{-s1}\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}$$

$$g_{1,j1^+,\#1}\,g_{2,j1^+,\#1}\,g_{3,\#1,i1} \to -\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0} - \chi_{i0==i1}\,g_{3,j0,i0} +$$
$$\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} + \chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} + \left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} +$$
$$\left(1-T_2^{s0}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} + \left(1-T_1^{s0}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}$$

$$
\begin{aligned}
&g_{1,i1^+,\#1}\, g_{2,i1^+,\#1}\, g_{3,\#1,j1} \to \\
&\quad -T_1^{-s0}\, T_2^{-s0} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,i0,i0} + T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,i0,i0} - \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,i0,i0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,i0,i0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,i0,j0} - T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,i0,j0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right)^2 T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,i0,j0} - T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,i0,j0} - \\
&\quad T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right) \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,j0,i0} + \\
&\quad T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right)^2 \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,j0,i0} - \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,i0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right)^2 \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,i0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,i0}\, g_{2,j0,j0} - T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,j0}\, g_{2,j0,j0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right)^2 T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,j0} - \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,j0} - T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{3,i0,j0} + \\
&\quad T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{1,i0,i0}\, g_{3,i0,j0} + T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,i0}\, g_{3,i0,j0} + \\
&\quad T_1^{-s0}\, T_2^{-s0}\, \chi_{i0==i1}\, g_{2,i0,i0}\, g_{3,i0,j0} + T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{2,j0,i0}\, g_{3,i0,j0} - \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{3,j0,j0} + T_1^{-s0}\, T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,i0,j0}\, g_{3,j0,j0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{3,j0,j0} + \\
&\quad T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0}\, \chi_{i0==i1}\, g_{2,i0,j0}\, g_{3,j0,j0} + T_1^{-s0} \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{2,j0,j0}\, g_{3,j0,j0} + \\
&\quad T_1^{-s1}\, T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) g_{1,i1,i0}\, g_{2,i1,i0}\, g_{3,j0,j1} - T_1^{-s1}\, T_2^{-s1} \left(-1 + T_2^{s0}\right) g_{1,i1,j0}\, g_{2,i1,i0}\, g_{3,j0,j1} + \\
&\quad T_1^{-s1} \left(-1 + T_1^{s1}\right) T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1} \left(-1 + T_1^{s1}\right) T_2^{-s1} \left(-1 + T_2^{s0}\right) g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1} \left(-1 + T_1^{s0}\right) T_2^{-s1}\, g_{1,i1,i0}\, g_{2,i1,j0}\, g_{3,j0,j1} - T_1^{-s1} \left(-1 + T_1^{s0}\right) \left(-1 + T_1^{s1}\right) T_2^{-s1}\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,j1} + \\
&\quad T_1^{-s1}\, T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,i1,i0}\, g_{2,j1,i0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1}\, T_2^{-s1} \left(-1 + T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,i1,j0}\, g_{2,j1,i0}\, g_{3,j0,j1} + \\
&\quad T_1^{-s1} \left(-1 + T_1^{s1}\right) T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1} \left(-1 + T_1^{s1}\right) T_2^{-s1} \left(-1 + T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1} \left(-1 + T_1^{s0}\right) T_2^{-s1} \left(-1 + T_2^{s1}\right) g_{1,i1,i0}\, g_{2,j1,j0}\, g_{3,j0,j1} - \\
&\quad T_1^{-s1} \left(-1 + T_1^{s0}\right) \left(-1 + T_1^{s1}\right) T_2^{-s1} \left(-1 + T_2^{s1}\right) g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,j1}
\end{aligned}
$$

$$
\begin{aligned}
&g_{1,j1^+,\#1}\, g_{2,i1^+,\#1}\, g_{3,\#1,j1} \to \\
&\quad -T_2^{-s0} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,i0,i0} + T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,i0,i0} + \\
&\quad \left(-1 + T_1^{s0}\right) T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,i0,j0} - T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,i0,j0} - \\
&\quad T_2^{-s0} \left(-1 + T_2^{s0}\right) \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,i0} + T_2^{-s0} \left(-1 + T_2^{s0}\right)^2 \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,i0} + \\
&\quad \left(-1 + T_1^{s0}\right) T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,i0}\, g_{2,j0,j0} - T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{2,j0,j0} + \\
&\quad T_2^{-s0}\, \chi_{i0==i1}\, g_{1,j0,i0}\, g_{3,i0,j0} - T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{3,j0,j0} + T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{1,j0,j0}\, g_{3,j0,j0} + \\
&\quad T_2^{-s0}\, \chi_{i0==i1}\, g_{2,j0,i0}\, g_{3,j0,j0} + T_2^{-s0} \left(-1 + T_2^{s0}\right) \chi_{i0==i1}\, g_{2,j0,j0}\, g_{3,j0,j0} + \\
&\quad T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) g_{1,j1,i0}\, g_{2,i1,i0}\, g_{3,j0,j1} - T_2^{-s1} \left(-1 + T_2^{s0}\right) g_{1,j1,j0}\, g_{2,i1,i0}\, g_{3,j0,j1} - \\
&\quad \left(-1 + T_1^{s0}\right) T_2^{-s1}\, g_{1,j1,i0}\, g_{2,i1,j0}\, g_{3,j0,j1} + T_2^{-s1} \left(-T_1^{s0} - T_2^{s0} + 2\, T_1^{s0}\, T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,j1,i0}\, g_{2,j1,i0}\, g_{3,j0,j1} - \\
&\quad T_2^{-s1} \left(-1 + T_2^{s0}\right) \left(-1 + T_2^{s1}\right) g_{1,j1,j0}\, g_{2,j1,i0}\, g_{3,j0,j1} - \left(-1 + T_1^{s0}\right) T_2^{-s1} \left(-1 + T_2^{s1}\right) g_{1,j1,i0}\, g_{2,j1,j0}\, g_{3,j0,j1}
\end{aligned}
$$

$$g_{1,i1^+,\#1}\,g_{2,j1^+,\#1}\,g_{3,\#1,j1} \to -T_1^{-s0}\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,j0,i0}+ T_1^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,j0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}+ T_1^{-s0}\left(-1+T_1^{s0}\right)\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}+T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,j0,j0}- T_1^{-s0}\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,j0,j0}+T_1^{-s0}\left(-1+T_1^{s0}\right)^2\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0}- T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0}+T_1^{-s0}\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,j0}- T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,j0}+T_1^{-s0}\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,j0}+T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0}+ T_1^{-s0}\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}+T_1^{-s1}\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,j1}- T_1^{-s1}\left(-1+T_2^{s0}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,j1}+T_1^{-s1}\left(-1+T_1^{s1}\right)\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,j1}- T_1^{-s1}\left(-1+T_1^{s1}\right)\left(-1+T_2^{s0}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,j1}- T_1^{-s1}\left(-1+T_1^{s0}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,j1}-T_1^{-s1}\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,j1}$$

$$g_{1,j1^+,\#1}\,g_{2,j1^+,\#1}\,g_{3,\#1,j1} \to \left(T_1^{s0}+T_2^{s0}-2\,T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}+\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}+ \left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0}-\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0}-\chi_{i0==i1}\,g_{3,j0,j0}+ \chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0}+\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}+\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,j1}+ \left(1-T_2^{s0}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,j1}+\left(1-T_1^{s0}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,j1}$$

In[•]:=
```
Column[(# → sRules@D_{s1,i1,j1}[#]) & /@ bas1]
```

Out[•]=

$$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,i0,i0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}+ T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,i0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,i0}- T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,i0}+ T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}- T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+ T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+ T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}$$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,i0,i0}+T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}+ T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,i0,i0}- T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,i0,i0}+ T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,i0}+T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,i0,i1}+ T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}-T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}- T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,i0,j0}+T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}-T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}- T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}+ T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}+ T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+ T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}- T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,i0}\,g_{2,i0,j0} + T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,i0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,i0,i0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,i0} + T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,i0,i1} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1} - T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1} -$$
$$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,i0,j0} + T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0}\,g_{3,j0,j0}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{2,i0,j0} + T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,i0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,i0} - T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\,\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,i0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,i0} + T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,i0,i1} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1} - T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1} +$$
$$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{3,i0,j0} - T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,i0,j0} -$$
$$T_1^{-s0}\,T_2^{-s0}\,\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,i0,j0} - T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} -$$
$$T_1^{-s0}\left(-1+T_1^{s0}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-2+T_1^{s0}+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} +$$
$$T_1^{-s0}\left(-1+T_1^{s1}\right)T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1} +$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} + T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,j0} -$$
$$T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0} - T_1^{-s0}\,T_2^{-s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}$$

$$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to$$
$$\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0} + \chi_{i0==i1}\,g_{3,j0,i0} - \chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0} + \left(1-T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0} -$$
$$\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0} + \left(1-T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0} + \left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1} +$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} + \left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}$$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to \chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0} + \left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} -$$
$$\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0} + \left(1-T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} + \left(1-T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0} +$$
$$\left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0} + \left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1} +$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} + \left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1} - \chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to \chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,j0} + \left(-1+T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} +$$
$$\left(1-T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0} + \left(-2+T_1^{s0}+T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0} -$$
$$\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0} + \left(1-T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} + \left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1} +$$
$$\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1} - \chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,j0}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to \chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,j0} + \left(2 - T_1^{s0} - T_2^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0} + \left(1 - T_2^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0} + \left(-2 + T_1^{s0} + T_2^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0} + \left(1 - T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,j0}\,g_{3,j0,i0} + \left(-2 + T_1^{s0} + T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,i0} + \left(-1 + T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1} + \left(-1 + T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \left(2 - T_1^{s1} - T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1} + \chi_{i0==i1}\,g_{3,j0,j0} - \chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,j0} - \chi_{i0==i1}\,g_{2,j0,j0}\,g_{3,j0,j0}$$

In[•]:=

```
tw0 = Table[a_i, {i, 8}].bas0 / (T_2 - 1);
tw1 = Table[b_i, {i, 8}].bas1 / (T_2 - 1);
{D_{s0,i0,j0}[tw0], D_{s1,i1,j1}[tw1]}
Θ[Knot[7, 6], F1 → 0, F2 → D_{s0,i0,j0}[tw0] + D_{s1,i1,j1}[tw1], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 6], F2 → F2i + D_{s0,i0,j0}[(T_2 - 1) tw0] + D_{s1,i1,j1}[(T_2 - 1) tw1]] // Echo // PolyPlot
```

Out[•]=

$$\left\{\langle\!\langle 305 \rangle\!\rangle + \frac{\langle\!\langle 1 \rangle\!\rangle}{\langle\!\langle 1 \rangle\!\rangle} - \frac{a_7\left(-1+T_2^{s0}\right) g_{\langle\!\langle 1 \rangle\!\rangle}\, g_{2,\langle\!\langle 1 \rangle\!\rangle}\, g_{\langle\!\langle 1 \rangle\!\rangle}\, g_{3,j0,j1}}{-1+T_2} + \frac{a_8\left(-T_1^{s0}-T_2^{s0}+2\,T_1^{s0}\,T_2^{s0}\right) g_{1,j1^+,i0}\, g_{2,j1^+,i0}\, g_{3,j0,j1}}{-1+T_2} - \frac{a_8\left(-1+T_2^{s0}\right) g_{1,j1^+,j0}\, g_{2,j1^+,i0}\, g_{3,j0,j1}}{-1+T_2} + \frac{a_7\,\chi_{i1==j0}\, g_{2,j1^+,j0}\, g_{3,j0,j1}}{-1+T_2} + \frac{a_8\,\chi_{j0==j1}\, g_{2,j1^+,j0}\, g_{3,j0,j1}}{-1+T_2} - \frac{a_7\left(-1+T_1^{s0}\right) g_{1,i1^+,i0}\, g_{2,j1^+,j0}\, g_{3,j0,j1}}{-1+T_2} - \frac{a_8\left(-1+T_1^{s0}\right) g_{1,j1^+,i0}\, g_{2,j1^+,j0}\, g_{3,j0,j1}}{-1+T_2},$$

$$\frac{b_1\,T_1^{s1}\,T_2^{s1}\,\chi_{i0==i1}}{-1+T_2} + \frac{b_1\,\chi_{i0==j1}}{-1+T_2} + \frac{b_1\,T_1^{-s1}\,T_2^{-s1}\left(-1+T_1^{s1}\,T_2^{s1}\right)\chi_{i0==i1}\,\chi_{i0==j1}}{-1+T_2} + \frac{b_8\,T_1^{\langle\!\langle 1 \rangle\!\rangle}\,T_2^{\langle\!\langle 1 \rangle\!\rangle}\,\chi_{i1==j0}}{-1+T_2} + \langle\!\langle 394 \rangle\!\rangle\right\}$$

Full expression not available (original memory size: 487.4 kB)

KnotTheory: Loading precomputed data in PD4Knots`.

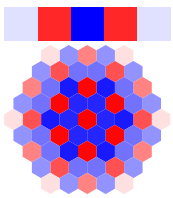
» $$\left\{-\frac{1 - 5\,T + 7\,T^2 - 5\,T^3 + T^4}{T^2},\ 0\right\}$$

Out[•]=

» $$\left\{-\frac{1 - 5\,T + 7\,T^2 - 5\,T^3 + T^4}{T^2},\right.$$

$$\frac{1}{T_1^4\,T_2^4}\left(1 - 5\,T_1 + 7\,T_1^2 - 5\,T_1^3 + T_1^4 - 5\,T_2 + 20\,T_1\,T_2 - 10\,T_1^2\,T_2 - 10\,T_1^3\,T_2 + 20\,T_1^4\,T_2 - 5\,T_1^5\,T_2 + 7\,T_2^2 - 10\,T_1\,T_2^2 - 64\,T_1^2\,T_2^2 + 98\,T_1^3\,T_2^2 - 64\,T_1^4\,T_2^2 - 10\,T_1^5\,T_2^2 + 7\,T_1^6\,T_2^2 - 5\,T_2^3 - 10\,T_1\,T_2^3 + 98\,T_1^2\,T_2^3 - 50\,T_1^3\,T_2^3 - 50\,T_1^4\,T_2^3 + 98\,T_1^5\,T_2^3 - 10\,T_1^6\,T_2^3 - 5\,T_1^7\,T_2^3 + T_2^4 + 20\,T_1\,T_2^4 - 64\,T_1^2\,T_2^4 - 50\,T_1^3\,T_2^4 + 108\,T_1^4\,T_2^4 - 50\,T_1^5\,T_2^4 - 64\,T_1^6\,T_2^4 + 20\,T_1^7\,T_2^4 + T_1^8\,T_2^4 - 5\,T_1\,T_2^5 - 10\,T_1^2\,T_2^5 + 98\,T_1^3\,T_2^5 - 50\,T_1^4\,T_2^5 - 50\,T_1^5\,T_2^5 + 98\,T_1^6\,T_2^5 - 10\,T_1^7\,T_2^5 - 5\,T_1^8\,T_2^5 + 7\,T_2^2\,T_2^6 - 10\,T_1^3\,T_2^6 - 64\,T_1^4\,T_2^6 + 98\,T_1^5\,T_2^6 - 64\,T_1^6\,T_2^6 - 10\,T_1^7\,T_2^6 + 7\,T_1^8\,T_2^6 - 5\,T_1^3\,T_2^7 + 20\,T_1^4\,T_2^7 - 10\,T_1^5\,T_2^7 - 10\,T_1^6\,T_2^7 + 20\,T_1^7\,T_2^7 - 5\,T_1^8\,T_2^7 + T_1^4\,T_2^8 - 5\,T_1^5\,T_2^8 + 7\,T_1^6\,T_2^8 - 5\,T_1^7\,T_2^8 + T_1^8\,T_2^8\right)\Bigg\}$$

Out[•]=



```
In[•]:= res = sRules[Residue[CF[χ_{i0==i1} F1i + F2i + D_{{s0,i0,j0}}[tw0] + D_{{s1,i1,j1}}[tw1]], {T_2, 1}] /.
          {g_{3,α_,β_} :> g_{1,α,β}, g_{2,α_,β_} :> χ_{α≤β}} ];

In[•]:= sRules[res /. {b_5 → 1 - T_1^{s0}, b_7 → T_1^{s0} - 1} /. (a | b)_ → 0]

Out[•]=
0

In[•]:= vars = Union@Cases[res, a__ | b__, ∞]

Out[•]=
{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8}

In[•]:= cvs = Union@Cases[res, g__ | χ__, ∞]

Out[•]=
{χ_{i0==i1}, χ_{i0≤i1}, χ_{i0≤j0}, χ_{i0≤j1}, χ_{i1≤j0}, χ_{j0≤j1}, g_{1,i0,i0}, g_{1,i0,i1},
 g_{1,i0,j0}, g_{1,i1,i0}, g_{1,j0,i0}, g_{1,j0,i1}, g_{1,j0,j0}, g_{1,j0,j1}, g_{1,j1,i0}, g_{1,j1,j0}}
```

```
In[•]:= eqns = Sort[CoefficientRules[res, cvs] /. (_ → c_) :> (c == 0)]
```

Out[•]=

$$\Big\{a_5 - a_7 - a_5\,T_1^{-s0} + a_7\,T_1^{-s0} == 0,\ a_1\,T_1^{-s0} - b_1\,T_1^{-s0} == 0,\ a_5\,T_1^{-s0} - b_2\,T_1^{-s0} == 0,$$
$$-a_5 + b_2 - a_1\,T_1^{-s0} + a_5\,T_1^{-s0} + b_1\,T_1^{-s0} - b_2\,T_1^{-s0} == 0,\ a_5 + a_6 - b_2 - b_6 + a_3\,T_1^{-s0} - a_5\,T_1^{-s0} + b_2\,T_1^{-s0} - b_3\,T_1^{-s0} == 0,$$
$$a_7\,T_1^{-s0} - b_4\,T_1^{-s0} == 0,\ -a_7 - a_8 + b_4 + b_8 + a_7\,T_1^{-s0} - b_4\,T_1^{-s0} == 0,\ a_7 + a_8 - b_4 - b_8 - a_7\,T_1^{-s0} + b_4\,T_1^{-s0} == 0,$$
$$-a_5\,T_1^{-s0} - a_7\,T_1^{-s0} + b_2\,T_1^{-s0} + b_4\,T_1^{-s0} == 0,\ a_5 + a_7 - b_2 - b_4 - a_5\,T_1^{-s0} - a_7\,T_1^{-s0} + b_2\,T_1^{-s0} + b_4\,T_1^{-s0} == 0,$$
$$1 + 2\,a_1 + a_2 - a_3 - b_1 - b_5 - 2\,a_1\,T_1^{-s0} + a_3\,T_1^{-s0} + b_1\,T_1^{-s0} - T_1^{-s0} == 0,$$
$$-2\,a_1 - a_2 + 2\,a_3 + a_4 + a_1\,T_1^{-s0} - a_3\,T_1^{-s0} + a_1\,T_1^{s0} + a_2\,T_1^{s0} - a_3\,T_1^{s0} - a_4\,T_1^{s0} == 0,$$
$$-1 + a_3 + a_4 - 2\,a_5 - a_6 + 2\,a_7 + a_8 - b_3 - b_7 - a_3\,T_1^{-s0} + a_5\,T_1^{-s0} - a_7\,T_1^{-s0} + b_3\,T_1^{-s0} + T_1^{s0} + a_5\,T_1^{s0} +$$
$$a_6\,T_1^{s0} - a_7\,T_1^{s0} - a_8\,T_1^{s0} == 0,\ -a_1 - a_2 - a_3 - a_4 + 2\,a_5 + a_6 + b_1 - 2\,b_2 + b_3 + b_5 - b_6 + b_7 +$$
$$a_1\,T_1^{-s0} + a_3\,T_1^{-s0} - a_5\,T_1^{-s0} - b_1\,T_1^{-s0} + b_2\,T_1^{-s0} - b_3\,T_1^{-s0} - a_5\,T_1^{s0} - a_6\,T_1^{s0} + b_2\,T_1^{s0} + b_6\,T_1^{s0} == 0,$$
$$a_1 + a_2 + a_3 + a_4 - 2\,a_5 - a_6 - 2\,a_7 - a_8 - b_1 + 2\,b_2 - b_3 + 2\,b_4 - b_5 + b_6 - b_7 + b_8 - a_1\,T_1^{-s0} -$$
$$a_3\,T_1^{-s0} + a_5\,T_1^{-s0} + a_7\,T_1^{-s0} + b_1\,T_1^{-s0} - b_2\,T_1^{-s0} + b_3\,T_1^{-s0} - b_4\,T_1^{-s0} + a_5\,T_1^{s0} + a_6\,T_1^{s0} +$$
$$a_7\,T_1^{s0} + a_8\,T_1^{s0} - b_2\,T_1^{s0} - b_4\,T_1^{s0} - b_6\,T_1^{s0} - b_8\,T_1^{s0} == 0,\ a_1\,T_1^{s0-s1} - a_1\,T_1^{-s1} == 0,$$
$$-a_1\,T_1^{s0-s1} + a_1\,T_1^{-s1} == 0,\ -a_1\,T_1^{s0-s1} + a_1\,T_1^{-s1} == 0,\ a_3\,T_1^{s0-s1} - a_3\,T_1^{-s1} == 0,\ -a_3\,T_1^{s0-s1} + a_3\,T_1^{-s1} == 0,$$
$$a_5\,T_1^{s0-s1} - a_5\,T_1^{-s1} == 0,\ a_5 + a_6 - a_5\,T_1^{s0} - a_6\,T_1^{s0} + a_5\,T_1^{s0-s1} - a_5\,T_1^{-s1} == 0,$$
$$a_5 + a_6 - a_5\,T_1^{s0} - a_6\,T_1^{s0} + a_5\,T_1^{s0-s1} - a_5\,T_1^{-s1} == 0,\ -a_5\,T_1^{s0-s1} + a_5\,T_1^{-s1} == 0,$$
$$-a_5\,T_1^{s0-s1} + a_5\,T_1^{-s1} == 0,\ -a_5 - a_6 + a_5\,T_1^{s0} + a_6\,T_1^{s0} - a_5\,T_1^{s0-s1} + a_5\,T_1^{-s1} == 0,$$
$$a_7\,T_1^{s0-s1} - a_7\,T_1^{-s1} == 0,\ a_7 + a_8 - a_7\,T_1^{s0} - a_8\,T_1^{s0} + a_7\,T_1^{s0-s1} - a_7\,T_1^{-s1} == 0,\ -a_7\,T_1^{s0-s1} + a_7\,T_1^{-s1} == 0,$$
$$-a_7 - a_8 + a_7\,T_1^{s0} + a_8\,T_1^{s0} - a_7\,T_1^{s0-s1} + a_7\,T_1^{-s1} == 0,\ b_1\,T_1^{-s0} - b_1\,T_1^{-s0+s1} == 0,\ -b_1\,T_1^{-s0} + b_1\,T_1^{-s0+s1} == 0,$$
$$b_2\,T_1^{-s0} - b_2\,T_1^{-s0+s1} == 0,\ -b_2 - b_6 + b_2\,T_1^{-s0} + b_2\,T_1^{s1} + b_6\,T_1^{s1} - b_2\,T_1^{-s0+s1} == 0,$$
$$-b_2\,T_1^{-s0} + b_2\,T_1^{-s0+s1} == 0,\ b_2 + b_6 - b_2\,T_1^{-s0} - b_2\,T_1^{s1} - b_6\,T_1^{s1} + b_2\,T_1^{-s0+s1} == 0,$$
$$b_3\,T_1^{-s0} - b_3\,T_1^{-s0+s1} == 0,\ -b_3\,T_1^{-s0} + b_3\,T_1^{-s0+s1} == 0,\ -b_3\,T_1^{-s0} + b_3\,T_1^{-s0+s1} == 0,$$
$$b_4\,T_1^{-s0} - b_4\,T_1^{-s0+s1} == 0,\ -b_4 - b_8 + b_4\,T_1^{-s0} + b_4\,T_1^{s1} + b_8\,T_1^{s1} - b_4\,T_1^{-s0+s1} == 0,$$
$$-b_4 - b_8 + b_4\,T_1^{-s0} + b_4\,T_1^{s1} + b_8\,T_1^{s1} - b_4\,T_1^{-s0+s1} == 0,\ -b_4\,T_1^{-s0} + b_4\,T_1^{-s0+s1} == 0,$$
$$-b_4\,T_1^{s0} + b_4\,T_1^{s0+s1} == 0,\ b_4 + b_8 - b_4\,T_1^{-s0} - b_4\,T_1^{s1} - b_8\,T_1^{s1} + b_4\,T_1^{s0+s1} == 0,$$
$$-1 + a_1 + a_2 + b_1 + b_5 - b_1\,T_1^{-s0} + T_1^{s0} - a_1\,T_1^{s0} - a_2\,T_1^{s0} + a_1\,T_1^{s0-s1} - a_1\,T_1^{-s1} +$$
$$T_1^{s1} - b_1\,T_1^{s1} - b_5\,T_1^{s1} + b_1\,T_1^{-s0+s1} - T_1^{s0+s1} == 0,\ -1 + a_1 + a_2 - b_3 - b_7 + b_3\,T_1^{-s0} + T_1^{s0} -$$
$$a_1\,T_1^{s0} - a_2\,T_1^{s0} + a_1\,T_1^{s0-s1} - a_1\,T_1^{-s1} + T_1^{s1} + b_3\,T_1^{s1} + b_7\,T_1^{s1} - b_3\,T_1^{-s0+s1} - T_1^{s0+s1} == 0,$$
$$-1 - a_3 - a_4 - b_3 - b_7 + b_3\,T_1^{-s0} + T_1^{s0} + a_3\,T_1^{s0} + a_4\,T_1^{s0} - a_3\,T_1^{s0-s1} + a_3\,T_1^{-s1} + T_1^{s1} + b_3\,T_1^{s1} + b_7\,T_1^{s1} -$$
$$b_3\,T_1^{-s0+s1} - T_1^{s0+s1} == 0,\ 1 + a_3 + a_4 - b_1 - b_5 + b_1\,T_1^{-s0} - T_1^{s0} - a_3\,T_1^{s0} - a_4\,T_1^{s0} + a_3\,T_1^{s0-s1} -$$
$$a_3\,T_1^{-s1} - T_1^{s1} + b_1\,T_1^{s1} + b_5\,T_1^{s1} - b_1\,T_1^{-s0+s1} + T_1^{s0+s1} == 0,\ 1 - a_1 - a_2 + b_3 + b_7 - b_3\,T_1^{-s0} -$$
$$T_1^{s0} + a_1\,T_1^{s0} + a_2\,T_1^{s0} - a_1\,T_1^{s0-s1} + a_1\,T_1^{-s1} - T_1^{s1} - b_3\,T_1^{s1} - b_7\,T_1^{s1} + b_3\,T_1^{-s0+s1} + T_1^{s0+s1} == 0\Big\}$$

```
In[•]:= {sol} = Solve[eqns, vars]
```

Out[•]=

$$\Big\{\Big\{a_1 \to 0,\ a_2 \to 0,\ a_3 \to 0,\ a_4 \to 0,\ a_5 \to 0,\ a_6 \to 0,\ a_7 \to 0,\ a_8 \to 0,$$
$$b_1 \to 0,\ b_2 \to 0,\ b_3 \to 0,\ b_4 \to 0,\ b_5 \to 1 - T_1^{s0},\ b_6 \to 0,\ b_7 \to -1 + T_1^{s0},\ b_8 \to 0\Big\}\Big\}$$

```
In[•]:= nF2 = CF[CF[(χ_{i0==i1} F1i + F2i + D_{s0,i0,j0}[tw0] + D_{s1,i1,j1}[tw1]) /. sol]];
        {nF1, nF2} = Simplify@{Coefficient[nF2, χ_{i0==i1}], nF2 /. χ_{i0==i1} → 0}
        Table[Θ[K] == Θ[K, F1 → nF1, F2 → nF2], {K, AllKnots[{3, 8}]}]
```

Out[•]=

$$\Big\{\frac{s0}{2} - \frac{T_1^{-s1}\left(-1 + T_1^{s0}\right) T_2^{-s1}\left(-2 + T_1^{s1} + T_2^{s1}\right) \chi_{i0==j1}\,\chi_{i1==j0}}{-1 + T_2} + \frac{T_1^{-s1}\left(-1 + T_1^{s0}\right) T_2^{-s1}\left(-1 + T_2^{s1}\right) \chi_{i1==j0}\,\chi_{j0==j1}}{-1 + T_2} +$$

$$\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)T_2^{-s1}\chi_{i1==j0}\,g_{1,i1,i0}}{-1+T_2}+\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)T_2^{-s1}\chi_{i1==j0}\,g_{1,j1,i0}}{-1+T_2}+$$
$$\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)T_2^{-s1}\chi_{i1==j0}\,g_{2,i1,i0}}{-1+T_2}-\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)T_2^{-s1}\chi_{i1==j0}\,g_{2,i1,j0}}{-1+T_2}+$$
$$s0\,T_2^{s0}\,g_{1,i0,i0}\,g_{2,j0,i0}+\frac{s0\left(-1+T_1^{s0}\right)T_2^{2\,s0}\,g_{1,j0,i0}\,g_{2,j0,i0}}{-1+T_2^{s0}}-s0\,g_{1,i0,i0}\,g_{2,j0,j0}-$$
$$\frac{s0\left(-1+T_1^{s0}\right)T_2^{s0}\,g_{1,j0,i0}\,g_{2,j0,j0}}{-1+T_2^{s0}}+\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)T_2^{-s1}\left(-1+T_2^{s1}\right)\chi_{i1==j0}\,g_{2,j1,i0}}{-1+T_2}-$$
$$\frac{T_1^{-s1}\left(-1+T_1^{s0}\right)T_2^{-s1}\left(-1+T_2^{s1}\right)\chi_{i1==j0}\,g_{2,j1,j0}}{-1+T_2}-s0\,g_{3,i0,i0}-s0\left(-1+T_2^{s0}\right)g_{2,j0,i0}\,g_{3,i0,i0}+$$
$$2\,s0\,g_{2,j0,j0}\,g_{3,i0,i0}+\frac{s0\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0}}{-1+T_2^{s0}}-\frac{s0\,T_2^{s0}\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,i0,i0}\,g_{3,j0,i0}}{-1+T_2^{s0}}-$$
$$\frac{s0\left(-1+T_1^{s0}\right)\left(1+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{1,j0,i0}\,g_{3,j0,i0}}{-1+T_2^{s0}}+\frac{s0\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{2,i0,j0}\,g_{3,j0,i0}}{-1+T_2^{s0}}+$$
$$s0\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{2,j0,i0}\,g_{3,j0,i0}+\frac{s0\left(-2+T_2^{s0}\right)\left(-1+T_1^{s0}\,T_2^{s0}\right)g_{2,j0,j0}\,g_{3,j0,i0}}{-1+T_2^{s0}}+$$
$$s0\,g_{1,i0,i0}\,g_{3,j0,j0}+\frac{s0\left(-1+T_1^{s0}\right)T_2^{s0}\,g_{1,j0,i0}\,g_{3,j0,j0}}{-1+T_2^{s0}}-s0\,g_{2,i0,i0}\,g_{3,j0,j0}-$$
$$s0\,T_2^{s0}\,g_{2,j0,i0}\,g_{3,j0,j0}-\frac{\left(-1+T_1^{s0}\right)g_{3,j0^+,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)\chi_{i0==j1}\,g_{3,j0^+,i1}}{-1+T_2}+$$
$$\frac{\left(-1+T_1^{s0}\right)\chi_{i1==j0}\,g_{3,j0^+,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\left(-1+T_2^{s1}\right)\chi_{j0==j1}\,g_{3,j0^+,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)g_{1,i1,i0}\,g_{3,j0^+,i1}}{-1+T_2}+$$
$$\frac{\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{3,j0^+,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)g_{2,i1,i0}\,g_{3,j0^+,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)g_{2,i1,j0}\,g_{3,j0^+,i1}}{-1+T_2}+$$
$$\frac{\left(-1+T_1^{s0}\right)\left(-1+T_2^{s1}\right)g_{2,j1,i0}\,g_{3,j0^+,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\left(-1+T_2^{s1}\right)g_{2,j1,j0}\,g_{3,j0^+,i1}}{-1+T_2},$$

$$\frac{1}{-1+T_2}\left(-1+T_1^{s0}\right)\Big(-T_1^{-s1}\,T_2^{-s1}\,\chi_{i1==j0}\left(g_{1,i1,i0}+\left(-1+T_1^{s1}\right)g_{1,j1,i0}\right)\left(1+\left(-1+T_2^{s1}\right)\chi_{j0==j1}+$$
$$g_{2,i1,i0}-g_{2,i1,j0}-g_{2,j1,i0}+T_2^{s1}\,g_{2,j1,i0}+g_{2,j1,j0}-T_2^{s1}\,g_{2,j1,j0}+T_1^{s1}\,T_2^{s1}\,g_{3,j0^+,i1}\Big)+$$
$$\frac{1}{-1+T_2^{s1}}\Big(-s1\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}+s1\,T_2^{s0}\,g_{1,j1,i0}\left(g_{2,i1,i0}-g_{2,j1,i0}\right)g_{3,j0,i1}-$$
$$s1\,T_2^{1+s0}\,g_{1,j1,i0}\left(g_{2,i1,i0}-g_{2,j1,i0}\right)g_{3,j0,i1}-s1\,T_1^{s1}\,T_2^{s0+s1}\,g_{1,j1,i0}\left(g_{2,i1,i0}-g_{2,j1,i0}\right)g_{3,j0,i1}+$$
$$s1\,T_1^{s1}\,T_2^{1+s0+s1}\,g_{1,j1,i0}\left(g_{2,i1,i0}-g_{2,j1,i0}\right)g_{3,j0,i1}+s1\,T_2\,g_{1,j1,i0}\left(g_{2,i1,j0}-g_{2,j1,j0}\right)g_{3,j0,i1}-$$
$$s1\,T_1^{s1}\,T_2^{1+s1}\,g_{1,j1,i0}\left(g_{2,i1,j0}-g_{2,j1,j0}\right)g_{3,j0,i1}+s1\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}-$$
$$g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0^+,i1}+T_1^{s1}\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0^+,i1}+g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0^+,i1}-$$
$$T_1^{s1}\,g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0^+,i1}-g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0^+,i1}+2\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0^+,i1}-$$
$$T_1^{s1}\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0^+,i1}-T_2^{2\,s1}\left(g_{1,i1,i0}-g_{1,j1,i0}\right)\left(g_{2,j1,i0}-g_{2,j1,j0}\right)g_{3,j0^+,i1}+$$
$$g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0^+,i1}-2\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0^+,i1}+T_1^{s1}\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0^+,i1}+$$

$$T_2^{s1}\Big((2\,g_{1,i1,i0}\,(g_{2,j1,i0}-g_{2,j1,j0})+g_{1,j1,i0}\,(g_{2,i1,i0}-g_{2,i1,j0}-3\,g_{2,j1,i0}+3\,g_{2,j1,j0}))\,g_{3,j0^+,i1}+T_1^{s1}\,g_{1,j1,i0}\,((-g_{2,i1,i0}+g_{2,j1,i0})\,g_{3,j0^+,i1}+g_{2,i1,j0}\,(s1\,g_{3,j0,i1}+g_{3,j0^+,i1})-g_{2,j1,j0}\,(s1\,g_{3,j0,i1}+g_{3,j0^+,i1}))\Big)+\Big(-1+T_2^{s1}\Big)\,\chi_{j0==j1}\Big(-\Big(\Big(-1+T_2^{s1}\Big)\,g_{1,i1,i0}\,g_{3,j0^+,i1}\Big)+g_{1,j1,i0}\Big(-1-g_{2,j1,i0}+g_{2,j1,j0}-2\,g_{3,j0^+,i1}+T_1^{s1}\,g_{3,j0^+,i1}+T_2^{s1}\,g_{3,j0^+,i1}-g_{3,j0^+,j1}\Big)\Big)\Big)+\chi_{i0==j1}\Big(\Big(-2-g_{1,i1,i0}+2\,g_{1,j1,i0}-g_{2,i1,i0}+g_{2,i1,j0}+2\,g_{2,j1,i0}-T_1^{s1}\,(-1+g_{1,j1,i0}-g_{2,i1,i0}+g_{2,i1,j0}+g_{2,j1,i0}-g_{2,j1,j0})-2\,g_{2,j1,j0}+T_2^{s1}\,(1+g_{1,i1,i0}-g_{1,j1,i0}-g_{2,j1,i0}+g_{2,j1,j0})\Big)\,g_{3,j0^+,i1}+T_1^{-s1}\,T_2^{-s1}\,\chi_{i1==j0}\Big(-2+\chi_{j0==j1}-g_{1,i1,i0}+g_{1,j1,i0}-g_{2,i1,i0}+g_{2,i1,j0}+g_{2,j1,i0}-T_2^{s1}\,(-1+\chi_{j0==j1}-g_{1,i1,i0}+g_{1,j1,i0}+g_{2,j1,i0}-g_{2,j1,j0})-g_{2,j1,j0}+T_1^{s1}\Big(2-\chi_{j0==j1}-g_{1,j1,i0}+g_{2,i1,i0}-g_{2,i1,j0}-g_{2,j1,i0}+g_{2,j1,j0}+T_2^{s1}\,(-1+\chi_{j0==j1}+g_{1,j1,i0}+g_{2,j1,i0}-g_{2,j1,j0}-g_{3,j0^+,i1})\Big)+T_1^{2\,s1}\,T_2^{s1}\,g_{3,j0^+,i1}\Big)+(-1+g_{1,j1,i0}+g_{2,j1,i0}-g_{2,j1,j0})\,g_{3,j0^+,j1}+\chi_{j0==j1}\Big(g_{1,j1,i0}+g_{2,j1,i0}-g_{2,j1,j0}+2\,g_{3,j0^+,i1}-T_1^{s1}\,g_{3,j0^+,i1}-T_2^{s1}\,g_{3,j0^+,i1}+g_{3,j0^+,j1}\Big)\Big)\Big)\Big\}$$

Out[•]=

{True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True}