Hidden Utilities and TEX

Headers

pdf

Pensieve header: Quantum sl_n following Rosso and Yamane. Continuing Doubling.nb from Pensieve 1708?

pdf

Notes.
1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$’’, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso’s X_i and Y_j. Rosso’s $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow$ xij. Likewise Rosso’s $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow$ yij.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i.

pdf

```
In[•]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= MakePDF[]
```

pdf

```
In[•]:= δij__ := KroneckerDelta[ij];
```

pdf

```
In[•]:= CF = Expand;
```

pdf

```
In[•]:= x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
y[i_, j_] := y <> i <> j; b[i_] := b <> i; B[i_] := B <> i;
```

pdf

```
In[•]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

```
In[•]:= {τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[•]=

```
List["x", List[3, 4]]
```

pdf

```
In[∘]:= UCF[ℰ_] :=
  Module[{F}, Expand@Collect[ℰ //. U_i_[l___, _[0], r___] :→ U_i[l, r], U_[___], F] /. F → CF]
```

pdf

The following ordering agrees with Rosso’s; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

pdf

```
In[∘]:= (*q=ℯ^(ℏ γ ϵ); *)
```

pdf

```
In[∘]:= $n = 4;
$PBWGens = Flatten@{
    Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
    Table[As[i, 1, $n - 1], A[i][_]], (* Ai[n]=ℯ^(-ℏϵai) *)
    Table[As[i, 1, $n - 1], a[i]],
    Table[As[i, 1, $n - 1], As[j, i + 1, $n], y[i, j]],
    Table[As[i, 1, $n - 1], B[i][_]], (* Bi[n]=ℯ^(-ℏγai) *)
    Table[As[i, 1, $n - 1], b[i]]
   }
(#_U = U_0@#) & /@ $PBWGens;
$PBWRule = Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $PBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
```

Time: 0.06 seconds Out[∘]=

pdf

```
{x12, x13, x14, x23, x24, x34, A1[_], A2[_], A3[_], a1, a2,
 a3, y12, y13, y14, y23, y24, y34, B1[_], B2[_], B3[_], b1, b2, b3}
```

pdf

```
In[∘]:= m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
  ];
m_s__[ℰ_] := m_s[UCF[ℰ]];
```

pdf

```
In[∘]:= Δ_{i_→j_,k_}[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
          U_i[] → U_j[] U_k[],
          U_i[f_, fs___] :→ m_{k,k1→k}@ m_{j,j1→j}[Δ_{j,k}[f] Δ_{i→j1,k1}[U_i[fs]]]
        }];
        Δ_{i_→j_,k_,l__}[ℰ_] := ℰ // Δ_{i→j,k} // Δ_{k→k,l}
```

```
In[∘]:= S_{i_}[ℰ_] := UCF@Module[{i1}, ℰ /. {
          U_i[f_, fs___] :→ m_{i1,i→i}[S_i[f] S_{i1}[U_{i1}[fs]]]
        }];
```

The 𝔸-algebra

pdf

```
In[∘]:= RR_{k_}[ai_, aj_] /; τ[ai] == τ[aj] == "a" := U_k[aj, ai];
RR_{k_}[ai_, Aj_[n_]] /;  τ[ai] == "a" ∧ τ[Aj] == "A" := U_k[Aj[n], ai];
RR_{k_}[Ai_[m_], Aj_[n_]] /;  τ[Ai] == τ[Aj] == "A" := U_k[Aj[n], Ai[m]];
RR_{k_}[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = ι@xij; {m} = ι@am;
    U_k[xij, am] + γ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[xij] / 2
  ];
RR_{k_}[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@Am; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[xij, Am[n]]
  ]
```

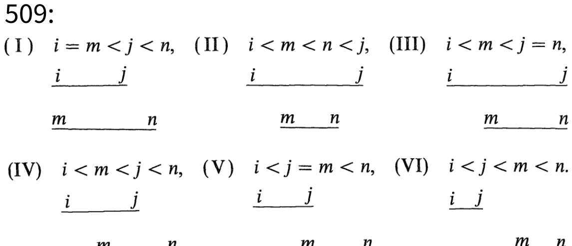
```
In[∘]:= RR_0[a[1], x[1, 2]]
```

Time: 0.02 seconds Out[∘]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

pdf

We take the reduction rules RR for xij's from Yamane page 509:



$$
\begin{array}{ll}
q^{-2}e_{ij}e_{mn}-e_{mn}e_{ij}=0 & \text{if } ((i,j),\,(m,n))\in C_{(\mathrm{I})}\bigcup C_{(\mathrm{III})}.\\
[e_{ij},\,e_{mn}]=0 & \text{if } ((i,j),\,(m,n))\in C_{(\mathrm{II})}\bigcup C_{(\mathrm{VI})}.\\
[e_{ij},\,e_{mn}]=(q^2-q^{-2})e_{in}e_{mj} & \text{if } ((i,j),\,(m,n))\in C_{(\mathrm{IV})}.\\
q^2e_{ij}e_{mn}-e_{mn}e_{ij}=qe_{in} & \text{if } ((i,j),\,(m,n))\in C_{(\mathrm{V})}.
\end{array}
$$

pdf

In[∘]:=
```
RR_k_[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[xij, xmn], (* I∨III *)
   (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
   i < m < j ∧ n > j, U_k[xij, xmn] + (q^-2 - q^2) U_k[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
 ]
```

In[∘]:=
```
U_1[x34] U_2[x23] U_3[x12] // m_{1,2→2}
```

Time: 0.05 seconds Out[∘]=

$$-q\, U_2[x24]\, U_3[x12] + q^2\, U_2[x23, x34]\, U_3[x12]$$

In[∘]:=
```
U_1[x34] U_2[x23] U_3[x12] // m_{1,2→2} // m_{2,3→0}
```

Time: 0.06 seconds Out[∘]=

$$q^2\, U_0[x14] - q^3\, U_0[x12, x24] - q^3\, U_0[x13, x34] + q^4\, U_0[x12, x23, x34]$$

In[∘]:=
```
Module[{Bas = Echo@Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → (P // m_{1,2→2} // m_{2,3→0}) == (P // m_{2,3→2} // m_{1,2→0})
   ]]
]
```

» {x12, a1}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U_1[X] U_2[Y] U_3[Z];
    {X, Y, Z} → (P // m_{1,2→2} // m_{2,3→0}) == (P // m_{2,3→2} // m_{1,2→0})
   ]]
]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
 A4[-1], A4[1], A4[0], A3[2], A2[-2], A4[1], A2[2], A4[-2], A4[0], A4[-2]}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = Echo@Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
  DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[W] U2[X] U3[Y] U4[Z];
     {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
       (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
       (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
    ]]
]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4}

Out[∘]=

{}

pdf

The Co-Product and Co-Associativity

pdf

In[∘]:=
```
Do[As[i, 1, $n - 1],
   Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
   Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
   Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
  ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
     q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
    ]]
]
```

In[∘]:=
```
Short[{Δ4,5[a2], Δ7,9[A2[77]], Δ1,2[x12], Δ1,2[x15]}, 8]
```

Time: 0.02 seconds Out[∘]=

$$\left\{\Delta_{4,5}[a2],\ \Delta_{7,9}[A2[77]],\ U_1\left[A1\left[-\frac{1}{2}\right]\right] U_2[x12] + U_1[x12]\, U_2\left[A1\left[\frac{1}{2}\right]\right],\ \Delta_{1,2}[x15]\right\}$$

In[∘]:=
```
Δ1→1,2[U1[x13, a2] U0[x34]]
```

Time: 0.05 seconds Out[∘]=

$$U_0[x34]\, U_1[A1[-1], A2[-1], a2]\, U_2[x13] + U_0[x34]\, U_1[A1[-1], A2[-1]]\, U_2[x13, a2] +$$
$$\left(-\frac{1}{q^2}+q^2\right) U_0[x34]\, U_1[x12, A2[-1], a2]\, U_2[x23, A1[1]] + U_0[x34]\, U_1[x13, a2]\, U_2[A1[1], A2[1]] +$$
$$\left(-\frac{1}{q^2}+q^2\right) U_0[x34]\, U_1[x12, A2[-1]]\, U_2[x23, A1[1], a2] + U_0[x34]\, U_1[x13]\, U_2[A1[1], A2[1], a2]$$

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
   ]]
]
```

» {x12, a1, A1[-1], A1[1], A1[-1], A1[0], A1[0], A1[2], A1[-1], A1[2], A1[2], A1[-1]}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} → (P // m1,2→1 // Δ1→1,2) == (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
   ]]
]
```

» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A1[0],
 A1[-1], A1[-1], A2[0], A3[1], A2[2], A1[1], A2[0], A1[-2], A2[1]}

Out[∘]=

{}

In[∘]:=
```
UCF@Module[{P = U1[x13] U2[x12]},
  (P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
```

Time: 0.03 seconds Out[∘]=

0

pdf

The antipode

In[∘]:=
```
Do[As[i, 1, $n - 1],
  Sj_[a[i]] = -Uj[a[i]];
  Sj_[A[i][n_]] = Uj[A[i][-n]];
  Sj_[x[i, i + 1]] = -q^-2 Uj[x[i, i + 1]] (*?*)
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Sj_[x[i, n]] = mj1,j→j[
   q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q^-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

In[∘]:=
```
U1[x12] // Δ1→2,3 // S2 // m2,3→1
```

Time: 0.03 seconds Out[∘]=

0

In[•]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "x" ∨ τ[#] == "a") &], P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U_1[X];
    X → (P // Δ_{1→2,3} // S_2 // m_{2,3→1})
   ]]
]
```

```
» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A1[-1], A1[-1],
  A2[-1], A1[-2], A1[-2], A2[2], A3[2], A3[2], A3[-2], A2[-2]}
```

Out[•]=
```
{x12 → 0, x13 → 0, x14 → 0, x23 → 0, x24 → 0, x34 → 0, a1 → 0, a2 → 0, a3 → 0,
 A1[-1] → U_1[], A1[-1] → U_1[], A2[-1] → U_1[], A1[-2] → U_1[], A1[-2] → U_1[],
 A2[2] → U_1[], A3[2] → U_1[], A3[2] → U_1[], A3[-2] → U_1[], A2[-2] → U_1[]}
```

The 𝔹-algebra

pdf

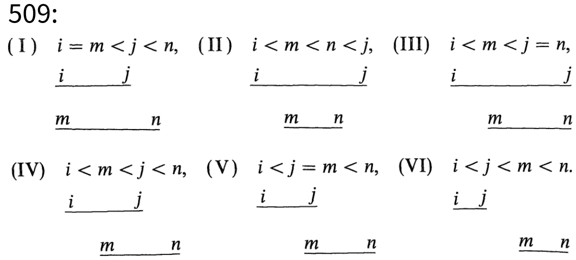
In[•]:=
```
RR_k_[bi_, bj_] /; τ[bi] == τ[bj] == "b" := U_k[bj, bi];
RR_k_[bi_, Bj_[n_]] /;  τ[bi] == "b" ∧ τ[Bj] == "B"  := U_k[Bj[n], bi];
RR_k_[Bi_[m_], Bj_[n_]] /;  τ[Bi] == τ[Bj] == "B"  := U_k[Bj[n], Bi[m]];
RR_k_[bm_, yij_] /; τ[bm] == "b" ∧ τ[yij] == "y" := Module[{i, j, m},
   {i, j} = ι@yij; {m} = ι@bm;
   U_k[yij, bm] + ϵ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[yij] / 2
  ];
RR_k_[Bm_[n_], yij_] /; τ[Bm] == "B" ∧ τ[yij] == "y" := Module[{i, j, m},
  {i, j} = ι@yij; {m} = ι@Bm; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[yij, Bm[n]]
 ]
```

In[•]:=
```
RR_0[b[1], y[1, 2]]
```

Time: 0.05 seconds Out[•]=
```
ϵ U_0[y12] + U_0[y12, b1]
```

pdf

We take the reduction rules RR for xij’s from Yamane page 509:



$$\begin{array}{ll} q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 & \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}. \\ [e_{ij},\ e_{mn}] = 0 & \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}. \\ [e_{ij},\ e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} & \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{IV})}. \\ q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} & \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{V})}. \end{array}$$

pdf

```
In[∘]:= RR_k_[ymn_, yij_] /; τ[yij] == τ[ymn] == "y" := Module[{i, j, m, n},
   {i, j} = ι@yij; {m, n} = ι@ymn;
   Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[yij, ymn], (* I∨III *)
    (i < m < j ∧ n < j) ∨ (j < m), U_k[yij, ymn], (* II∨VI *)
    i < m < j ∧ n > j, U_k[yij, ymn] + (q^-2 - q^2) U_k[y[i, n], y[m, j]], (* IV *)
    j == m, q^2 U_k[yij, ymn] - q U_k[y[i, n]] (* V *)
   ]
  ]
```

```
In[∘]:= U1[y34] U2[y23] U3[y12] // m1,2→2
```

Time: 0.02 seconds Out[∘]=

$-q\,U_2[y24]\,U_3[y12] + q^2\,U_2[y23,\ y34]\,U_3[y12]$

```
In[∘]:= U1[y34] U2[y23] U3[y12] // m1,2→2 // m2,3→0
```

Time: 0.02 seconds Out[∘]=

$q^2\,U_0[y14] - q^3\,U_0[y12,\ y24] - q^3\,U_0[y13,\ y34] + q^4\,U_0[y12,\ y23,\ y34]$

```
In[∘]:= Module[{Bas = Echo@Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
    ]]
 ]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3}

Out[∘]=

{}

```
In[∘]:= Module[{Bas = Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
  Bas = Echo@Join[Bas, Table[B[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
    ]]
 ]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3, B2[1],
 B3[0], B2[0], B2[-2], B1[1], B2[2], B1[1], B1[1], B3[-2], B1[1]}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = Echo@Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
 DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
  ]]
]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3}

Out[∘]=

{}

pdf

The Co-Product and Co-Associativity

pdf

In[∘]:=
```
Do[As[i, 1, $n - 1],
  Δj_,k_[b[i]] = Uj[b[i]] Uk[] + Uj[] Uk[b[i]];
  Δj_,k_[B[i][n_]] = Uj[B[i][n]] Uk[B[i][n]];
  Δj_,k_[y[i, i + 1]] = Uj[y[i, i + 1]] Uk[B[i][1]] + Uj[B[i][-1]] Uk[y[i, i + 1]]
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Δj_,k_[y[i, n]] = mk,k1→k[mj,j1→j[
    q Δj,k[y[i, n - 1]] Δj1,k1[y[n - 1, n]] - q^-1 Δj,k[y[n - 1, n]] Δj1,k1[y[i, n - 1]]
   ]]
]
```

In[∘]:=
```
Short[{Δ4,5[b2], Δ7,9[B2[77]], Δ1,2[y12], Δ1,2[y15]}, 8]
```

Time: 0.01 seconds Out[∘]=

$\{U_4[b2]\,U_5[\,]+U_4[\,]\,U_5[b2],\ U_7[B2[77]]\,U_9[B2[77]],\ U_1[B1[-1]]\,U_2[y12]+U_1[y12]\,U_2[B1[1]],\ \Delta_{1,2}[y15]\}$

In[∘]:=
```
Δ1→1,2[U1[y13, b2] U0[y34]]
```

Time: 0.05 seconds Out[∘]=

$$U_0[y34]\,U_1[B1[-1],B2[-1],b2]\,U_2[y13]+U_0[y34]\,U_1[B1[-1],B2[-1]]\,U_2[y13,b2]+\left(-\frac{1}{q^2}+q^2\right)U_0[y34]\,U_1[y12,B2[-1],b2]\,U_2[y23,B1[1]]+U_0[y34]\,U_1[y13,b2]\,U_2[B1[1],B2[1]]+\left(-\frac{1}{q^2}+q^2\right)U_0[y34]\,U_1[y12,B2[-1]]\,U_2[y23,B1[1],b2]+U_0[y34]\,U_1[y13]\,U_2[B1[1],B2[1],b2]$$

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
 Bas = Echo@Join[Bas, Table[B[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
   ]]
]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3, B3[2], B2[1], B1[0], B3[-1], B1[-2], B2[-1], B3[1], B2[0], B3[2], B3[-2]}

Out[∘]=
{}

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
 Bas = Echo@Join[Bas, Table[B[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} → (P // m1,2→1 // Δ1→1,2) == (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
   ]]
]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3, B3[1], B2[-1], B2[-2], B3[-2], B2[0], B3[0], B1[1], B2[0], B3[1], B3[1]}

Out[∘]=
{}

pdf

The antipode

In[∘]:=
```
Do[As[i, 1, $n - 1],
  Sj_[b[i]] = -Uj[b[i]];
  Sj_[B[i][n_]] = Uj[B[i][-n]];
  Sj_[y[i, i + 1]] = -q^-2 Uj[y[i, i + 1]] (*?*)
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Sj_[y[i, n]] = mj1,j→j[
   q Sj[y[i, n - 1]] Sj1[y[n - 1, n]] - q^-1 Sj[y[n - 1, n]] Sj1[y[i, n - 1]]
  ]
]
```

In[∘]:=
```
U1[y12] // Δ1→2,3 // S2 // m2,3→1
```

Time: 0.00 seconds Out[∘]=
0

In[∘]:=
```
Module[{Bas = Select[$PBWGens, (τ[#] == "y" ∨ τ[#] == "b") &], P},
 Bas = Echo@Join[Bas, Table[B[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
   ]]
]
```

» {y12, y13, y14, y23, y24, y34, b1, b2, b3, B2[2], B2[2], B3[-1], B2[1], B2[1], B1[0], B3[-1], B1[2], B3[0], B1[-1]}

Out[∘]=
{y12 → 0, y13 → 0, y14 → 0, y23 → 0, y24 → 0, y34 → 0, b1 → 0, b2 → 0, b3 → 0, B2[2] → U_1[], B2[2] → U_1[], B3[-1] → U_1[], B2[1] → U_1[], B2[1] → U_1[], B1[0] → U_1[], B3[-1] → U_1[], B1[2] → U_1[], B3[0] → U_1[], B1[-1] → U_1[]}

The pairing

In[∘]:= $\delta_{1,2}$

Time: 0.06 seconds Out[∘]=

0

In[∘]:=
```
Do[As[i, $n - 1], P[U[], U[A[i][_]]] = P[U[B[i][_]], U[]] = 1];
P[U[], U[__]] = P[U[__], U[]] = 0;
Do[As[i, $n - 1], As[j, $n - 1], P[U[b[i]], U[a[j]]] = δi,j ℏ^-1];
Do[As[i, $n - 1], As[j, $n - 1], P[U[B[i][n_]], U[a[j]]] = -n γ];
Do[As[i, $n - 1], As[j, i + 1, $n], As[k, $n - 1], P[U[y[i, j]], U[a[k]]] = 0];
Do[As[i, $n - 1], As[j, $n - 1], P[U[b[i]], U[A[j][n_]]] = -n ϵ δi,j];
Do[As[i, $n - 1], As[j, $n - 1], P[U[B[i][n_]], U[A[j][m_]]] = δi,j ⅇ^(n m ℏ ϵ γ)];
Do[As[i, $n - 1], As[j, i + 1, $n], As[k, $n - 1], P[U[y[i, j]], U[A[k][_]]] = 0];
Do[As[i, $n - 1], As[j, i + 1, $n], As[k, $n - 1], P[U[b[k]], U[x[i, j]]] = 0];
Do[As[i, $n - 1], As[j, i + 1, $n], As[k, $n - 1], P[U[B[k][_]], U[x[i, j]]] = 0];
Do[As[i, $n - 1], As[j, i + 1, $n], As[k, $n - 1],
  As[l, k + 1, $n], P[U[y[i, j]], U[x[k, l]]] = δi,k δj,l ℏ^-1];
```

In[∘]:=
```
P[U[], U[]] = 1;
Pi_,j_[ℰ_] := UCF[ℰ /. Ui[ys___] Uj[xs___] → P[U[ys], U[xs]]];
```

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

In[∘]:=
```
P[U[y_, ys__], U[x_]] := P[U[y, ys], U[x]] =
   Module[{i, j, k, l}, UCF[Ui[y] Uj[ys] Δk→k,l[Uk[x]]] // Pi,k // Pj,l];
P[U[ys__], U[x_, xs__]] := P[U[ys], U[x, xs]] =
   Module[{i, j, k, l}, UCF[Δi→i,j[Ui[ys]] Uk[x] Ul[xs]] // Pi,k // Pj,l];
```

In[ ]:=
```
P[U[b[1]], U[a[1], a[1]]]
```
Time: 0.02 seconds Out[ ]=

0

In[ ]:=
```
P[U[b[1], b[1], b[1], b[1]], U[a[1], a[1], a[1], a[1]]]
```
Time: 0.03 seconds Out[ ]=

$$\frac{24}{\hbar^4}$$

In[ ]:=
```
lhs =
  Factor@Table[ℏ^n P[U @@ Table[y12, {n}], U @@ Table[x12, {n}]], {n, 7}] // Simplify // Expand
```
Time: 0.05 seconds Out[ ]=

$$\left\{1,\ \frac{1}{q^2}+q^2,\ \frac{1}{q^6}+\frac{2}{q^2}+2q^2+q^6,\ 6+\frac{1}{q^{12}}+\frac{3}{q^8}+\frac{5}{q^4}+5q^4+3q^8+q^{12},\ 22+\frac{1}{q^{20}}+\frac{4}{q^{16}}+\frac{9}{q^{12}}+\frac{15}{q^8}+\frac{20}{q^4}+20q^4+15q^8+9q^{12}+4q^{16}+q^{20},\ \frac{1}{q^{30}}+\frac{5}{q^{26}}+\frac{14}{q^{22}}+\frac{29}{q^{18}}+\frac{49}{q^{14}}+\frac{71}{q^{10}}+\frac{90}{q^6}+\frac{101}{q^2}+101q^2+90q^6+71q^{10}+49q^{14}+29q^{18}+14q^{22}+5q^{26}+q^{30},\ \frac{1}{q^{42}}+\frac{6}{q^{38}}+\frac{20}{q^{34}}+\frac{49}{q^{30}}+\frac{98}{q^{26}}+\frac{169}{q^{22}}+\frac{259}{q^{18}}+\frac{359}{q^{14}}+\frac{455}{q^{10}}+\frac{531}{q^6}+\frac{573}{q^2}+573q^2+531q^6+455q^{10}+359q^{14}+259q^{18}+169q^{22}+98q^{26}+49q^{30}+20q^{34}+6q^{38}+q^{42}\right\}$$

In[ ]:=
```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, q^4] q^(-n (n-1)), {n, 7}] // Expand
```
Time: 0.01 seconds Out[ ]=

$$\left\{1,\ \frac{1}{q^2}+q^2,\ \frac{1}{q^6}+\frac{2}{q^2}+2q^2+q^6,\ 6+\frac{1}{q^{12}}+\frac{3}{q^8}+\frac{5}{q^4}+5q^4+3q^8+q^{12},\ 22+\frac{1}{q^{20}}+\frac{4}{q^{16}}+\frac{9}{q^{12}}+\frac{15}{q^8}+\frac{20}{q^4}+20q^4+15q^8+9q^{12}+4q^{16}+q^{20},\ \frac{1}{q^{30}}+\frac{5}{q^{26}}+\frac{14}{q^{22}}+\frac{29}{q^{18}}+\frac{49}{q^{14}}+\frac{71}{q^{10}}+\frac{90}{q^6}+\frac{101}{q^2}+101q^2+90q^6+71q^{10}+49q^{14}+29q^{18}+14q^{22}+5q^{26}+q^{30},\ \frac{1}{q^{42}}+\frac{6}{q^{38}}+\frac{20}{q^{34}}+\frac{49}{q^{30}}+\frac{98}{q^{26}}+\frac{169}{q^{22}}+\frac{259}{q^{18}}+\frac{359}{q^{14}}+\frac{455}{q^{10}}+\frac{531}{q^6}+\frac{573}{q^2}+573q^2+531q^6+455q^{10}+359q^{14}+259q^{18}+169q^{22}+98q^{26}+49q^{30}+20q^{34}+6q^{38}+q^{42}\right\}$$

In[ ]:=
```
MapThread[Equal, {lhs, rhs}]
```
Time: 0.00 seconds Out[ ]=

{True, True, True, True, True, True, True}

Pairing axioms:

In[ ]:=
```
(U_1[x12] S_-1[U_-1[y12]] // Expand // P_-1,1) - (S_1[U_1[x12]] U_-1[y12] // Expand // P_-1,1)
(U_1[a1] S_-1[U_-1[b1]] // Expand // P_-1,1) - (S_1[U_1[a1]] U_-1[b1] // Expand // P_-1,1)
```
Time: 0.00 seconds Out[ ]=

0

Out[ ]=

0

In[∘]:= $(U_1[x12, x12, a1]\ S_{-1}[U_{-1}[y12, y12, b1]]$ // Expand // $P_{-1,1})$ -
$(S_1[U_1[x12, x12, a1]]\ U_{-1}[y12, y12, b1]$ // Expand // $P_{-1,1})$

Time: 0.00 seconds Out[∘]=

0

In[∘]:= $((U_1[x12, x12, a1]\ U_2[A1[1], a1, x12]\ U_{-1}[y12, y12, y12, b1, b1, b1]$ // $m_{1,2\to 1}$ // Expand) //
$P_{-1,1})$ -
$((U_1[x12, x12, a1]\ U_2[A1[1], a1, x12]\ U_{-1}[y12, y12, y12, b1, b1, b1]$ // $\Delta_{-1\to -1,-2}$ // Expand) //
$P_{-1,1}$ // $P_{-2,2})$

Time: 0.30 seconds Out[∘]=

0

In[∘]:= {$((U_1[x12]\ U_{-2}[y12]\ U_{-1}[b1]$ // $m_{-1,-2\to -1}$ // Expand) // $PP_{-1,1})$,
$((U_1[x12]\ U_{-2}[y12]\ U_{-1}[b1]$ // $\Delta_{1\to 1,2}$ // Expand) // $P_{-1,1}$ // $P_{-2,2})$}

Time: 0.02 seconds Out[∘]=

$$\left\{PP_{-1,1}[\epsilon\, U_{-1}[y12]\, U_1[x12] + U_{-1}[y12, b1]\, U_1[x12]],\ \frac{\epsilon}{\hbar}\right\}$$

In[∘]:= $P_{-1,1}[\epsilon\, U_{-1}[y12]\, U_1[x12] + zz\, U_{-1}[y12, b1]\, U_1[x12]]$

Time: 0.00 seconds Out[∘]=

$$\frac{\epsilon}{\hbar} - \frac{zz\,\epsilon}{\hbar}$$

In[∘]:= $\epsilon\, P_{-1,1}[U_{-1}[y12]\, U_1[x12]]$

Time: 0.00 seconds Out[∘]=

$$\frac{\epsilon}{\hbar}$$

In[∘]:= $P_{-1,1}[U_{-1}[y12, b1]\, U_1[x12]]$

Time: 0.00 seconds Out[∘]=

$$-\frac{\epsilon}{\hbar}$$

In[∘]:= $U_{-2}[y12]\, U_{-1}[b1]$ // $m_{-1,-2\to -1}$

Time: 0.00 seconds Out[∘]=

$$\epsilon\, U_{-1}[y12] + U_{-1}[y12, b1]$$

In[∘]:= $((U_1[x12]\ U_{-2}[y12]\ U_{-1}[b1]$ // $m_{-1,-2\to -1}$ // Expand) // $P_{-1,1})$ -
$((U_1[x12]\ U_{-2}[y12]\ U_{-1}[b1]$ // $\Delta_{1\to 1,2}$ // Expand) // $P_{-1,1}$ // $P_{-2,2})$

Time: 0.05 seconds Out[∘]=

$$-\frac{\epsilon}{\hbar}$$

In[∘]:= $\Delta_{1\to 1,2}[U_1[x12]]$

Time: 0.02 seconds Out[∘]=

$$U_1[A1[-1]]\, U_2[x12] + U_1[x12]\, U_2[A1[1]]$$

In[ ]:= ((U₁[x12, x12] U₋₂[y12] U₋₁[y12, b1] // m₋₁,₋₂→₋₁ // Expand) // P₋₁,₁) - ((U₁[x12, x12] U₋₂[y12] U₋₁[y12, b1] // Δ₁→₁,₂ // Expand) // P₋₁,₁ // P₋₂,₂)

Time: 0.02 seconds Out[ ]=

$$-\frac{\epsilon}{q^2 \hbar^2}-\frac{q^2 \epsilon}{\hbar^2}$$

In[ ]:= ((U₁[x12, x12, x12, x12, a1] U₋₂[y12, B1[1], b1] U₋₁[y12, y12, y12, b1, b1, b1] // m₋₁,₋₂→₋₁ // Expand) // P₋₁,₁) - ((U₁[x12, x12, x12, x12, a1] U₋₂[y12, B1[1], b1] U₋₁[y12, y12, y12, b1, b1, b1] // Δ₁→₁,₂ // Expand) // P₋₁,₁ // P₋₂,₂)

Time: 0.05 seconds Out[ ]=

$$-\frac{474\, e^{4\gamma\epsilon\hbar}\epsilon^3}{\hbar^5}-\frac{79\, e^{4\gamma\epsilon\hbar}\epsilon^3}{q^{12}\hbar^5}-\frac{237\, e^{4\gamma\epsilon\hbar}\epsilon^3}{q^8\hbar^5}-\frac{395\, e^{4\gamma\epsilon\hbar}\epsilon^3}{q^4\hbar^5}-\frac{395\, e^{4\gamma\epsilon\hbar}q^4\epsilon^3}{\hbar^5}-\frac{237\, e^{4\gamma\epsilon\hbar}q^8\epsilon^3}{\hbar^5}-\frac{79\, e^{4\gamma\epsilon\hbar}q^{12}\epsilon^3}{\hbar^5}-\frac{2280\, e^{4\gamma\epsilon\hbar}\gamma\epsilon^4}{\hbar^4}-\frac{380\, e^{4\gamma\epsilon\hbar}\gamma\epsilon^4}{q^{12}\hbar^4}-\frac{1140\, e^{4\gamma\epsilon\hbar}\gamma\epsilon^4}{q^8\hbar^4}-\frac{1900\, e^{4\gamma\epsilon\hbar}\gamma\epsilon^4}{q^4\hbar^4}-\frac{1900\, e^{4\gamma\epsilon\hbar}q^4\gamma\epsilon^4}{\hbar^4}-\frac{1140\, e^{4\gamma\epsilon\hbar}q^8\gamma\epsilon^4}{\hbar^4}-\frac{380\, e^{4\gamma\epsilon\hbar}q^{12}\gamma\epsilon^4}{\hbar^4}$$