

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
Once[<< Theta.m]
SetOptions[PolyPlot, ImageSize -> Tiny];
Clear[Θ]
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[*]:= RandomVK[n_] := {
  Prepend[#, 2 RandomInteger[1] - 1] & /@
  Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
  Table[RandomInteger[{-1, 1}], 2 n + 1]
};
```

```
In[*]:= RandomVK[5]
```

```
Out[*]=
```

```
{{{1, 7, 3}, {-1, 6, 2}, {1, 4, 9}, {-1, 10, 1}, {1, 5, 8}},
{-1, -1, 0, 1, 0, -1, 1, 1, -1, 1, 0}}
```

```
In[*]:= CF[ε_] := Expand@Collect[ε, g__ | χ__, F] /. F -> Factor@*PowerExpand;
```

```
In[*]:= Short[Options[Θ] = {F1 -> (F1i = F1[{s0, i0, j0}]),
  F2 -> (F2i = F2[{s0, i0, j0}, {s1, i1, j1}]), F3 -> (F3i = F3[φ, k])}]
```

```
Out[*]//Short=
```

$$\left\{ F1 \rightarrow \frac{s_0}{2} + s_0 T_2^{s_0} g_{1,i_0,i_0} g_{2,j_0,i_0} + \langle\langle 19 \rangle\rangle + \frac{\langle\langle 1 \rangle\rangle}{\langle\langle 1 \rangle\rangle} - s_0 g_{2,i_0,i_0} g_{3,j_0,j_0} - s_0 T_2^{s_0} g_{2,j_0,i_0} g_{3,j_0,j_0}, \right. \\ \left. F2 \rightarrow \langle\langle 1 \rangle\rangle, F3 \rightarrow -\frac{\varphi}{2} + \varphi g_{\langle\langle 1 \rangle\rangle} \right\}$$

```

In[ ]:=  $\Theta[K\_ , opts\_ Rule] := Module[ \{X, \varphi\varphi, n, A, \Delta, G, ev, \emptyset, kk, k0, k1, f1, f2, f3\},
  f1 = F1 /. \{opts\} /. Options[\emptyset];
  f2 = F2 /. \{opts\} /. Options[\emptyset];
  f3 = F3 /. \{opts\} /. Options[\emptyset];
  \{X, \varphi\varphi\} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  Cases[X, \{s\_ , i\_ , j\_ \} \Rightarrow \left( A[[\{i, j\}, \{i + 1, j + 1\}]] += \begin{pmatrix} -T^s & T^s & -1 \\ 0 & & -1 \end{pmatrix} \right) ];
  \Delta = T^{(-Total[\varphi\varphi] - Total[X[[All, 1]]]) / 2} Det[A];
  G = Inverse[A];
  ev[\mathcal{E}\_] := Factor[
    \mathcal{E} /. \{k\_+ \Rightarrow k + 1, \$ \Rightarrow 2 n + 1\} /. \{g_{\nu, \alpha, \beta} \Rightarrow (G[[\alpha, \beta]] /. T \rightarrow T_{\nu}), X_{True} \rightarrow 1, X_{False} \rightarrow 0\}];
  \emptyset = ev@Sum[f1 /. Thread[\{s0, i0, j0\} \rightarrow X[[kk]]], \{kk, n\}];
  \emptyset += ev@Sum[f2 /. Thread[\{s0, i0, j0\} \rightarrow X[[k0]]] /.
    Thread[\{s1, i1, j1\} \rightarrow X[[k1]]], \{k0, n\}, \{k1, n\}];
  \emptyset += ev@Sum[f3 /. \{\varphi \rightarrow \varphi\varphi[[kk]], k \rightarrow kk\}, \{kk, Length@\varphi\varphi\}];
  Factor@\{\Delta, (\Delta /. T \rightarrow T_1) (\Delta /. T \rightarrow T_2) (\Delta /. T \rightarrow T_3) \emptyset\}
];$ 
```

```

In[ ]:=  $\Theta[Knot[7, 3]]$ 

```

 KnotTheory: Loading precomputed data in PD4Knots`.

```

Out[ ]:=

```

$$\left\{ \frac{2 - 3 T + 3 T^2 - 3 T^3 + 2 T^4}{T^2}, \right. \\ \left. \frac{1}{T_1^4 T_2^4} \left(17 - 25 T_1 + 25 T_1^2 - 25 T_1^3 + 17 T_1^4 - 25 T_2 + 12 T_1 T_2 + 12 T_1^4 T_2 - 25 T_1^5 T_2 + 25 T_2^2 - \right. \right. \\ T_1^2 T_2^2 - 7 T_1^3 T_2^2 - T_1^4 T_2^2 + 25 T_1^6 T_2^2 - 25 T_2^3 - 7 T_1^2 T_2^3 + 6 T_1^3 T_2^3 + 6 T_1^4 T_2^3 - 7 T_1^5 T_2^3 - 25 T_1^7 T_2^3 + \\ 17 T_2^4 + 12 T_1 T_2^4 - T_1^2 T_2^4 + 6 T_1^3 T_2^4 - 12 T_1^4 T_2^4 + 6 T_1^5 T_2^4 - T_1^6 T_2^4 + 12 T_1^7 T_2^4 + 17 T_1^8 T_2^4 - 25 T_1 T_2^5 - \\ 7 T_1^3 T_2^5 + 6 T_1^4 T_2^5 + 6 T_1^5 T_2^5 - 7 T_1^6 T_2^5 - 25 T_1^8 T_2^5 + 25 T_1^2 T_2^6 - T_1^4 T_2^6 - 7 T_1^5 T_2^6 - T_1^6 T_2^6 + 25 T_1^8 T_2^6 - \\ \left. \left. 25 T_1^3 T_2^7 + 12 T_1^4 T_2^7 + 12 T_1^7 T_2^7 - 25 T_1^8 T_2^7 + 17 T_1^4 T_2^8 - 25 T_1^5 T_2^8 + 25 T_1^6 T_2^8 - 25 T_1^7 T_2^8 + 17 T_1^8 T_2^8 \right) \right\}$$

```

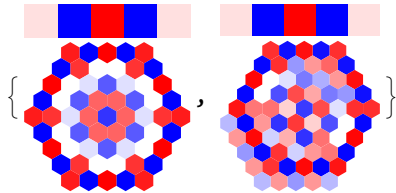
In[ ]:= PolyPlot /@ \{\Theta[Knot[7, 3]], \Theta[Knot[7, 3]], F3 \rightarrow 0\}

```

```

Out[ ]:=

```



```

In[*]:=  $\delta_{i,j} := x_{i=j}; x_{ph}^p /; p > 1 \wedge := x_{ph};$ 
bRules[{s_, i_, j_}] := { (* b for "push indices backwards" *)
   $g_{v,j^+, \beta} \mapsto g_{v,j, \beta} - \delta_{j, \beta}, g_{v,i^+, \beta} \mapsto T_v^{-s} g_{v,i, \beta} + (1 - T_v^{-s}) g_{v,j, \beta} - T_v^{-s} \delta_{i, \beta} - (1 - T_v^{-s}) \delta_{j, \beta},$ 
   $g_{v, \alpha, i^+} \mapsto T_v^s g_{v, \alpha, i} + \delta_{\alpha, i^+}, g_{v, \alpha, j^+} \mapsto g_{v, \alpha, j} + (1 - T_v^s) g_{v, \alpha, i} + \delta_{\alpha, j^+}$ 
};
bRules[X___List] := Union @@ Table[bRules[c], {c, {X}}]

```

```

In[*]:= {gv,i,β, gv,j,β} /. gRules[{s, i, j}]

```

```

Out[*]= {Xi=β + Tvs gv,i+,β + (1 - Tvs) gv,j+,β, Xj=β + gv,j+,β}

```

```

In[*]:= Expand[{gv,i,β, gv,j,β} /. gRules[{s, i, j}] /. bRules[{s, i, j}]]

```

```

Out[*]= {gv,i,β, gv,j,β}

```

```

In[*]:= D[{s_, i_, j_}][ $\mathcal{E}$ _] := CF[Expand[Plus[
   $\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, i^+, \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, i^+},$ 
   $\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, j^+, \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, j^+},$ 
   $-\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, i, \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, i},$ 
   $-\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, j, \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, j}\}$ 
] /. bRules[{s, i, j}] /. bRules[{s0, i0, j0}]] /. Xeq_Equal  $\mapsto$  XSort@eq // . {
  Xj0=j1  $\rightarrow$  Xi0=i1, Xj0+=j1+  $\rightarrow$  Xj0=j1,
  Xi0=j1  $\rightarrow$  0, Xi1=j0  $\rightarrow$  0, Xi0+=j1+  $\rightarrow$  0, Xi0+=i1+  $\rightarrow$  Xi0=i1, Xi1+=j0+  $\rightarrow$  0, Xi0+=j0+  $\rightarrow$  0,
  Xi0=i1 A_  $\mapsto$  Xi0=i1 (A /. {s1  $\rightarrow$  s0, i1  $\rightarrow$  i0, j1  $\rightarrow$  j0})
} /. {Xi0=j0  $\rightarrow$  0, Xi0+=j0+  $\rightarrow$  0, XTrue  $\rightarrow$  1}
];
B[ $\mathcal{E}$ _] := CF[
  ( $\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, \$ , \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, \$}\}$ ) - ( $\mathcal{E} / . \{g_{v, \# , \beta} \mapsto g_{v, 1, \beta}, g_{v, \alpha, \#} \mapsto g_{v, \alpha, 1}\}$ ) /. {
    g_, $, β  $\rightarrow$  X$=β, g_, $  $\rightarrow$  1, g_, α, 1  $\rightarrow$  Xα=1} /. {X$=i0  $\rightarrow$  0, Xi0+=1  $\rightarrow$  0, X$=j0  $\rightarrow$  0, Xj0+=1  $\rightarrow$  0}
];

```

```

In[*]:= D[{s1, i1, j1}][g1, #, i0]

```

```

Out[*]= -T1-s0 Xi0=i1 - T1-s1 (-1 + T1s1) g1, i1, i0 + T1-s1 (-1 + T1s1) g1, j1, i0

```

```

In[*]:= D[{s1, i1, j1}][g3, j0, #]

```

```

Out[*]= Xj0=i1+ + Xj0=j1+

```

```

In[*]:= B[g3, j0, #]

```

```

Out[*]= 1 - Xj0=1

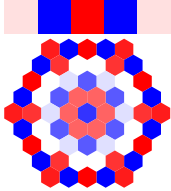
```

In[]:= **tw = g_{1,*,i0} ;**

Θ[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot

$$\gg \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, \frac{1}{T_1^4 T_2^4} \right. \\ \left. \left(17 - 25T_1 + 25T_1^2 - 25T_1^3 + 17T_1^4 - 25T_2 + 12T_1T_2 + 12T_1^4T_2 - 25T_1^5T_2 + 25T_2^2 - T_1^2T_2^2 - 7T_1^3T_2^2 - T_1^4T_2^2 + 25T_1^6T_2^2 - \right. \right. \\ \left. 25T_2^3 - 7T_1^2T_2^3 + 6T_1^3T_2^3 + 6T_1^4T_2^3 - 7T_1^5T_2^3 - 25T_1^7T_2^3 + 17T_2^4 + 12T_1T_2^4 - T_1^2T_2^4 + 6T_1^3T_2^4 - 12T_1^4T_2^4 + 6T_1^5T_2^4 - \right. \\ \left. T_1^6T_2^4 + 12T_1^7T_2^4 + 17T_1^8T_2^4 - 25T_1T_2^5 - 7T_1^3T_2^5 + 6T_1^4T_2^5 + 6T_1^5T_2^5 - 7T_1^6T_2^5 - 25T_1^8T_2^5 + 25T_1^2T_2^6 - T_1^4T_2^6 - 7T_1^5T_2^6 - \right. \\ \left. T_1^6T_2^6 + 25T_1^8T_2^6 - 25T_1^3T_2^7 + 12T_1^4T_2^7 + 12T_1^7T_2^7 - 25T_1^8T_2^7 + 17T_1^4T_2^8 - 25T_1^5T_2^8 + 25T_1^6T_2^8 - 25T_1^7T_2^8 + 17T_1^8T_2^8 \right) \Big\}$$

Out[]:=



In[]:= **tw = g_{1,j0,*} ;**

Θ[Knot[7, 3], F1 → -B[tw], F2 → D_{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot

$$\gg \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, 0 \right\}$$

Out[]:=

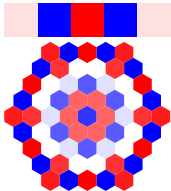


In[]:= **tw = g_{1,j0,*} ;**

Θ[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D_{s1,i1,j1}[tw]] // Echo // PolyPlot

$$\gg \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, \frac{1}{T_1^4 T_2^4} \right. \\ \left(17 - 25T_1 + 25T_1^2 - 25T_1^3 + 17T_1^4 - 25T_2 + 12T_1T_2 + 12T_1^4T_2 - 25T_1^5T_2 + 25T_2^2 - T_1^2T_2^2 - 7T_1^3T_2^2 - T_1^4T_2^2 + 25T_1^6T_2^2 - \right. \\ \left. 25T_2^3 - 7T_1^2T_2^3 + 6T_1^3T_2^3 + 6T_1^4T_2^3 - 7T_1^5T_2^3 - 25T_1^7T_2^3 + 17T_2^4 + 12T_1T_2^4 - T_1^2T_2^4 + 6T_1^3T_2^4 - 12T_1^4T_2^4 + 6T_1^5T_2^4 - \right. \\ \left. T_1^6T_2^4 + 12T_1^7T_2^4 + 17T_1^8T_2^4 - 25T_1T_2^5 - 7T_1^3T_2^5 + 6T_1^4T_2^5 + 6T_1^5T_2^5 - 7T_1^6T_2^5 - 25T_1^8T_2^5 + 25T_1^2T_2^6 - T_1^4T_2^6 - 7T_1^5T_2^6 - \right. \\ \left. T_1^6T_2^6 + 25T_1^8T_2^6 - 25T_1^3T_2^7 + 12T_1^4T_2^7 + 12T_1^7T_2^7 - 25T_1^8T_2^7 + 17T_1^4T_2^8 - 25T_1^5T_2^8 + 25T_1^6T_2^8 - 25T_1^7T_2^8 + 17T_1^8T_2^8 \right) \Big\}$$

Out[]:=



```
In[*]:= tw = g1, #1, j0 g2, #1, j0 g3, i0, #1;
          @Knot[7, 3], F1 → -B[tw], F2 → D_{s1, i1, j1}[tw], F3 → 0 // Echo // PolyPlot
          @Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D_{s1, i1, j1}[tw] // Echo // PolyPlot
```

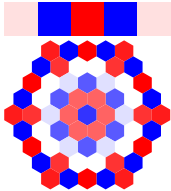
$$\gg \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, 0 \right\}$$

Out[*]=



$$\gg \left\{ \frac{2 - 3T + 3T^2 - 3T^3 + 2T^4}{T^2}, \frac{1}{T_1^4 T_2^4} \right. \\ \left. (17 - 25T_1 + 25T_1^2 - 25T_1^3 + 17T_1^4 - 25T_2 + 12T_1T_2 + 12T_1^4T_2 - 25T_1^5T_2 + 25T_2^2 - T_1^2T_2^2 - 7T_1^3T_2^2 - T_1^4T_2^2 + 25T_1^6T_2^2 - \right. \\ 25T_2^3 - 7T_1^2T_2^3 + 6T_1^3T_2^3 + 6T_1^4T_2^3 - 7T_1^5T_2^3 - 25T_1^7T_2^3 + 17T_2^4 + 12T_1T_2^4 - T_1^2T_2^4 + 6T_1^3T_2^4 - 12T_1^4T_2^4 + 6T_1^5T_2^4 - \\ T_1^6T_2^4 + 12T_1^7T_2^4 + 17T_1^8T_2^4 - 25T_1T_2^5 - 7T_1^3T_2^5 + 6T_1^4T_2^5 + 6T_1^5T_2^5 - 7T_1^6T_2^5 - 25T_1^8T_2^5 + 25T_1^2T_2^6 - T_1^4T_2^6 - 7T_1^5T_2^6 - \\ T_1^6T_2^6 + 25T_1^8T_2^6 - 25T_1^3T_2^7 + 12T_1^4T_2^7 + 12T_1^7T_2^7 - 25T_1^8T_2^7 + 17T_1^4T_2^8 - 25T_1^5T_2^8 + 25T_1^6T_2^8 - 25T_1^7T_2^8 + 17T_1^8T_2^8) \left. \right\}$$

Out[*]=



```
In[*]:= bas = List @@ Expand[(g1, #, i0 + g1, #, j0) (g2, #, i0 + g2, #, j0) (g3, i0, # + g3, j0, #)]
```

Out[*]=

```
{g1, #1, i0 g2, #1, i0 g3, i0, #1, g1, #1, j0 g2, #1, i0 g3, i0, #1, g1, #1, i0 g2, #1, j0 g3, i0, #1, g1, #1, j0 g2, #1, j0 g3, i0, #1,
g1, #1, i0 g2, #1, i0 g3, j0, #1, g1, #1, j0 g2, #1, i0 g3, j0, #1, g1, #1, i0 g2, #1, j0 g3, j0, #1, g1, #1, j0 g2, #1, j0 g3, j0, #1}
```

```
In[*]:= Column[(# → @Knot[7, 6], F1 → -B[#], F2 → D_{s1, i1, j1}[#], F3 → 0][[2]]) & /@ bas]
Column[(# → {B[#], D_{s1, i1, j1}[#]}) & /@ bas]
```

Out[*]=

```
g1, #1, i0 g2, #1, i0 g3, i0, #1 → 0
g1, #1, j0 g2, #1, i0 g3, i0, #1 → 0
g1, #1, i0 g2, #1, j0 g3, i0, #1 → 0
g1, #1, j0 g2, #1, j0 g3, i0, #1 → 0
g1, #1, i0 g2, #1, i0 g3, j0, #1 → 0
g1, #1, j0 g2, #1, i0 g3, j0, #1 → 0
g1, #1, i0 g2, #1, j0 g3, j0, #1 → 0
g1, #1, j0 g2, #1, j0 g3, j0, #1 → 0
```

Out[*]=

```
g1, #1, i0 g2, #1, i0 g3, i0, #1 →
{0, T1^s0 T2^s0 χ_{i0=i1} g1, i0, i0 g2, i0, i0 + T1^s0 T2^s0 (-1 + T1^s0 T2^s0) χ_{i0=i1} g1, j0, i0 g2, j0, i0 +
T1^s0 T2^s0 χ_{i0=i1} g3, i0, i0 - T1^s0 T2^s0 χ_{i0=i1} g1, i0, i0 g3, i0, i0 -
T1^s0 (-1 + T1^s0) T2^s0 χ_{i0=i1} g1, j0, i0 g3, i0, i0 - T1^s0 T2^s0 χ_{i0=i1} g2, i0, i0 g3, i0, i0 -
T1^s0 T2^s0 (-1 + T2^s0) χ_{i0=i1} g2, j0, i0 g3, i0, i0 + T1^s0 (-1 + T1^s0) T2^s0 g1, j1, i0 g2, i1, i0 g3, i0, i1 +
T1^s0 T2^s0 (-1 + T2^s0) g1, i1, i0 g2, j1, i0 g3, i0, i1 - T1^s0 T2^s0 (-2 + T1^s0 + T2^s0) g1, j1, i0 g2, j1, i0 g3, i0, i1 +
T1^s0 T2^s0 (-1 + T1^s0 T2^s0) χ_{i0=i1} g1, j0, i0 g2, j0, i0 - T1^s0 T2^s0 χ_{i0=i1} g1, i0, i0 g2, j0, i0 -
T1^s0 (-1 + T1^s0) T2^s0 (-1 + T1^s0 T2^s0) χ_{i0=i1} g1, j0, i0 g3, j0, i0 - T1^s0 T2^s0 (-1 + T1^s0 T2^s0)
χ_{i0=i1} g2, i0, i0 g3, j0, i0 - T1^s0 T2^s0 (-1 + T2^s0) (-1 + T1^s0 T2^s0) χ_{i0=i1} g2, i0, i0 g3, j0, i0 +
```

$\mathbf{g}_{1,\#1,j0} \mathbf{g}_{2,\#1,i0} \mathbf{g}_{3,i0^*,\#1} \rightarrow$

$$\left\{ \begin{aligned} & \mathbf{0}, T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{2,i0,i0} + T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{2,j0,i0} + \\ & T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{3,i0,i0} - T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,i0,i0} - \\ & T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,i0,i0} - T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{2,i0,i0} \mathbf{g}_{3,i0,i0} + \\ & T_1^{-s0} T_2^{-s0} (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,i0,i0} + T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,i0} \mathbf{g}_{3,i0,i1} + \\ & T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,i0,i1} - T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,i0,i1} - \\ & T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,i0,j0} + T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} - \\ & T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,j0,i0} - T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \\ & \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,i0} - T_1^{-s0} (-1 + T_1^{s0}) T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,i0,i0} \mathbf{g}_{3,j0,i0} + \\ & T_1^{-s0} T_2^{-s0} (-2 + T_1^{s0} + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,j0,i0} + \\ & T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,i0} \mathbf{g}_{3,j0,i1} + \\ & T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) (-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} - T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \\ & (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} - T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,j0,j0} \end{aligned} \right\}$$

 $\mathbf{g}_{1,\#1,i0} \mathbf{g}_{2,\#1,j0} \mathbf{g}_{3,i0^*,\#1} \rightarrow$

$$\left\{ \begin{aligned} & \mathbf{0}, T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{1,i0,i0} \mathbf{g}_{2,i0,j0} + T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{2,j0,j0} + \\ & T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,i0,i0} - T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,i0} \mathbf{g}_{3,i0,i0} + \\ & T_1^{-s0} T_2^{-s0} (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,i0,i0} - T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,i0,i0} - \\ & T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,i0,i0} + T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,i0,i1} + \\ & T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) \mathbf{g}_{1,i1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,i0,i1} - T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,i0,i1} - \\ & T_1^{-s0} T_2^{-s0} \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,i0,j0} + T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} - \\ & T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,i0} \mathbf{g}_{3,j0,i0} + \\ & T_1^{-s0} T_2^{-s0} (-2 + T_1^{s0} + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,j0,i0} - \\ & T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,j0,i0} - T_1^{-s0} T_2^{-s0} (-1 + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \\ & \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,i0} + T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,j0,i1} + \\ & T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) (-1 + T_2^{s1}) \mathbf{g}_{1,i1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} - T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \\ & (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} - T_1^{-s0} T_2^{-s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,j0,j0} \end{aligned} \right\}$$

$\mathbf{g}_{1,\#1,j0} \mathbf{g}_{2,\#1,j0} \mathbf{g}_{3,i0^*,\#1} \rightarrow$
 $\{0, T_1^{s0} T_2^{s0} \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{2,i0,j0} + T_1^{s0} T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{2,j0,j0} -$
 $T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,i0,i0} - T_1^{s0} T_2^{s0} (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,i0,i0} +$
 $T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,i0,i0} - T_1^{s0} (-1 + T_1^{s0}) T_2^{s0} \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,i0,i0} +$
 $T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,i0,i0} + T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,i0,i1} +$
 $T_1^{s0} T_2^{s0} (-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,i0,i1} - T_1^{s0} T_2^{s0} (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,i0,i1} +$
 $T_1^{s0} T_2^{s0} \chi_{i0=i1} \mathbf{g}_{3,i0,j0} - T_1^{s0} T_2^{s0} \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,i0,j0} -$
 $T_1^{s0} T_2^{s0} \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,i0,j0} - T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} -$
 $T_1^{s0} T_2^{s0} (-1 + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,j0,i0} +$
 $T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,i0} -$
 $T_1^{s0} (-1 + T_1^{s0}) T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,j0,i0} +$
 $T_1^{s0} T_2^{s0} (-2 + T_1^{s0} + T_2^{s0}) (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,i0} +$
 $T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,j0,i1} +$
 $T_1^{s0} T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) (-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} -$
 $T_1^{s0} T_2^{s0} (-1 + T_1^{s1} T_2^{s0}) (-2 + T_1^{s1} + T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} + T_1^{s0} T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,j0} -$
 $T_1^{s0} T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,j0} - T_1^{s0} T_2^{s0} (-1 + T_1^{s0} T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,j0} \}$
 $\mathbf{g}_{1,\#1,i0} \mathbf{g}_{2,\#1,i0} \mathbf{g}_{3,j0^*,\#1} \rightarrow$
 $\{0, \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{2,j0,i0} + \chi_{i0=i1} \mathbf{g}_{3,j0,i0} - \chi_{i0=i1} \mathbf{g}_{1,i0,i0} \mathbf{g}_{3,j0,i0} - (-1 + T_1^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,j0,i0} -$
 $\chi_{i0=i1} \mathbf{g}_{2,i0,i0} \mathbf{g}_{3,j0,i0} - (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,j0,i0} + (-1 + T_1^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,i1,i0} \mathbf{g}_{3,j0,i1} +$
 $(-1 + T_2^{s1}) \mathbf{g}_{1,i1,i0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} + (2 - T_1^{s1} - T_2^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} \}$
 $\mathbf{g}_{1,\#1,j0} \mathbf{g}_{2,\#1,i0} \mathbf{g}_{3,j0^*,\#1} \rightarrow \{0, \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{2,j0,i0} + (-1 + T_1^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} -$
 $\chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,j0,i0} - (-1 + T_1^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,i0} - (-1 + T_1^{s0}) \chi_{i0=i1} \mathbf{g}_{2,i0,i0} \mathbf{g}_{3,j0,i0} +$
 $(-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,j0,i0} + (-1 + T_1^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,i0} \mathbf{g}_{3,j0,i1} +$
 $(-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} + (2 - T_1^{s1} - T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,i0} \mathbf{g}_{3,j0,i1} - \chi_{i0=i1} \mathbf{g}_{2,j0,i0} \mathbf{g}_{3,j0,j0} \}$
 $\mathbf{g}_{1,\#1,i0} \mathbf{g}_{2,\#1,j0} \mathbf{g}_{3,j0^*,\#1} \rightarrow$
 $\{0, \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{2,j0,j0} + (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} - (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,i0} \mathbf{g}_{3,j0,i0} +$
 $(-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,j0,i0} - \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,j0,i0} - (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,i0} +$
 $(-1 + T_1^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,j0,i1} + (-1 + T_2^{s1}) \mathbf{g}_{1,i1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} +$
 $(2 - T_1^{s1} - T_2^{s1}) \mathbf{g}_{1,j1,i0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} - \chi_{i0=i1} \mathbf{g}_{1,j0,i0} \mathbf{g}_{3,j0,j0} \}$
 $\mathbf{g}_{1,\#1,j0} \mathbf{g}_{2,\#1,j0} \mathbf{g}_{3,j0^*,\#1} \rightarrow$
 $\{0, \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{2,j0,j0} - (-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{3,j0,i0} - (-1 + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,i0,j0} \mathbf{g}_{3,j0,i0} +$
 $(-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,i0} - (-1 + T_1^{s0}) \chi_{i0=i1} \mathbf{g}_{2,i0,j0} \mathbf{g}_{3,j0,i0} +$
 $(-2 + T_1^{s0} + T_2^{s0}) \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,i0} + (-1 + T_1^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,i1,j0} \mathbf{g}_{3,j0,i1} +$
 $(-1 + T_2^{s1}) \mathbf{g}_{1,i1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} + (2 - T_1^{s1} - T_2^{s1}) \mathbf{g}_{1,j1,j0} \mathbf{g}_{2,j1,j0} \mathbf{g}_{3,j0,i1} +$
 $\chi_{i0=i1} \mathbf{g}_{3,j0,j0} - \chi_{i0=i1} \mathbf{g}_{1,j0,j0} \mathbf{g}_{3,j0,j0} - \chi_{i0=i1} \mathbf{g}_{2,j0,j0} \mathbf{g}_{3,j0,j0} \}$

$\text{In}[*]:= \text{eq} = T_2 - 1 == s0 T_2^{(1-s0)/2} (T_2^{s0} - 1); \text{Simplify}[\{\text{eq} /. s0 \rightarrow 1, \text{eq} /. s0 \rightarrow -1\}]$
 $\text{Out}[*]=$
 $\{\text{True}, \text{True}\}$

```
In[*]:= tw = Table[ai, {i, 8}].bas / (T2 - 1)
{D{s1,i1,j1}[tw], B[tw]}
θ[Knot[7, 6], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
θ[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

Out[*]=

$$\frac{1}{-1 + T_2} (a_1 g_{1,\#1,i0} g_{2,\#1,i0} g_{3,i0^+,\#1} + a_2 g_{1,\#1,j0} g_{2,\#1,i0} g_{3,i0^+,\#1} +$$

$$a_3 g_{1,\#1,i0} g_{2,\#1,j0} g_{3,i0^+,\#1} + a_4 g_{1,\#1,j0} g_{2,\#1,j0} g_{3,i0^+,\#1} + a_5 g_{1,\#1,i0} g_{2,\#1,i0} g_{3,j0^+,\#1} +$$

$$a_6 g_{1,\#1,j0} g_{2,\#1,i0} g_{3,j0^+,\#1} + a_7 g_{1,\#1,i0} g_{2,\#1,j0} g_{3,j0^+,\#1} + a_8 g_{1,\#1,j0} g_{2,\#1,j0} g_{3,j0^+,\#1})$$

Out[*]=

$$\left\{ \frac{a_1 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,i0,i0} g_{2,i0,i0}}{-1 + T_2} + \frac{a_2 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,i0,j0} g_{2,i0,i0}}{-1 + T_2} + \right.$$

$$\frac{a_3 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,i0,i0} g_{2,i0,j0}}{-1 + T_2} + \frac{a_4 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,i0,j0} g_{2,i0,j0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (-a_1 + a_1 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0}) \chi_{i0=i1} g_{1,j0,i0} g_{2,j0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (-a_2 + a_2 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0}) \chi_{i0=i1} g_{1,j0,j0} g_{2,j0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (-a_3 + a_3 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0}) \chi_{i0=i1} g_{1,j0,i0} g_{2,j0,j0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0}) \chi_{i0=i1} g_{1,j0,j0} g_{2,j0,j0}}{-1 + T_2} -$$

$$\frac{T_1^{-s0} T_2^{-s0} (-a_1 + a_2 + a_3 - 2 a_4 - a_2 T_1^{s0} + a_4 T_1^{s0} - a_3 T_2^{s0} + a_4 T_2^{s0}) \chi_{i0=i1} g_{3,i0,i0}}{-1 + T_2} -$$

$$\frac{T_1^{-s0} T_2^{-s0} (a_1 - a_3 + a_3 T_2^{s0}) \chi_{i0=i1} g_{1,i0,i0} g_{3,i0,i0}}{-1 + T_2} - \frac{T_1^{-s0} T_2^{-s0} (a_2 - a_4 + a_4 T_2^{s0}) \chi_{i0=i1} g_{1,i0,j0} g_{3,i0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (a_1 - 2 a_3 - a_1 T_1^{s0} + a_3 T_1^{s0} + a_3 T_2^{s0}) \chi_{i0=i1} g_{1,j0,i0} g_{3,i0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (a_2 - 2 a_4 - a_2 T_1^{s0} + a_4 T_1^{s0} + a_4 T_2^{s0}) \chi_{i0=i1} g_{1,j0,j0} g_{3,i0,i0}}{-1 + T_2} -$$

$$\frac{T_1^{-s0} (a_1 - a_2 + a_2 T_1^{s0}) T_2^{-s0} \chi_{i0=i1} g_{2,i0,i0} g_{3,i0,i0}}{-1 + T_2} - \frac{T_1^{-s0} (a_3 - a_4 + a_4 T_1^{s0}) T_2^{-s0} \chi_{i0=i1} g_{2,i0,j0} g_{3,i0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (a_1 - 2 a_2 + a_2 T_1^{s0} - a_1 T_2^{s0} + a_2 T_2^{s0}) \chi_{i0=i1} g_{2,j0,i0} g_{3,i0,i0}}{-1 + T_2} +$$

$$\frac{T_1^{-s0} T_2^{-s0} (a_3 - 2 a_4 + a_4 T_1^{s0} - a_3 T_2^{s0} + a_4 T_2^{s0}) \chi_{i0=i1} g_{2,j0,j0} g_{3,i0,i0}}{-1 + T_2} +$$

$$\frac{a_1 T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} g_{1,j1,i0} g_{2,i1,i0} g_{3,i0,i1}}{-1 + T_2} + \frac{a_2 T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} g_{1,j1,j0} g_{2,i1,i0} g_{3,i0,i1}}{-1 + T_2} +$$

$$\begin{aligned}
& \frac{a_3 T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} g_{1,j1,i0} g_{2,i1,j0} g_{3,i0,i1}}{-1 + T_2} + \frac{a_4 T_1^{-s0} (-1 + T_1^{s1}) T_2^{-s0} g_{1,j1,j0} g_{2,i1,j0} g_{3,i0,i1}}{-1 + T_2} + \\
& \frac{a_1 T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) g_{1,i1,i0} g_{2,j1,i0} g_{3,i0,i1}}{-1 + T_2} + \frac{a_2 T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) g_{1,i1,j0} g_{2,j1,i0} g_{3,i0,i1}}{-1 + T_2} - \\
& \frac{a_1 T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,i0} g_{2,j1,i0} g_{3,i0,i1}}{-1 + T_2} - \\
& \frac{a_2 T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,j0} g_{2,j1,i0} g_{3,i0,i1}}{-1 + T_2} + \frac{a_3 T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) g_{1,i1,i0} g_{2,j1,j0} g_{3,i0,i1}}{-1 + T_2} + \\
& \frac{a_4 T_1^{-s0} T_2^{-s0} (-1 + T_2^{s1}) g_{1,i1,j0} g_{2,j1,j0} g_{3,i0,i1}}{-1 + T_2} - \frac{a_3 T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,i0} g_{2,j1,j0} g_{3,i0,i1}}{-1 + T_2} - \\
& \frac{a_4 T_1^{-s0} T_2^{-s0} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,j0} g_{2,j1,j0} g_{3,i0,i1}}{-1 + T_2} + \frac{a_4 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{3,i0,j0}}{-1 + T_2} - \\
& \frac{a_3 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,j0,i0} g_{3,i0,j0}}{-1 + T_2} - \frac{a_4 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{1,j0,j0} g_{3,i0,j0}}{-1 + T_2} - \\
& \frac{a_2 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{2,j0,i0} g_{3,i0,j0}}{-1 + T_2} - \frac{a_4 T_1^{-s0} T_2^{-s0} \chi_{i0=i1} g_{2,j0,j0} g_{3,i0,j0}}{-1 + T_2} + \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (-a_1 + a_2 + a_3 - 2 a_4 - a_2 T_1^{s0} + a_4 T_1^{s0} - a_3 T_2^{s0} + a_4 T_2^{s0} + a_1 T_1^{s0} T_2^{s0} - a_2 T_1^{s0} T_2^{s0} - a_3 T_1^{s0} T_2^{s0} + \\
& 2 a_4 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0} - a_6 T_1^{s0} T_2^{s0} - a_7 T_1^{s0} T_2^{s0} + 2 a_8 T_1^{s0} T_2^{s0} + a_2 T_1^{2s0} T_2^{s0} - a_4 T_1^{2s0} T_2^{s0} + a_6 T_1^{2s0} T_2^{s0} - \\
& a_8 T_1^{2s0} T_2^{s0} + a_3 T_1^{s0} T_2^{2s0} - a_4 T_1^{s0} T_2^{2s0} + a_7 T_1^{s0} T_2^{2s0} - a_8 T_1^{s0} T_2^{2s0}) \chi_{i0=i1} g_{3,j0,i0} - \frac{1}{-1 + T_2} \\
& T_1^{-s0} T_2^{-s0} (-a_1 + a_3 - a_3 T_2^{s0} + a_1 T_1^{s0} T_2^{s0} - a_3 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0} - a_7 T_1^{s0} T_2^{s0} + a_3 T_1^{s0} T_2^{2s0} + a_7 T_1^{s0} T_2^{2s0}) \\
& \chi_{i0=i1} g_{1,i0,i0} g_{3,j0,i0} - \frac{1}{-1 + T_2} \\
& T_1^{-s0} T_2^{-s0} (-a_2 + a_4 - a_4 T_2^{s0} + a_2 T_1^{s0} T_2^{s0} - a_4 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0} - a_8 T_1^{s0} T_2^{s0} + a_4 T_1^{s0} T_2^{2s0} + a_8 T_1^{s0} T_2^{2s0}) \\
& \chi_{i0=i1} g_{1,i0,j0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (a_1 - 2 a_3 - a_1 T_1^{s0} + a_3 T_1^{s0} + a_3 T_2^{s0} - a_1 T_1^{s0} T_2^{s0} + 2 a_3 T_1^{s0} T_2^{s0} - a_5 T_1^{s0} T_2^{s0} + 2 a_7 T_1^{s0} T_2^{s0} + \\
& a_1 T_1^{2s0} T_2^{s0} - a_3 T_1^{2s0} T_2^{s0} + a_5 T_1^{2s0} T_2^{s0} - a_7 T_1^{2s0} T_2^{s0} - a_3 T_1^{s0} T_2^{2s0} - a_7 T_1^{s0} T_2^{2s0}) \chi_{i0=i1} g_{1,j0,i0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (a_2 - 2 a_4 - a_2 T_1^{s0} + a_4 T_1^{s0} + a_4 T_2^{s0} - a_2 T_1^{s0} T_2^{s0} + 2 a_4 T_1^{s0} T_2^{s0} - a_6 T_1^{s0} T_2^{s0} + 2 a_8 T_1^{s0} T_2^{s0} + \\
& a_2 T_1^{2s0} T_2^{s0} - a_4 T_1^{2s0} T_2^{s0} + a_6 T_1^{2s0} T_2^{s0} - a_8 T_1^{2s0} T_2^{s0} - a_4 T_1^{s0} T_2^{2s0} - a_8 T_1^{s0} T_2^{2s0}) \chi_{i0=i1} g_{1,j0,j0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (-a_1 + a_2 - a_2 T_1^{s0} + a_1 T_1^{s0} T_2^{s0} - a_2 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0} - a_6 T_1^{s0} T_2^{s0} + \\
& a_2 T_1^{2s0} T_2^{s0} + a_6 T_1^{2s0} T_2^{s0}) \chi_{i0=i1} g_{2,i0,i0} g_{3,j0,i0} - \frac{1}{-1 + T_2} \\
& T_1^{-s0} T_2^{-s0} (-a_3 + a_4 - a_4 T_1^{s0} + a_3 T_1^{s0} T_2^{s0} - a_4 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0} - a_8 T_1^{s0} T_2^{s0} + a_4 T_1^{2s0} T_2^{s0} + a_8 T_1^{2s0} T_2^{s0}) \\
& \chi_{i0=i1} g_{2,i0,j0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (a_1 - 2 a_2 + a_2 T_1^{s0} - a_1 T_2^{s0} + a_2 T_2^{s0} - a_1 T_1^{s0} T_2^{s0} + 2 a_2 T_1^{s0} T_2^{s0} - a_5 T_1^{s0} T_2^{s0} + 2 a_6 T_1^{s0} T_2^{s0} - \\
& a_1 T_1^{2s0} T_2^{s0} - a_2 T_1^{2s0} T_2^{s0} + a_5 T_1^{2s0} T_2^{s0} - a_6 T_1^{2s0} T_2^{s0} - a_1 T_1^{s0} T_2^{2s0} - a_2 T_1^{s0} T_2^{2s0}) \chi_{i0=i1} g_{2,i0,i0} g_{3,j0,i0} - \frac{1}{-1 + T_2} \\
& T_1^{-s0} T_2^{-s0} (-a_3 + a_4 - a_4 T_1^{s0} + a_3 T_1^{s0} T_2^{s0} - a_4 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0} - a_8 T_1^{s0} T_2^{s0} + a_4 T_1^{2s0} T_2^{s0} + a_8 T_1^{2s0} T_2^{s0}) \\
& \chi_{i0=i1} g_{2,i0,j0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{-s0} T_2^{-s0} (a_1 - 2 a_2 + a_2 T_1^{s0} - a_1 T_2^{s0} + a_2 T_2^{s0} - a_1 T_1^{s0} T_2^{s0} + 2 a_2 T_1^{s0} T_2^{s0} - a_5 T_1^{s0} T_2^{s0} + 2 a_6 T_1^{s0} T_2^{s0} - \\
& a_1 T_1^{2s0} T_2^{s0} - a_2 T_1^{2s0} T_2^{s0} + a_5 T_1^{2s0} T_2^{s0} - a_6 T_1^{2s0} T_2^{s0} - a_1 T_1^{s0} T_2^{2s0} - a_2 T_1^{s0} T_2^{2s0}) \chi_{i0=i1} g_{2,i0,i0} g_{3,j0,i0} - \frac{1}{-1 + T_2}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{-1 + T_2} \left(a_2 T_1^{s0} T_2^{s0} - a_6 T_1^{s0} T_2^{s0} + a_1 T_1^{s0} T_2^{s0} - a_2 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0} - a_6 T_1^{s0} T_2^{s0} \right) \chi_{i0=i1} g_{2,j0,i0} g_{3,j0,i0} - \\
& \frac{1}{-1 + T_2} T_1^{s0} T_2^{s0} \left(a_3 - 2 a_4 + a_4 T_1^{s0} - a_3 T_2^{s0} + a_4 T_2^{s0} - a_3 T_1^{s0} T_2^{s0} + 2 a_4 T_1^{s0} T_2^{s0} - a_7 T_1^{s0} T_2^{s0} + 2 a_8 T_1^{s0} T_2^{s0} - \right. \\
& \left. a_4 T_1^{s0} T_2^{s0} - a_8 T_1^{s0} T_2^{s0} + a_3 T_1^{s0} T_2^{s0} - a_4 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0} - a_8 T_1^{s0} T_2^{s0} \right) \chi_{i0=i1} g_{2,j0,j0} g_{3,j0,i0} + \\
& \frac{T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} (-a_1 + a_1 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0})}{-1 + T_2} g_{1,j1,i0} g_{2,i1,i0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} (-a_2 + a_2 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0})}{-1 + T_2} g_{1,j1,j0} g_{2,i1,i0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} (-a_3 + a_3 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0})}{-1 + T_2} g_{1,j1,i0} g_{2,i1,j0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} (-1 + T_1^{s1}) T_2^{s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0})}{-1 + T_2} g_{1,j1,j0} g_{2,i1,j0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} T_2^{s0} (-a_1 + a_1 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0})}{-1 + T_2} (-1 + T_2^{s1}) g_{1,i1,i0} g_{2,j1,i0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} T_2^{s0} (-a_2 + a_2 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0})}{-1 + T_2} (-1 + T_2^{s1}) g_{1,i1,j0} g_{2,j1,i0} g_{3,j0,i1} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_1 + a_1 T_1^{s0} T_2^{s0} + a_5 T_1^{s0} T_2^{s0})}{-1 + T_2} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,i0} g_{2,j1,i0} g_{3,j0,i1} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_2 + a_2 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0})}{-1 + T_2} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,j0} g_{2,j1,i0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} T_2^{s0} (-a_3 + a_3 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0})}{-1 + T_2} (-1 + T_2^{s1}) g_{1,i1,i0} g_{2,j1,j0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} T_2^{s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0})}{-1 + T_2} (-1 + T_2^{s1}) g_{1,i1,j0} g_{2,j1,j0} g_{3,j0,i1} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_3 + a_3 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0})}{-1 + T_2} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,i0} g_{2,j1,j0} g_{3,j0,i1} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0})}{-1 + T_2} (-2 + T_1^{s1} + T_2^{s1}) g_{1,j1,j0} g_{2,j1,j0} g_{3,j0,i1} + \\
& \frac{T_1^{s0} T_2^{s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0})}{-1 + T_2} \chi_{i0=i1} g_{3,j0,j0} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_3 + a_3 T_1^{s0} T_2^{s0} + a_7 T_1^{s0} T_2^{s0})}{-1 + T_2} \chi_{i0=i1} g_{1,j0,i0} g_{3,j0,j0} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_4 + a_4 T_1^{s0} T_2^{s0} + a_8 T_1^{s0} T_2^{s0})}{-1 + T_2} \chi_{i0=i1} g_{1,j0,j0} g_{3,j0,j0} - \\
& \frac{T_1^{s0} T_2^{s0} (-a_2 + a_2 T_1^{s0} T_2^{s0} + a_6 T_1^{s0} T_2^{s0})}{-1 + T_2} \chi_{i0=i1} g_{2,j0,i0} g_{3,j0,j0} -
\end{aligned}$$

$$\frac{T_1^{-s_0} T_2^{-s_0} \left(-a_4 + a_4 T_1^{s_0} T_2^{s_0} + a_8 T_1^{s_0} T_2^{s_0} \right) \chi_{i0=i1} g_{2,j0,j0} g_{3,j0,j0}}{-1 + T_2}, 0 \}$$

 KnotTheory: Loading precomputed data in PD4Knots`.

$$\gg \left\{ -\frac{1 - 5 T + 7 T^2 - 5 T^3 + T^4}{T^2}, 0 \right\}$$

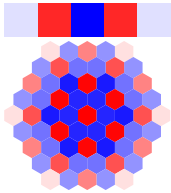
Out[*]=



$$\gg \left\{ -\frac{1 - 5 T + 7 T^2 - 5 T^3 + T^4}{T^2}, \right.$$

$$\frac{1}{T_1^4 T_2^4} \left(1 - 5 T_1 + 7 T_1^2 - 5 T_1^3 + T_1^4 - 5 T_2 + 20 T_1 T_2 - 10 T_1^2 T_2 - 10 T_1^3 T_2 + 20 T_1^4 T_2 - 5 T_1^5 T_2 + 7 T_2^2 - \right. \\ \left. 10 T_1 T_2^2 - 64 T_1^2 T_2^2 + 98 T_1^3 T_2^2 - 64 T_1^4 T_2^2 - 10 T_1^5 T_2^2 + 7 T_1^6 T_2^2 - 5 T_2^3 - 10 T_1 T_2^3 + 98 T_1^2 T_2^3 - 50 T_1^3 T_2^3 - \right. \\ \left. 50 T_1^4 T_2^3 + 98 T_1^5 T_2^3 - 10 T_1^6 T_2^3 - 5 T_1^7 T_2^3 + T_2^4 + 20 T_1 T_2^4 - 64 T_1^2 T_2^4 - 50 T_1^3 T_2^4 + 108 T_1^4 T_2^4 - 50 T_1^5 T_2^4 - \right. \\ \left. 64 T_1^6 T_2^4 + 20 T_1^7 T_2^4 + T_1^8 T_2^4 - 5 T_1 T_2^5 - 10 T_1^2 T_2^5 + 98 T_1^3 T_2^5 - 50 T_1^4 T_2^5 - 50 T_1^5 T_2^5 + 98 T_1^6 T_2^5 - \right. \\ \left. 10 T_1^7 T_2^5 - 5 T_1^8 T_2^5 + 7 T_1^2 T_2^6 - 10 T_1^3 T_2^6 - 64 T_1^4 T_2^6 + 98 T_1^5 T_2^6 - 64 T_1^6 T_2^6 - 10 T_1^7 T_2^6 + 7 T_1^8 T_2^6 - 5 T_1^3 T_2^7 + \right. \\ \left. 20 T_1^4 T_2^7 - 10 T_1^5 T_2^7 - 10 T_1^6 T_2^7 + 20 T_1^7 T_2^7 - 5 T_1^8 T_2^7 + T_1^4 T_2^8 - 5 T_1^5 T_2^8 + 7 T_1^6 T_2^8 - 5 T_1^7 T_2^8 + T_1^8 T_2^8 \right) \}$$

Out[*]=



```
In[*]:= XTrue = 1; XFalse = 0;
X_{\alpha \leq \alpha} = 1; X_{ph}^p /; p > 1 ^:= X_{ph}; X_{\alpha == \beta} /; OrderedQ[{ \beta, \alpha}] := X_{\beta == \alpha};
X_{\alpha \neq \beta} := X_{\alpha == \beta};
\delta_{i,j} := X_{i==j};
```

```

In[ ]:= sRules[ $\mathcal{E}$ ] := FixedPoint[CF[# /. bRules[{s0, i0, j0}]  $\cup$  bRules[{s1, i1, j1}]  $\cup$  {
    s02  $\rightarrow$  1,
    g3, $\alpha$ , $\beta$   $\mapsto$  g1, $\alpha$ , $\beta$ , g2, $\alpha$ , $\beta$   $\mapsto$   $\chi_{\alpha \leq \beta}$ ,
     $\chi_{i0=j1} \rightarrow 0$ ,  $\chi_{i1=j0} \rightarrow 0$ ,  $\chi_{i0=j0} \rightarrow 0$ ,
     $\chi_{j0=j1} \rightarrow \chi_{i0=i1}$ ,
     $\chi_{i1' \leq i0} \rightarrow \chi_{i1 \leq i0} - \chi_{i0=i1}$ ,  $\chi_{i0' = i1'} \rightarrow \chi_{i0=i1}$ ,
     $\chi_{i0=i1} \chi_{i1 \leq i0} \rightarrow \chi_{i0=i1}$ ,  $\chi_{i0=i1} \chi_{j1 \leq j0} \rightarrow \chi_{i0=i1}$ ,
     $\chi_{j0 \leq i0} \rightarrow 1 - \chi_{i0 \leq j0}$ ,  $\chi_{i1 \leq i0} \rightarrow 1 - \chi_{i0 \leq i1} + \chi_{i1=i0}$ ,
     $\chi_{j1 \leq i0} \rightarrow 1 - \chi_{i0 \leq j1} + \chi_{j1=i0}$ ,  $\chi_{i1 \leq i0} \rightarrow 1 - \chi_{i0 \leq i1} + \chi_{i1=i0}$ ,  $\chi_{j1 \leq j0} \rightarrow 1 - \chi_{j0 \leq j1} + \chi_{j1=j0}$ ,
     $\chi_{i0=i1} \gamma$  /; ! FreeQ[ $\gamma$ , s1 | i1 | j1]  $\mapsto \chi_{i0=i1} (\gamma /. \{s1 \rightarrow s0, i1 \rightarrow i0, j1 \rightarrow j0\})$ ,
     $\chi_{\alpha = \beta} g_{\beta, \gamma} \mapsto \chi_{\alpha = \beta} g_{\alpha, \gamma}$ ,  $\chi_{\alpha = \beta} g_{\gamma, \beta} \mapsto \chi_{\alpha = \beta} g_{\gamma, \alpha}$ 
}] &,
 $\mathcal{E}$ ]

```

```

In[ ]:= sRules[sRules[Residue[CF[ $\chi_{i0=i1} F1i + F2i + D_{\{s1, i1, j1\}}[tw]$ ], {T2, 1}]] /.
    {a1|2|3|4|6|8  $\rightarrow$  0, a7  $\rightarrow$  -a5} /. a5  $\rightarrow$  1 - T1s0]

```

```

Out[ ]:=
0

```

```

In[ ]:= eq = T2 - 1 == s0 T2(1-s0)/2 (T2s0 - 1); Simplify[{eq /. s0  $\rightarrow$  1, eq /. s0  $\rightarrow$  -1}]

```

```

Out[ ]:=
{True, True}

```

```

In[ ]:= nF2 = CF[CF[( $\chi_{i0=i1} F1i + F2i + D_{\{s1, i1, j1\}}[tw]$ ) /. {a1|2|3|4|6|8  $\rightarrow$  0, a7  $\rightarrow$  -a5} /. a5  $\rightarrow$  1 - T1s0]];
{nF1, nF2} = Simplify@{Coefficient[nF2,  $\chi_{i0=i1}$ ], nF2 /.  $\chi_{i0=i1} \rightarrow 0$ }
Table[ $\Theta[K]$  ==  $\Theta[K, F1 \rightarrow nF1, F2 \rightarrow nF2]$ , {K, AllKnots[{3, 8}]}]

```

Out[8]=

$$\begin{aligned}
& \left\{ \frac{s_0}{2} + s_0 T_2^{s_0} g_{1,i_0,i_0} g_{2,j_0,i_0} + \frac{(-1 + T_1^{s_0}) (1 - T_2^{s_0} - s_0 T_2^{2s_0} + s_0 T_2^{1+2s_0}) g_{1,j_0,i_0} g_{2,j_0,i_0}}{(-1 + T_2) (-1 + T_2^{s_0})} - \right. \\
& s_0 g_{1,i_0,i_0} g_{2,j_0,j_0} - \frac{(-1 + T_1^{s_0}) (1 - (1 + s_0) T_2^{s_0} + s_0 T_2^{1+s_0}) g_{1,j_0,i_0} g_{2,j_0,j_0}}{(-1 + T_2) (-1 + T_2^{s_0})} - \\
& s_0 g_{3,i_0,i_0} - s_0 (-1 + T_2^{s_0}) g_{2,j_0,i_0} g_{3,i_0,i_0} + 2 s_0 g_{2,j_0,j_0} g_{3,i_0,i_0} + \\
& \left. \frac{(-2 + s_0 - s_0 T_2 + 3 T_2^{s_0} - T_2^{2s_0} + T_1^{s_0} (2 - (3 + s_0) T_2^{s_0} + T_2^{2s_0} + s_0 T_2^{1+s_0})) g_{3,j_0,i_0}}{(-1 + T_2) (-1 + T_2^{s_0})} - \right. \\
& \frac{(-2 + (3 + s_0) T_2^{s_0} - T_2^{2s_0} - s_0 T_2^{1+s_0} + T_1^{s_0} (2 - 3 T_2^{s_0} - (-1 + s_0) T_2^{2s_0} + s_0 T_2^{1+2s_0})) g_{1,i_0,i_0} g_{3,j_0,i_0}}{(-1 + T_2) (-1 + T_2^{s_0})} - \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s_0})} (-1 + T_1^{s_0}) \\
& (-3 + s_0 - s_0 T_2 + (4 + s_0) T_2^{s_0} - T_2^{2s_0} - s_0 T_2^{1+s_0} + T_1^{s_0} (2 - (2 + s_0) T_2^{s_0} - s_0 T_2^{2s_0} + s_0 T_2^{1+s_0} + s_0 T_2^{1+2s_0})) \\
& g_{1,j_0,i_0} g_{3,j_0,i_0} + \frac{(-1 + T_1^{s_0}) g_{2,i_0,i_0} g_{3,j_0,i_0}}{-1 + T_2} + \\
& \frac{(-1 + s_0 - s_0 T_2 + T_2^{s_0} + T_1^{s_0} (1 - (1 + s_0) T_2^{s_0} + s_0 T_2^{1+s_0})) g_{2,i_0,j_0} g_{3,j_0,i_0}}{(-1 + T_2) (-1 + T_2^{s_0})} + \\
& \frac{(1 + s_0 - s_0 T_2 - T_2^{s_0} + T_1^{s_0} (-1 - (-1 + s_0) T_2^{s_0} + s_0 T_2^{1+s_0})) g_{2,j_0,i_0} g_{3,j_0,i_0}}{-1 + T_2} + \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s_0})} (1 - 2 s_0 + 2 s_0 T_2 + (-2 + s_0) T_2^{s_0} + T_2^{2s_0} - s_0 T_2^{1+s_0} + \\
& T_1^{s_0} (-1 + 2 (1 + s_0) T_2^{s_0} - (1 + s_0) T_2^{2s_0} - 2 s_0 T_2^{1+s_0} + s_0 T_2^{1+2s_0})) g_{2,j_0,j_0} g_{3,j_0,i_0} + \\
& s_0 g_{1,i_0,i_0} g_{3,j_0,j_0} + \frac{(-1 + T_1^{s_0}) (1 - (1 + s_0) T_2^{s_0} + s_0 T_2^{1+s_0}) g_{1,j_0,i_0} g_{3,j_0,j_0}}{(-1 + T_2) (-1 + T_2^{s_0})} - \\
& s_0 g_{2,i_0,i_0} g_{3,j_0,j_0} - s_0 T_2^{s_0} g_{2,j_0,i_0} g_{3,j_0,j_0}, \frac{1}{(-1 + T_2) (-1 + T_2^{s_1})} \\
& (-1 + T_1^{s_0}) \left((-1 + s_1 T_2^{s_0} - s_1 T_2^{1+s_0} + T_2^{s_1} + T_1^{s_1} (1 - T_2^{s_1} - s_1 T_2^{s_0+s_1} + s_1 T_2^{1+s_0+s_1})) g_{1,j_1,i_0} g_{2,i_1,i_0} - \right. \\
& (-1 + s_1 - s_1 T_2 + T_2^{s_1} + T_1^{s_1} (1 - (1 + s_1) T_2^{s_1} + s_1 T_2^{1+s_1})) g_{1,j_1,i_0} g_{2,i_1,j_0} - (-1 + T_2^{s_1})^2 g_{1,i_1,i_0} \\
& g_{2,j_1,i_0} + (2 - s_1 T_2^{s_0} + s_1 T_2^{1+s_0} - 3 T_2^{s_1} + T_2^{2s_1} + T_1^{s_1} (-1 + T_2^{s_1} + s_1 T_2^{s_0+s_1} - s_1 T_2^{1+s_0+s_1})) \\
& g_{1,j_1,i_0} g_{2,j_1,i_0} + (-1 + T_2^{s_1})^2 g_{1,i_1,i_0} g_{2,j_1,j_0} + \\
& \left. (-2 + s_1 - s_1 T_2 + 3 T_2^{s_1} - T_2^{2s_1} + T_1^{s_1} (1 - (1 + s_1) T_2^{s_1} + s_1 T_2^{1+s_1})) g_{1,j_1,i_0} g_{2,j_1,j_0} \right) g_{3,j_0,i_1} \}
\end{aligned}$$

Out[8]=

```

{True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True}

```

```

In[*]:= Table[Θ[K] == Θ[K,
  F1 → 
$$\frac{s0}{2} + s0 T_2^{s0} g_{1,i0,i0} g_{2,j0,i0} + \frac{(-1 + T_1^{s0}) (1 - T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+2s0}) g_{1,j0,i0} g_{2,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} -$$


$$s0 g_{1,i0,i0} g_{2,j0,j0} - \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{2,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} -$$


$$\frac{s0 g_{3,i0,i0} - s0 (-1 + T_2^{s0}) g_{2,j0,i0} g_{3,i0,i0} + 2 s0 g_{2,j0,j0} g_{3,i0,i0} + (-2 + s0 - s0 T_2 + 3 T_2^{s0} - T_2^{2s0} + T_1^{s0} (2 - (3 + s0) T_2^{s0} + T_2^{2s0} + s0 T_2^{1+s0})) g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} -$$


$$\frac{(-2 + (3 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + T_1^{s0} (2 - 3 T_2^{s0} - (-1 + s0) T_2^{2s0} + s0 T_2^{1+2s0})) g_{1,i0,i0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} -$$


$$\frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (-1 + T_1^{s0}) (-3 + s0 - s0 T_2 + (4 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + T_1^{s0}$$


$$(2 - (2 + s0) T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{1,j0,i0} g_{3,j0,i0} + \frac{(-1 + T_1^{s0}) g_{2,i0,i0} g_{3,j0,i0}}{-1 + T_2} +$$


$$\frac{(-1 + s0 - s0 T_2 + T_2^{s0} + T_1^{s0} (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,i0,j0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} +$$


$$\frac{(1 + s0 - s0 T_2 - T_2^{s0} + T_1^{s0} (-1 - (-1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,j0,i0} g_{3,j0,i0}}{-1 + T_2} +$$


$$\frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (1 - 2 s0 + 2 s0 T_2 + (-2 + s0) T_2^{s0} + T_2^{2s0} - s0 T_2^{1+s0} +$$


$$T_1^{s0} (-1 + 2 (1 + s0) T_2^{s0} - (1 + s0) T_2^{2s0} - 2 s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{2,j0,j0} g_{3,j0,i0} +$$


$$s0 g_{1,i0,i0} g_{3,j0,j0} + \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{3,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} -$$


$$s0 g_{2,i0,i0} g_{3,j0,j0} - s0 T_2^{s0} g_{2,j0,i0} g_{3,j0,j0},$$

  F2 → 
$$\frac{1}{(-1 + T_2) (-1 + T_2^{s1})} (-1 + T_1^{s0})$$


$$(( -1 + s1 T_2^{s0} - s1 T_2^{1+s0} + T_2^{s1} + T_1^{s1} (1 - T_2^{s1} - s1 T_2^{s0+s1} + s1 T_2^{1+s0+s1})) g_{1,j1,i0} g_{2,i1,i0} -$$


$$(-1 + s1 - s1 T_2 + T_2^{s1} + T_1^{s1} (1 - (1 + s1) T_2^{s1} + s1 T_2^{1+s1})) g_{1,j1,i0} g_{2,i1,j0} - (-1 + T_2^{s1})^2 g_{1,i1,i0}$$


$$g_{2,j1,i0} + (2 - s1 T_2^{s0} + s1 T_2^{1+s0} - 3 T_2^{s1} + T_2^{2s1} + T_1^{s1} (-1 + T_2^{s1} + s1 T_2^{s0+s1} - s1 T_2^{1+s0+s1}))$$


$$g_{1,j1,i0} g_{2,j1,i0} + (-1 + T_2^{s1})^2 g_{1,i1,i0} g_{2,j1,j0} +$$


$$(-2 + s1 - s1 T_2 + 3 T_2^{s1} - T_2^{2s1} + T_1^{s1} (1 - (1 + s1) T_2^{s1} + s1 T_2^{1+s1})) g_{1,j1,i0} g_{2,j1,j0} g_{3,j0,i1}$$

], {K, AllKnots[{3, 8}]}]

Out[*]=
{True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True}

```

$$\begin{aligned}
\text{In}[*]:= & \text{CF} \left[\frac{s0}{2} + s0 T_2^{s0} g_{1,i0,i0} g_{2,j0,i0} + \frac{(-1 + T_1^{s0}) (1 - T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+2s0}) g_{1,j0,i0} g_{2,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \right. \\
& s0 g_{1,i0,i0} g_{2,j0,j0} - \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{2,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& s0 g_{3,i0,i0} - s0 (-1 + T_2^{s0}) g_{2,j0,i0} g_{3,i0,i0} + 2 s0 g_{2,j0,j0} g_{3,i0,i0} + \\
& \frac{(-2 + s0 - s0 T_2 + 3 T_2^{s0} - T_2^{2s0} + T_1^{s0} (2 - (3 + s0) T_2^{s0} + T_2^{2s0} + s0 T_2^{1+s0})) g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& \frac{(-2 + (3 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + T_1^{s0} (2 - 3 T_2^{s0} - (-1 + s0) T_2^{2s0} + s0 T_2^{1+2s0})) g_{1,i0,i0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (-1 + T_1^{s0}) (-3 + s0 - s0 T_2 + (4 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + \\
& T_1^{s0} (2 - (2 + s0) T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{1,j0,i0} g_{3,j0,i0} + \frac{(-1 + T_1^{s0}) g_{2,i0,i0} g_{3,j0,i0}}{-1 + T_2} + \\
& \frac{(-1 + s0 - s0 T_2 + T_2^{s0} + T_1^{s0} (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,i0,j0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} + \\
& \frac{(1 + s0 - s0 T_2 - T_2^{s0} + T_1^{s0} (-1 - (-1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,j0,i0} g_{3,j0,i0}}{-1 + T_2} + \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (1 - 2 s0 + 2 s0 T_2 + (-2 + s0) T_2^{s0} + T_2^{2s0} - s0 T_2^{1+s0} + \\
& T_1^{s0} (-1 + 2 (1 + s0) T_2^{s0} - (1 + s0) T_2^{2s0} - 2 s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{2,j0,j0} g_{3,j0,i0} + \\
& s0 g_{1,i0,i0} g_{3,j0,j0} + \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{3,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& s0 g_{2,i0,i0} g_{3,j0,j0} - s0 T_2^{s0} g_{2,j0,i0} g_{3,j0,j0} /. s0 \rightarrow 1 \Big]
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& \frac{1}{2} + T_2 g_{1,i0,i0} g_{2,j0,i0} + (-1 + T_1) (1 + T_2) g_{1,j0,i0} g_{2,j0,i0} - \\
& g_{1,i0,i0} g_{2,j0,j0} + (1 - T_1) g_{1,j0,i0} g_{2,j0,j0} - g_{3,i0,i0} + (1 - T_2) g_{2,j0,i0} g_{3,i0,i0} + \\
& 2 g_{2,j0,j0} g_{3,i0,i0} + (-1 + 2 T_1) g_{3,j0,i0} + (2 - 2 T_1 - T_1 T_2) g_{1,i0,i0} g_{3,j0,i0} - \\
& (-1 + T_1) (-2 + 2 T_1 + T_1 T_2) g_{1,j0,i0} g_{3,j0,i0} + \frac{(-1 + T_1) g_{2,i0,i0} g_{3,j0,i0}}{-1 + T_2} + T_1 g_{2,i0,j0} g_{3,j0,i0} + \\
& (-2 + T_1 + T_1 T_2) g_{2,j0,i0} g_{3,j0,i0} + \frac{(1 + T_1 - 3 T_1 T_2 + T_1 T_2^2) g_{2,j0,j0} g_{3,j0,i0}}{-1 + T_2} + \\
& g_{1,i0,i0} g_{3,j0,j0} + (-1 + T_1) g_{1,j0,i0} g_{3,j0,j0} - g_{2,i0,i0} g_{3,j0,j0} - T_2 g_{2,j0,i0} g_{3,j0,j0}
\end{aligned}$$

$$\text{In}[*]:= \text{Factor} \left[(-1 + T_1) + (1 + T_1 - 3 T_1 T_2 + T_1 T_2^2) \right]$$

Out[*]=

$$T_1 (-2 + T_2) (-1 + T_2)$$

$$\begin{aligned}
In[*] := & CF \left[\frac{s0}{2} + s0 T_2^{s0} g_{1,i0,i0} g_{2,j0,i0} + \frac{(-1 + T_1^{s0}) (1 - T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+2s0}) g_{1,j0,i0} g_{2,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \right. \\
& s0 g_{1,i0,i0} g_{2,j0,j0} - \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{2,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& s0 g_{3,i0,i0} - s0 (-1 + T_2^{s0}) g_{2,j0,i0} g_{3,i0,i0} + 2 s0 g_{2,j0,j0} g_{3,i0,i0} + \\
& \frac{(-2 + s0 - s0 T_2 + 3 T_2^{s0} - T_2^{2s0} + T_1^{s0} (2 - (3 + s0) T_2^{s0} + T_2^{2s0} + s0 T_2^{1+s0})) g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& \frac{(-2 + (3 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + T_1^{s0} (2 - 3 T_2^{s0} - (-1 + s0) T_2^{2s0} + s0 T_2^{1+2s0})) g_{1,i0,i0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (-1 + T_1^{s0}) (-3 + s0 - s0 T_2 + (4 + s0) T_2^{s0} - T_2^{2s0} - s0 T_2^{1+s0} + \\
& T_1^{s0} (2 - (2 + s0) T_2^{s0} - s0 T_2^{2s0} + s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{1,j0,i0} g_{3,j0,i0} + \frac{(-1 + T_1^{s0}) g_{2,i0,i0} g_{3,j0,i0}}{-1 + T_2} + \\
& \frac{(-1 + s0 - s0 T_2 + T_2^{s0} + T_1^{s0} (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,i0,j0} g_{3,j0,i0}}{(-1 + T_2) (-1 + T_2^{s0})} + \\
& \frac{(1 + s0 - s0 T_2 - T_2^{s0} + T_1^{s0} (-1 - (-1 + s0) T_2^{s0} + s0 T_2^{1+s0})) g_{2,j0,i0} g_{3,j0,i0}}{-1 + T_2} + \\
& \frac{1}{(-1 + T_2) (-1 + T_2^{s0})} (1 - 2 s0 + 2 s0 T_2 + (-2 + s0) T_2^{s0} + T_2^{2s0} - s0 T_2^{1+s0} + \\
& T_1^{s0} (-1 + 2 (1 + s0) T_2^{s0} - (1 + s0) T_2^{2s0} - 2 s0 T_2^{1+s0} + s0 T_2^{1+2s0})) g_{2,j0,j0} g_{3,j0,i0} + \\
& s0 g_{1,i0,i0} g_{3,j0,j0} + \frac{(-1 + T_1^{s0}) (1 - (1 + s0) T_2^{s0} + s0 T_2^{1+s0}) g_{1,j0,i0} g_{3,j0,j0}}{(-1 + T_2) (-1 + T_2^{s0})} - \\
& s0 g_{2,i0,i0} g_{3,j0,j0} - s0 T_2^{s0} g_{2,j0,i0} g_{3,j0,j0} /. s0 \rightarrow -1 \Big]
\end{aligned}$$

$$\begin{aligned}
Out[*] = & \frac{1}{2} - \frac{g_{1,i0,i0} g_{2,j0,i0}}{T_2} + \frac{(-1 + T_1) g_{1,j0,i0} g_{2,j0,i0}}{T_1 T_2} + g_{1,i0,i0} g_{2,j0,j0} + \\
& g_{3,i0,i0} - \frac{(-1 + T_2) g_{2,j0,i0} g_{3,i0,i0}}{T_2} - 2 g_{2,j0,j0} g_{3,i0,i0} - \frac{(1 - T_1 + T_1 T_2) g_{3,j0,i0}}{T_1 T_2} - \\
& \frac{(-2 + T_1) g_{1,i0,i0} g_{3,j0,i0}}{T_1 T_2} - \frac{(-1 + T_1) (1 - T_1 + T_1 T_2) g_{1,j0,i0} g_{3,j0,i0}}{T_1^2 T_2} - \\
& \frac{(-1 + T_1) g_{2,i0,i0} g_{3,j0,i0}}{T_1 (-1 + T_2)} - g_{2,i0,j0} g_{3,j0,i0} + \frac{(-2 + T_1 + T_1 T_2) g_{2,j0,i0} g_{3,j0,i0}}{T_1 T_2} + \\
& \frac{(T_1 - T_2 - 2 T_1 T_2 + 2 T_1 T_2^2) g_{2,j0,j0} g_{3,j0,i0}}{T_1 (-1 + T_2) T_2} - g_{1,i0,i0} g_{3,j0,j0} + g_{2,i0,i0} g_{3,j0,j0} + \frac{g_{2,j0,i0} g_{3,j0,j0}}{T_2}
\end{aligned}$$

$$In[*] := \text{Factor} \left[-T_2 (T_1 - 1) + (T_1 - T_2 - 2 T_1 T_2 + 2 T_1 T_2^2) \right]$$

$$Out[*] = T_1 (-1 + T_2) (-1 + 2 T_2)$$