```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Theta"];
        Once[<< Theta.m]
        SetOptions[PolyPlot, ImageSize → Tiny];
        Clear[Θ]
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= RandomVK[n_] := {
          Prepend[#, 2 RandomInteger[1] - 1] & /@
           Partition[PermutationList[RandomPermutation[2 n], 2 n], 2],
          Table[RandomInteger[{-1, 1}], 2 n + 1]
         };
```

```
In[•]:= RandomVK[5]
```

```
Out[•]=
{{{1, 7, 3}, {-1, 6, 2}, {1, 4, 9}, {-1, 10, 1}, {1, 5, 8}},
 {-1, -1, 0, 1, 0, -1, 1, 1, -1, 1, 0}}
```

```
In[•]:= CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor@*PowerExpand;
```

```
In[•]:= Short[Options[Θ] = {F1 → (F1i = F1[{s0, i0, j0}]),
          F2 → (F2i = F2[{s0, i0, j0}, {s1, i1, j1}]), F3 → (F3i = F3[φ, k])}]
```

```
Out[•]//Short=
{F1 → s0/2 + s0 T2^s0 g1,i0,i0 g2,j0,i0 + <<19>> + <<1>>/<<1>> - s0 g2,i0,i0 g3,j0,j0 - s0 T2^s0 g2,j0,i0 g3,j0,j0,
 F2 → <<1>>, F3 → -φ/2 + φ g<<1>>}
```

In[•]:=
```
Θ[K_, opts___Rule] := Module[{X, φφ, n, A, Δ, G, ev, Θ, kk, k0, k1, f1, f2, f3},
  f1 = F1 /. {opts} /. Options[Θ];
  f2 = F2 /. {opts} /. Options[Θ];
  f3 = F3 /. {opts} /. Options[Θ];
  {X, φφ} = Rot[K]; n = Length[X]; A = IdentityMatrix[2 n + 1];
  Cases[X, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += {{-T^s, T^s - 1}, {0, -1}})];
  Δ = T^((-Total[φφ] - Total[X[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[
    ℰ /. {k_^+ :> k + 1, $ -> 2 n + 1} /. {g_ν_, α_, β_ :> (G[[α, β]] /. T -> T_ν), χ_True -> 1, χ_False -> 0}];
  Θ = ev@Sum[f1 /. Thread[{s0, i0, j0} -> X[[kk]]], {kk, n}];
  Θ += ev@Sum[f2 /. Thread[{s0, i0, j0} -> X[[k0]]] //.
      Thread[{s1, i1, j1} -> X[[k1]]], {k0, n}, {k1, n}];
  Θ += ev@Sum[f3 /. {φ -> φφ[[kk]], k -> kk}, {kk, Length@φφ}];
  Factor@{Δ, (Δ /. T -> T1) (Δ /. T -> T2) (Δ /. T -> T3) Θ}
 ];
```

In[•]:=
```
Θ[Knot[7, 3]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

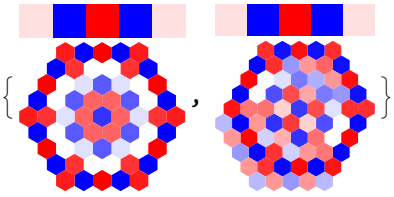
Out[•]=

$$\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\right.$$

$$\frac{1}{T_1^4\,T_2^4}\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-\right.$$
$$T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+$$
$$17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-$$
$$\left.\left.7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-T_1^6\,T_2^6+25\,T_1^8\,T_2^6-\right.\right.$$
$$\left.\left.25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\right\}$$

In[•]:=
```
PolyPlot /@ {Θ[Knot[7, 3]], Θ[Knot[7, 3], F3 -> 0]}
```

Out[•]=



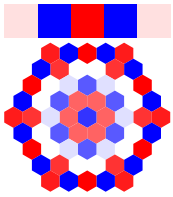
```
In[•]:= δ_i_,j_ := χ_i==j; χ_ph_^p_ /; p > 1 ^:= χ_ph;
        bRules[{s_, i_, j_}] := {  (* b for "push indices backwards" *)
            g_ν_,j^+,β_ :> g_ν,j,β - δ_j,β, g_ν_,i^+,β_ :> T_ν^-s g_ν,i,β + (1 - T_ν^-s) g_ν,j,β - T_ν^-s δ_i,β - (1 - T_ν^-s) δ_j,β,
            g_ν_,α_,i^+ :> T_ν^s g_ν,α,i + δ_α,i^+, g_ν_,α_,j^+ :> g_ν,α,j + (1 - T_ν^s) g_ν,α,i + δ_α,j^+
          };
        bRules[X___List] := Union @@ Table[bRules[c], {c, {X}}]

In[•]:= {g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}]

Out[•]=
        {χ_i==β + T_ν^s g_ν,i^+,β + (1 - T_ν^s) g_ν,j^+,β, χ_j==β + g_ν,j^+,β}

In[•]:= Expand[{g_ν,i,β, g_ν,j,β} /. gRules[{s, i, j}] /. bRules[{s, i, j}]]

Out[•]=
        {g_ν,i,β, g_ν,j,β}

In[•]:= D_{s_,i_,j_}[ℰ_] := CF[Expand[Plus[
                    ℰ /. {g_ν_,#,β_ :> g_ν,i^+,β, g_ν_,α_,# :> g_ν,α,i^+},
                    ℰ /. {g_ν_,#,β_ :> g_ν,j^+,β, g_ν_,α_,# :> g_ν,α,j^+},
                    -ℰ /. {g_ν_,#,β_ :> g_ν,i,β, g_ν_,α_,# :> g_ν,α,i},
                    -ℰ /. {g_ν_,#,β_ :> g_ν,j,β, g_ν_,α_,# :> g_ν,α,j}
                  ] /. bRules[{s, i, j}] /. bRules[{s0, i0, j0}]] /. χ_eq_Equal :> χ_Sort@eq //. {
              χ_j0==j1 → χ_i0==i1, χ_j0^+==j1^+ → χ_j0==j1,
              χ_i0==j1 → 0, χ_i1==j0 → 0, χ_i0^+==j1^+ → 0, χ_i0^+==i1^+ → χ_i0==i1, χ_i1^+==j0^+ → 0, χ_i0^+==j0^+ → 0,
              χ_i0==i1 A_ :> χ_i0==i1 (A /. {s1 → s0, i1 → i0, j1 → j0})
            } /. {χ_i0==j0 → 0, χ_i0^+==j0^+ → 0, χ_True → 1}
        ];
        B[ℰ_] := CF[
            (ℰ /. {g_ν_,#,β_ :> g_ν,$,β, g_ν_,α_,# :> g_ν,α,$}) - (ℰ /. {g_ν_,#,β_ :> g_ν,1,β, g_ν_,α_,# :> g_ν,α,1}) /. {
              g_,$,β_ → χ_$==β, g_,_,$ → 1, g_,α_,1 → χ_α==1} /. {χ_$==i0 → 0, χ_i0^+==1 → 0, χ_$==j0 → 0, χ_j0^+==1 → 0}
        ];

In[•]:= D_{s1,i1,j1}[g_1,#,i0]

Out[•]=
        -T_1^-s0 χ_i0==i1 - T_1^-s1 (-1 + T_1^s1) g_1,i1,i0 + T_1^-s1 (-1 + T_1^s1) g_1,j1,i0

In[•]:= D_{s1,i1,j1}[g_3,j0,#]

Out[•]=
        χ_j0==i1^+ + χ_j0==j1^+

In[•]:= B[g_3,j0,#]

Out[•]=
        1 - χ_j0==1
```

In[•]:=
```
tw = g1,#,i0 ;
⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$

$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-\right.$
$25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-$
$T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-$
$\left.T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ 0\right\}$

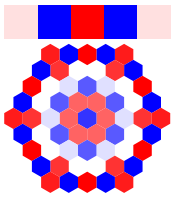
Out[•]=

In[•]:=
```
tw = g1,j0,#;
⊖[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2-3\,T+3\,T^2-3\,T^3+2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$

$\left(17-25\,T_1+25\,T_1^2-25\,T_1^3+17\,T_1^4-25\,T_2+12\,T_1\,T_2+12\,T_1^4\,T_2-25\,T_1^5\,T_2+25\,T_2^2-T_1^2\,T_2^2-7\,T_1^3\,T_2^2-T_1^4\,T_2^2+25\,T_1^6\,T_2^2-\right.$
$25\,T_2^3-7\,T_1^2\,T_2^3+6\,T_1^3\,T_2^3+6\,T_1^4\,T_2^3-7\,T_1^5\,T_2^3-25\,T_1^7\,T_2^3+17\,T_2^4+12\,T_1\,T_2^4-T_1^2\,T_2^4+6\,T_1^3\,T_2^4-12\,T_1^4\,T_2^4+6\,T_1^5\,T_2^4-$
$T_1^6\,T_2^4+12\,T_1^7\,T_2^4+17\,T_1^8\,T_2^4-25\,T_1\,T_2^5-7\,T_1^3\,T_2^5+6\,T_1^4\,T_2^5+6\,T_1^5\,T_2^5-7\,T_1^6\,T_2^5-25\,T_1^8\,T_2^5+25\,T_1^2\,T_2^6-T_1^4\,T_2^6-7\,T_1^5\,T_2^6-$
$\left.T_1^6\,T_2^6+25\,T_1^8\,T_2^6-25\,T_1^3\,T_2^7+12\,T_1^4\,T_2^7+12\,T_1^7\,T_2^7-25\,T_1^8\,T_2^7+17\,T_1^4\,T_2^8-25\,T_1^5\,T_2^8+25\,T_1^6\,T_2^8-25\,T_1^7\,T_2^8+17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



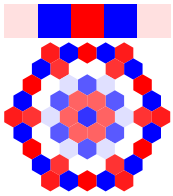
In[•]:=
```
tw = g1,#1,j0 g2,#1,j0 g3,i0,#1;
Θ[Knot[7, 3], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 3], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ 0\right\}$

Out[•]=

» $\left\{\frac{2 - 3\,T + 3\,T^2 - 3\,T^3 + 2\,T^4}{T^2},\ \frac{1}{T_1^4\,T_2^4}\right.$
$\left(17 - 25\,T_1 + 25\,T_1^2 - 25\,T_1^3 + 17\,T_1^4 - 25\,T_2 + 12\,T_1\,T_2 + 12\,T_1^4\,T_2 - 25\,T_1^5\,T_2 + 25\,T_2^2 - T_1^2\,T_2^2 - 7\,T_1^3\,T_2^2 - T_1^4\,T_2^2 + 25\,T_1^6\,T_2^2 - 25\,T_2^3 - 7\,T_1^2\,T_2^3 + 6\,T_1^3\,T_2^3 + 6\,T_1^4\,T_2^3 - 7\,T_1^5\,T_2^3 - 25\,T_1^7\,T_2^3 + 17\,T_2^4 + 12\,T_1\,T_2^4 - T_1^2\,T_2^4 + 6\,T_1^3\,T_2^4 - 12\,T_1^4\,T_2^4 + 6\,T_1^5\,T_2^4 - T_1^6\,T_2^4 + 12\,T_1^7\,T_2^4 + 17\,T_1^8\,T_2^4 - 25\,T_1\,T_2^5 - 7\,T_1^3\,T_2^5 + 6\,T_1^4\,T_2^5 + 6\,T_1^5\,T_2^5 - 7\,T_1^6\,T_2^5 - 25\,T_1^8\,T_2^5 + 25\,T_1^2\,T_2^6 - T_1^4\,T_2^6 - 7\,T_1^5\,T_2^6 - T_1^6\,T_2^6 + 25\,T_1^8\,T_2^6 - 25\,T_1^3\,T_2^7 + 12\,T_1^4\,T_2^7 + 12\,T_1^7\,T_2^7 - 25\,T_1^8\,T_2^7 + 17\,T_1^4\,T_2^8 - 25\,T_1^5\,T_2^8 + 25\,T_1^6\,T_2^8 - 25\,T_1^7\,T_2^8 + 17\,T_1^8\,T_2^8\right)\Big\}$

Out[•]=



In[•]:=
```
bas = List @@ Expand[(g1,#,i0 + g1,#,j0) (g2,#,i0 + g2,#,j0) (g3,i0⁺,# + g3,j0⁺,#)]
```

Out[•]=

$\{g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1},$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1},\ g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1},\ g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\}$

In[•]:=
```
Column[(# → Θ[Knot[7, 3], F1 → -B[#], F2 → D{s1,i1,j1}[#], F3 → 0][[2]]) & /@ bas]
Column[(# → {B[#], D{s1,i1,j1}[#]}) & /@ bas]
```

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$
$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1} \to 0$

Out[•]=

$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1} \to$
$\{0,\ -T_1^{-s0}\,T_2^{-s0}\left(-2 + T_1^{s1} + T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,i0,i1} + \chi_{i0=i1}\left(T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} + T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0} + g_{2,i0,i0}\left(-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0}\right) + g_{1,i0,i0}\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,i0} - T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0}\right) + g_{1,j0,i0}\left(T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{2,j0,i0} - T_1^{-s0}\left(-1 + T_1^{s0}\right)T_2^{-s0}\,g_{3,i0,i0} - T_1^{-s0}\left(-1 + T_1^{s0}\right)T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0}\right) + g_{2,j0,i0}\left(-T_1^{-s0}\,T_2^{-s0}\left(-1 + T_2^{s0}\right)g_{3,i0,i0} - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_2^{s0}\right)\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{3,j0,i0}\right)\right) - T_1^{-s0}\,T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)\left(-2 + T_1^{s1} + T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1} + g_{2,i1,i0}\left(T_1^{-s0}\left(-1 + T_1^{s1}\right)T_2^{-s0}\,g_{1,j1,i0}\,g_{3,i0,i1} + T_1^{-s0}\left(-1 + T_1^{s1}\right)T_2^{-s0}\left(-1 + T_1^{s0}\,T_2^{s0}\right)g_{1,j1,i0}\,g_{3,j0,i1}\right) +$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$g_{2,i1,i0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,j0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,i0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}+T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+\right.$$
$$g_{1,i0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{2,i0,i0}\,\left(-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{1,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\right.$$
$$\left.\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{2,j0,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$g_{2,i1,j0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,i0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,i0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+\right.$$
$$g_{2,i0,j0}\,\left(-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{2,j0,j0}\,\left(-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{1,i0,i0}$$
$$\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$g_{1,j0,i0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,j0}+T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,i0^+,\#1}\to\Big\{0,\;-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,i0,i1}-$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-2+T_1^{s1}+T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+$$
$$g_{2,i1,j0}\,\left(T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,g_{1,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,\left(-1+T_1^{s1}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{1,j1,j0}\,g_{3,j0,i1}\right)+$$
$$g_{1,i1,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,i0,i1}+T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,\left(-1+T_2^{s1}\right)\,g_{2,j1,j0}\,g_{3,j0,i1}\right)+$$
$$\chi_{i0=i1}\,\left(-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}+\right.$$
$$T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}+$$
$$g_{2,i0,j0}\,\left(-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,g_{3,i0,i0}-T_1^{-s0}\,\left(-1+T_1^{s0}\right)\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+g_{1,i0,j0}$$
$$\left(T_1^{-s0}\,T_2^{-s0}\,g_{2,i0,j0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}\right)+$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}+g_{2,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)+$$
$$g_{1,j0,j0}\,\left(T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{2,j0,j0}+T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,i0,i0}-T_1^{-s0}\,T_2^{-s0}\,g_{3,i0,j0}+$$
$$\left.\left.T_1^{-s0}\,T_2^{-s0}\,\left(-2+T_1^{s0}+T_2^{s0}\right)\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,i0}-T_1^{-s0}\,T_2^{-s0}\,\left(-1+T_1^{s0}\,T_2^{s0}\right)\,g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to\Big\{0,\;\chi_{i0=i1}$$
$$\left(\left(1-g_{1,i0,i0}-g_{2,i0,i0}\right)\,g_{3,j0,i0}+\left(1-T_2^{s0}\right)\,g_{2,j0,i0}\,g_{3,j0,i0}+g_{1,j0,i0}\,\left(g_{2,j0,i0}+\left(1-T_1^{s0}\right)\,g_{3,j0,i0}\right)\right)+$$
$$\left(-1+T_1^{s1}\right)\,g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)\,g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$\left(2-T_1^{s1}-T_2^{s1}\right)\,g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}\Big\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}\to\Big\{0,\;\left(-1+T_1^{s1}\right)\,g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)\,g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+$$
$$\left(2-T_1^{s1}-T_2^{s1}\right)\,g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}+\chi_{i0=i1}\,\left(\left(-1+T_1^{s0}-g_{1,i0,j0}+\left(1-T_1^{s0}\right)\,g_{2,i0,i0}\right)\,g_{3,j0,i0}+\right.$$
$$\left.g_{1,j0,j0}\,\left(g_{2,j0,i0}+\left(1-T_1^{s0}\right)\,g_{3,j0,i0}\right)+g_{2,j0,i0}\,\left(\left(-2+T_1^{s0}+T_2^{s0}\right)\,g_{3,j0,i0}-g_{3,j0,j0}\right)\right)\Big\}$$

$$g_{1,\#1,i0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\to \left\{0,\ \left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,j0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}+\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,j0}\,g_{3,j0,i1}+\chi_{i0==i1}\left(\left(-1+T_2^{s0}+\left(1-T_2^{s0}\right)g_{1,i0,i0}-g_{2,i0,j0}\right)g_{3,j0,i0}+\left(1-T_2^{s0}\right)g_{2,j0,j0}\,g_{3,j0,i0}+g_{1,j0,i0}\left(g_{2,j0,j0}+\left(-2+T_1^{s0}+T_2^{s0}\right)g_{3,j0,i0}-g_{3,j0,j0}\right)\right)\right\}$$

$$g_{1,\#1,j0}\,g_{2,\#1,j0}\,g_{3,j0^+,\#1}\to \left\{0,\ \left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,j0}\,g_{3,j0,i1}+\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+\left(2-T_1^{s1}-T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,j0}\,g_{3,j0,i1}+\chi_{i0==i1}\left(\left(2-T_1^{s0}-T_2^{s0}+\left(1-T_2^{s0}\right)g_{1,i0,j0}+\left(1-T_1^{s0}\right)g_{2,i0,j0}\right)g_{3,j0,i0}+g_{2,j0,j0}\left(\left(-2+T_1^{s0}+T_2^{s0}\right)g_{3,j0,i0}-g_{3,j0,j0}\right)+g_{1,j0,j0}\left(g_{2,j0,j0}+\left(-2+T_1^{s0}+T_2^{s0}\right)g_{3,j0,i0}-g_{3,j0,j0}\right)+g_{3,j0,j0}\right)\right\}$$

In[•]:=

```
tw = ((1 - T1^s0) g1,#1,i0 g2,#1,i0 g3,j0^+,#1 + (-1 + T1^s0) g1,#1,j0 g2,#1,i0 g3,j0^+,#1) / (T2 - 1)
{D{s1,i1,j1}[tw], B[tw]}
Θ[Knot[7, 6], F1 → -B[tw], F2 → D{s1,i1,j1}[tw], F3 → 0] // Echo // PolyPlot
Θ[Knot[7, 6], F1 → F1i - B[tw], F2 → F2i + D{s1,i1,j1}[tw]] // Echo // PolyPlot
```

Out[•]=

$$\frac{\left(1-T_1^{s0}\right)g_{1,\#1,i0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}+\left(-1+T_1^{s0}\right)g_{1,\#1,j0}\,g_{2,\#1,i0}\,g_{3,j0^+,\#1}}{-1+T_2}$$

Out[•]=

$$\left\{-\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,i0}\,g_{2,j0,i0}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,j0,j0}\,g_{2,j0,i0}}{-1+T_2}+\frac{\left(-2+T_1^{s0}\right)\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{3,j0,i0}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,i0,i0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{1,i0,j0}\,g_{3,j0,i0}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)^2\chi_{i0==i1}\,g_{1,j0,i0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)^2\chi_{i0==i1}\,g_{1,j0,j0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-2+T_1^{s0}\right)\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,i0,i0}\,g_{3,j0,i0}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\left(-3+T_1^{s0}+2\,T_2^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,i0}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)g_{1,j1,i0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\left(-1+T_1^{s1}\right)g_{1,j1,j0}\,g_{2,i1,i0}\,g_{3,j0,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\left(-1+T_2^{s1}\right)g_{1,i1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}+\frac{\left(-1+T_1^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,i0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\left(-2+T_1^{s1}+T_2^{s1}\right)g_{1,j1,j0}\,g_{2,j1,i0}\,g_{3,j0,i1}}{-1+T_2}-\frac{\left(-1+T_1^{s0}\right)\chi_{i0==i1}\,g_{2,j0,i0}\,g_{3,j0,j0}}{-1+T_2},\ 0\right\}$$

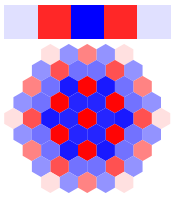
» $$\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\ 0\right\}$$

Out[•]=

» $$\left\{-\frac{1-5\,T+7\,T^2-5\,T^3+T^4}{T^2},\right.$$

$$\frac{1}{T_1^4\,T_2^4}\,\left(1-5\,T_1+7\,T_1^2-5\,T_1^3+T_1^4-5\,T_2+20\,T_1\,T_2-10\,T_1^2\,T_2-10\,T_1^3\,T_2+20\,T_1^4\,T_2-5\,T_1^5\,T_2+7\,T_2^2-\right.$$
$$10\,T_1\,T_2^2-64\,T_1^2\,T_2^2+98\,T_1^3\,T_2^2-64\,T_1^4\,T_2^2-10\,T_1^5\,T_2^2+7\,T_1^6\,T_2^2-5\,T_2^3-10\,T_1\,T_2^3+98\,T_1^2\,T_2^3-50\,T_1^3\,T_2^3-$$
$$50\,T_1^4\,T_2^3+98\,T_1^5\,T_2^3-10\,T_1^6\,T_2^3-5\,T_1^7\,T_2^3+T_2^4+20\,T_1\,T_2^4-64\,T_1^2\,T_2^4-50\,T_1^3\,T_2^4+108\,T_1^4\,T_2^4-50\,T_1^5\,T_2^4-$$
$$64\,T_1^6\,T_2^4+20\,T_1^7\,T_2^4+T_1^8\,T_2^4-5\,T_1\,T_2^5-10\,T_1^2\,T_2^5+98\,T_1^3\,T_2^5-50\,T_1^4\,T_2^5-50\,T_1^5\,T_2^5+98\,T_1^6\,T_2^5-$$
$$\left.\left.10\,T_1^7\,T_2^5-5\,T_1^8\,T_2^5+7\,T_1^2\,T_2^6-10\,T_1^3\,T_2^6-64\,T_1^4\,T_2^6+98\,T_1^5\,T_2^6-64\,T_1^6\,T_2^6-10\,T_1^7\,T_2^6+7\,T_1^8\,T_2^6-5\,T_1^3\,T_2^7+\right.\right.$$
$$\left.\left.20\,T_1^4\,T_2^7-10\,T_1^5\,T_2^7-10\,T_1^6\,T_2^7+20\,T_1^7\,T_2^7-5\,T_1^8\,T_2^7+T_1^4\,T_2^8-5\,T_1^5\,T_2^8+7\,T_1^6\,T_2^8-5\,T_1^7\,T_2^8+T_1^8\,T_2^8\right)\right\}$$

Out[•]=



In[•]:=
```
CF[Residue[CF[χ_{i0==i1} F1i + F2i + D_{s1,i1,j1}[tw]], {T_2, 1}] /.
  {g_{3,α_,β_} :> g_{1,α,β}, g_{2,α_,β_} :> χ_{α≤β}} /. {χ_{i0≤i0} → 1, χ_{j0≤j0} → 1} ]
```

Out[•]=

$$\left(-1+T_1^{s0}\right)\,\chi_{i0==i1}\,\left(-1+\chi_{i0\le j0}-\chi_{j0\le i0}+T_1^{s0}\,\chi_{j0\le i0}\right)\,g_{1,j0,i0}-$$
$$\left(-1+T_1^{s0}\right)\,\chi_{i0==i1}\,g_{1,i0,j0}\,g_{1,j0,i0}-\left(-1+T_1^{s0}\right)^2\,\chi_{i0==i1}\,g_{1,j0,i0}^2-$$
$$\left(-2+T_1^{s0}\right)\,\left(-1+T_1^{s0}\right)\,\chi_{i0==i1}\,g_{1,j0,i0}\,g_{1,j0,j0}-\left(-1+T_1^{s0}\right)\,\left(-1+T_1^{s1}\right)\,\left(\chi_{i1\le j0}-\chi_{j1\le j0}\right)\,g_{1,j0,i1}\,g_{1,j1,i0}+$$
$$\left(-1+T_1^{s0}\right)\,\left(-1+T_1^{s1}\right)\,\left(\chi_{i1\le i0}-\chi_{j1\le i0}\right)\,g_{1,j0,i1}\,g_{1,j1,j0}$$