Hidden Utilities and TEX

As: Enables the likes of Table[As[n,6], As[k,0,n], Binomial[n,k]], as in pensieve://2017-01/As.nb.

```
In[ ]:= SetAttributes[As, HoldAll];
As /: (op : Do | Product | Sum | Table) [As[iter___], ℰ__] := op[ℰ, {iter}]
```

Have StringJoin handle Integer and Symbol inputs correctly:

```
In[ ]:= Unprotect[StringJoin];
StringJoin[l___, m_Integer, r___] := StringJoin[l, ToString[m], r];
StringJoin[x_Symbol, r___] := Symbol[SymbolName[x] <> StringJoin[r]];
Protect[StringJoin];
```

Time every Out line:

```
In[ ]:= CurrentValue[EvaluationNotebook[], EvaluationCompletionAction] = {"ShowTiming"};

SetOptions[EvaluationNotebook[], StyleDefinitions →
  Notebook[{Cell[StyleData[StyleDefinitions → "Default.nb"]], Cell[StyleData["Input"],
     CellEpilog :→ With[{time = AbsoluteCurrentValue["WindowStatusArea"]},
       If[CurrentValue[NextCell[], CellStyle] === {"Output"}, SetOptions[
         NextCell[], CellLabel → time <> " Out[" <> ToString[$Line - 1] <> "]="]]]]},
    StyleDefinitions → "PrivateStylesheetFormatting.nb"]]
```

```
In[ ]:= PP_ = Identity;
```

tex

```
\documentclass[12pt,reqno]{amsart}
\usepackage[export]{adjustbox} % This also includes graphicx.
\usepackage{needspace}
\usepackage[textwidth=7.5in,textheight=10in,centering]{geometry}
\pagestyle{empty}

\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfPrint#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfMessage#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfSubsubsection#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphInput#1{\vskip 1mm\par\noindent\includegraphics{#1}}
\def\nbpdfgraphOutput#1{\vskip 1mm\par\noindent\includegraphics[width=1.5in]{#1}}
```

\begin{document}

nb2tex

In[ ]:=
```
MakePDF[] := (
  Get["../nb2tex/nb2tex.m"];
  $NotebookName = FileBaseName@FileNameTake@NotebookFileName[EvaluationNotebook[]];
  nb2tex[$NotebookName, PDFWidth → 7.5, PDFFolder → "Snips/" <> $NotebookName];
  texfile = OpenAppend[$NotebookName <> ".tex"];
  WriteLine[texfile, "\\end{document}"]; Close[texfile];
  Run@
   ("\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.
      exe\" " <> $NotebookName <> ".tex");
 )
```

Headers

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Pensieve header: Quantum sl_n following Rosso and Yamane.

pdf

Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso’s X_i and Y_j. Rosso’s $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow$ xij. Likewise Rosso’s $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow$ yij.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i.

pdf

In[ ]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

pdf

```
This version: April 2020. Original version: July 1994.
```

In[ ]:=
```
MakePDF[]
```

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In[ ]:=
```
δ_{ij__} := KroneckerDelta[ij];
```

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In[∘]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
```

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In[∘]:=
```
$N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝒜[i_] := 𝒜 <> i;
y[i_, j_] := y <> i <> j; b[i_] := b <> i; B[i_] := B <> i;
η[i_, j_] := η <> i <> j; β[i_] := β <> i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

In[∘]:=
```
{τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[∘]=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso’s; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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In[∘]:=
```
Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
 ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
 η[i, j]* = y[i, j]]
Do[As[i, 1, $N - 1], A[i]* = 𝒜[i]; 𝒜[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
 b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_αβ__)* := (u*)αβ;
```

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In[∘]:=
```
LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
    Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ⋃ GreekVars))_;
```

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In[∘]:=
```
CF[ℰ_] := PP_CF@Module[{s1},
   s1 = If[$k == ∞,
     Expand[ℰ] /. 𝕖^p_ :→ 𝕖^Factor@p,
     Expand[ℰ] /. 𝕖^p_ :→ 𝕖^Factor@p /. q^n_· :→ Sum[(ℏ ϵ γ n)^k / k!, {k, 0, $k}] /. ϵ^k_· /; k > $k :→ 0
    ]
   (*Expand@Collect[s1,Cases[s1,VarPattern,∞],CCF]/.CCF→Factor*)
  ]
```

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Weights:

In[∘]:=
```
Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, i - j};
  Wt[ξ[i, j]] = {0, j - i}; Wt[η[i, j]] = {1, j - i};
 ];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[α[i]] = {0, 0}; Wt[β[i]] = {1, 0};
 ];
Wt[u_] := Wt[u];
```

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In[∘]:=
```
(* q~𝕖^(ℏ γ ϵ); Ai[n]~𝕖^(ℏ ϵ ai); *)
```

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In[∘]:=
```
$n = 3; $k = ∞; $E := {$n, $k}
```

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In[∘]:=
```
$EPBWGens := Flatten@{
    Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
    Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
   }
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

In[∘]:=
```
$PBWGens*
```

Time: 0.01 seconds Out[∘]=

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

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In[∘]:=
```
UCF[ℰ_] := PP_UCF@Module[{F, k}, Expand@Collect[
    ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
       U_i_[l___, AB_[n_], r___] :→
        Sum[As[k, 0, $k], ((-2)^k ℏ^k ϵ^k n^k)/(k!) U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
      }],
    U_[___], F
   ] /. F → CF]
```

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In[∘]:=
```
m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
  ];
m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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In[∘]:=
```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :→ i, ∞];
```

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In[∘]:=
```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := PP_NCM@ Module[{i, is = Supp[x] ⋂ Supp[y], σ, z, k},
    z = x; Do[z = m_i→σ@i[z], {i, is}];
    z = Expand[y z];
    If[$k < ∞, z = z /. ϵ^k_. /; k > $k :→ 0];
    Do[z = m_σ@i,i→i[z], {i, is}]; z];
UB[x_, y_] := PP_UB@UCF[x ⊠ y - y ⊠ x];
```

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In[∘]:=
```
Δ_i_→j_,k_[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[f_, fs___] :→ m_k,k1→k@ m_j,j1→j[Δ_j,k[f] Δ_i→j1,k1[U_i[fs]]]
     }];
Δ_i_→j_,k_,l__[ℰ_] := ℰ // Δ_i→j,k // Δ_k→k,l
```

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In[∘]:=
```
Sᵢ_[ℰ_] := UCF@Module[{i1}, ℰ /. {
     Uᵢ[f_, fs___] ⧴ m_{i1,i→i}[Sᵢ[f] S_{i1}[U_{i1}[fs]]]
   }];
```

In[∘]:=
```
Δ_{1,2→3}[U₁[x12]]
```

Time: 0.02 seconds Out[∘]=

$\Delta_{1,2\to 3}[U_1[x12]]$

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The 𝔸-algebra

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In[∘]:=
```
RR_k_[ai_, aj_] /; τ[ai] == τ[aj] == "a" := U_k[aj, ai];
RR_k_[ai_, Aj_[n_]] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := U_k[Aj[n], ai];
RR_k_[Aj_[n_], ai_] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := U_k[ai, Aj[n]];
RR_k_[Ai_[m_], Aj_[n_]] /;  τ[Ai] == τ[Aj] == "A"  := U_k[Aj[n], Ai[m]];
RR_k_[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@am;
   U_k[xij, am] + γ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[xij] / 2
  ];
RR_k_[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = ι@xij; {m} = ι@Am; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[xij, Am[n]]
 ]
```

In[∘]:=
```
RR₀[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[∘]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

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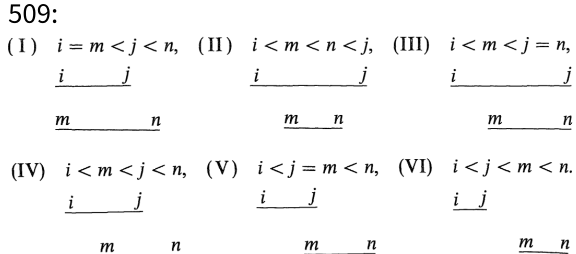
We take the reduction rules RR for xij’s from Yamane page 509:



$$
\begin{array}{ll}
q^{-2}e_{ij}e_{mn}-e_{mn}e_{ij}=0 & \text{if } ((i,j),(m,n))\in C_{(\mathrm{I})}\bigcup C_{(\mathrm{III})}.\\
[e_{ij},e_{mn}]=0 & \text{if } ((i,j),(m,n))\in C_{(\mathrm{II})}\bigcup C_{(\mathrm{VI})}.\\
[e_{ij},e_{mn}]=(q^2-q^{-2})e_{in}e_{mj} & \text{if } ((i,j),(m,n))\in C_{(\mathrm{IV})}.\\
q^2e_{ij}e_{mn}-e_{mn}e_{ij}=qe_{in} & \text{if } ((i,j),(m,n))\in C_{(\mathrm{V})}.
\end{array}
$$

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In[∘]:=
```
RR_k_[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[xij, xmn], (* I∨III *)
   (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
   i < m < j ∧ n > j, U_k[xij, xmn] + (q^-2 - q^2) U_k[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
 ]
```

In[∘]:=
```
U_1[x34] U_2[x23] U_3[x12] // m_{1,2→2}
```

Time: 0.06 seconds Out[∘]=

$$-q\,U_2[x24]\,U_3[x12] + q^2\,U_2[x23, x34]\,U_3[x12]$$

In[∘]:=
```
U_1[x34] U_2[x23] U_3[x12] // m_{1,2→2} // m_{2,3→0}
```

Time: 0.05 seconds Out[∘]=

$$q^2\,U_0[x14] - q^3\,U_0[x12, x24] - q^3\,U_0[x13, x34] + q^4\,U_0[x12, x23, x34]$$

In[∘]:=
```
Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U_1[X] U_2[Y] U_3[Z];
     {X, Y, Z} → (P // m_{1,2→2} // m_{2,3→0}) == (P // m_{2,3→2} // m_{1,2→0})
    ]]
 ]]
```

Time: 1.05 seconds Out[∘]=

{}

In[∘]:=
```
Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U_1[X] U_2[Y] U_3[Z];
     {X, Y, Z} → UCF[(P // m_{1,2→2} // m_{2,3→0}) - (P // m_{2,3→2} // m_{1,2→0})]
    ]]
 ]]
```

» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[∘]=

{}

```
Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
   ]]
]
```

» {x12, x13, x23, a1, a2}

Out[∘]=

{}

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The Co-Product, Co-Associativity, and the Hopf Axiom

pdf

According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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In[∘]:=

```
Do[As[i, 1, $n - 1],
  Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
   ]]
]
```

In[∘]:=

```
Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
 ]
```

Time: 0.03 seconds Out[∘]=

$U_i[x12] \to U_j[x12]\, U_k[\,] - 2\, \epsilon\, \hbar\, U_j[x12]\, U_k[a1] + U_j[\,]\, U_k[x12] + 2\, \epsilon\, \hbar\, U_j[a1]\, U_k[x12]$

$U_i[x13] \to U_j[x13]\, U_k[\,] - 2\, \epsilon\, \hbar\, U_j[x13]\, U_k[a1] - 2\, \epsilon\, \hbar\, U_j[x13]\, U_k[a2] +$
$\quad U_j[\,]\, U_k[x13] + 2\, \epsilon\, \hbar\, U_j[a1]\, U_k[x13] + 2\, \epsilon\, \hbar\, U_j[a2]\, U_k[x13] + 4\, \gamma\, \epsilon\, \hbar\, U_j[x12]\, U_k[x23]$

$U_i[x23] \to U_j[x23]\, U_k[\,] - 2\, \epsilon\, \hbar\, U_j[x23]\, U_k[a2] + U_j[\,]\, U_k[x23] + 2\, \epsilon\, \hbar\, U_j[a2]\, U_k[x23]$

$U_i[a1] \to U_j[a1]\, U_k[\,] + U_j[\,]\, U_k[a1]$

$U_i[a2] \to U_j[a2]\, U_k[\,] + U_j[\,]\, U_k[a2]$

```
In[∘]:= Module[{Bas = $PBWGens, P},
          Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
          DeleteCases[_ → True][Flatten@Table[As[X, Bas],
              P = U₁[X];
              X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
            ]]
        ]
      » {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}
Out[∘]=
        {}
```

```
In[∘]:= Module[{Bas = $PBWGens, P},
          Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
          DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas],
              P = U₁[X] U₂[Y];
              {X, Y} → UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
            ]]
        ]
      » {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[∘]=
        {}
```

Is Δ a quantization of the co-bracket?

```
In[∘]:= $k = 1
Time: 0.02 seconds Out[∘]=
        1
```

```
In[∘]:= Column@Module[{Bas = $PBWGens, P},
           Table[As[P, U₁ /@ Bas],
            P → UCF[(Δ1→1,2[P] - Δ1→2,1[P]) //.
                  {q^(n_·) :→ 1 + n ℏ / 2, U_i_[xs___, A_[n_], as___] :→ U_i[xs] + n ℏ U_i[xs, a @@ ι[A], as] / 2}
              ] /. ℏ^(n_) /; n > 1 :→ 0
           ]
         ]
Time: 0.05 seconds Out[∘]=
```

$U_1[x12] \to -4\,\epsilon\,\hbar\,U_1[x12]\,U_2[a1] + 4\,\epsilon\,\hbar\,U_1[a1]\,U_2[x12]$

$U_1[x13] \to -4\,\epsilon\,\hbar\,U_1[x13]\,U_2[a1] - 4\,\epsilon\,\hbar\,U_1[x13]\,U_2[a2] - 4\,\gamma\,\epsilon\,\hbar\,U_1[x23]\,U_2[x12] +$
$\quad 4\,\epsilon\,\hbar\,U_1[a1]\,U_2[x13] + 4\,\epsilon\,\hbar\,U_1[a2]\,U_2[x13] + 4\,\gamma\,\epsilon\,\hbar\,U_1[x12]\,U_2[x23]$

$U_1[x23] \to -4\,\epsilon\,\hbar\,U_1[x23]\,U_2[a2] + 4\,\epsilon\,\hbar\,U_1[a2]\,U_2[x23]$

$U_1[a1] \to 0$

$U_1[a2] \to 0$

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The Antipode

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```
In[◦]:= Do[As[i, 1, $n - 1],
   S_j_[a[i]] = -U_j[a[i]];
   S_j_[A[i][n_]] = U_j[A[i][-n]];
   S_j_[x[i, i + 1]] = -q^-2 U_j[x[i, i + 1]] (*?*)
  ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 S_j_[x[i, n]] = m_{j1,j→j}[
   q S_j[x[i, n - 1]] S_j1[x[n - 1, n]] - q^-1 S_j[x[n - 1, n]] S_j1[x[i, n - 1]]
  ]
]
```

```
In[◦]:= x[1, 2]
```

Time: 0.00 seconds Out[◦]=

x12

```
In[◦]:= S_1[x12]
```

Time: 0.00 seconds Out[◦]=

$$-\frac{U_1[x12]}{q^2}$$

```
In[◦]:= Δ_{1,2→3}[U_1[x12]]
```

Time: 0.00 seconds Out[◦]=

$\Delta_{1,2\to3}[U_1[x12]]$

```
In[◦]:= $k = 2
```

Time: 0.00 seconds Out[◦]=

2

```
In[◦]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas],
     P = U_1[X];
     X → (P // Δ_{1→2,3} // S_2 // m_{2,3→1})
    ]]
]
```

» {x12, x13, x23, a1, a2, A2[-1], A2[-1], A2[1], A1[2], A1[-2], A1[-1], A1[-2], A1[1], A1[0], A1[-2]}

Out[◦]=

{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[-1] → U_1[],
A2[-1] → U_1[], A2[1] → U_1[], A1[2] → U_1[], A1[-2] → U_1[],
A1[-1] → U_1[], A1[-2] → U_1[], A1[1] → U_1[], A1[0] → U_1[], A1[-2] → U_1[]}

```
In[◦]:= $n
```

Time: 0.02 seconds Out[◦]=

2

pdf

The Generating Functions for the 𝔸 operations

In[∘]:=
```
BeginProfile[]
```
Time: 0.03 seconds Out[∘]=
```
BeginProfile[]
```

In[∘]:=
```
$k = 1;
```

pdf

Ad, TT, Seepage

pdf

In[∘]:=
```
PotQ[0] = True;
PotQ[c_. U_i_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

pdf

In[∘]:=
```
Ad[x_][y_] /; PotQ[x] ∧ (Head[y] =!= Plus) := PP_AdPoty[ⅇ^(UB[y,x]/y) y];
Ad[x_][y_Plus] /; PotQ[x] := PP_AdPotPlus[Ad[x] /@ y];
```

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In[∘]:=
```
Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PP_AdNil@Module[{k = 1, s = y, t = y},
   While[(t = UB[t, x]) =!= 0, s += t / k!; ++k];
   UCF[s]
  ]
```

In[∘]:=
```
$k
```
Time: 0.03 seconds Out[∘]=

1

In[∘]:=
```
$k = 2
```
Time: 0.00 seconds Out[∘]=

2

In[∘]:= $\mathrm{Ad}[\epsilon\, \hbar\, U_j[x12]\, U_k[]][U_j[x23]\, U_k[a4]]$

Time: 0.03 seconds Out[∘]=

$-\epsilon\, \hbar\, U_j[x13]\, U_k[a4] + U_j[x23]\, U_k[a4]$

In[∘]:= $\mathrm{PotQ}[7\, \hbar\, U_j[x12]\, U_k[a3]]$

Time: 0.02 seconds Out[∘]=

$\mathrm{PotQ}[7\, \hbar\, U_j[x12]\, U_k[a3]]$

In[ ]:=
```
Ad[7 ℏ U_j[x12] U_k[a3]][U_j[x23] U_k[a4]]
```
Time: 0.09 seconds Out[ ]=

$$U_j[x23]\,U_k[a4] + \left(-7\hbar - 7\gamma\epsilon\hbar^2 - \frac{7}{2}\gamma^2\epsilon^2\hbar^3\right) U_j[x13]\,U_k[a3, a4] + \left(14\gamma\epsilon\hbar^2 + 14\gamma^2\epsilon^2\hbar^3\right) U_j[x12, x23]\,U_k[a3, a4] - 98\gamma^2\epsilon^2\hbar^4\,U_j[x12, x13]\,U_k[a3, a3, a4] + 98\gamma^2\epsilon^2\hbar^4\,U_j[x12, x12, x23]\,U_k[a3, a3, a4] - \frac{686}{3}\gamma^2\epsilon^2\hbar^5\,U_j[x12, x12, x13]\,U_k[a3, a3, a3, a4]$$

In[ ]:=
```
Ad[π i U_j[a1]][U_j[x23] U_k[a4]]
```
Time: 0.05 seconds Out[ ]=

$$e^{\frac{i\pi\gamma}{2}}\,U_j[x23]\,U_k[a4]$$

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In[ ]:=
```
TT_λ_[] = 0; (* Translated Tangent *)
TT_λ_[xs___, x_] := PP_TT@UCF[Ad[x][TT_λ[xs]] + ∂_λ x];
TT_λ_[xs_List] := TT_λ @@ xs
```

In[ ]:=
```
TT_λ[λ ξ12 U_0[x12] U_1[], λ ξ23 U_0[x23] U_1[]]
```
Time: 0.03 seconds Out[ ]=

$$\xi12\,U_0[x12]\,U_1[] + (\lambda\,\xi12\,\xi23 + \gamma\epsilon\lambda\,\xi12\,\xi23\,\hbar)\,U_0[x13]\,U_1[] + \xi23\,U_0[x23]\,U_1[] - 2\gamma\epsilon\lambda\,\xi12\,\xi23\,\hbar\,U_0[x12, x23]\,U_1[]$$

In[ ]:=
```
TT_λ[0]
```
Time: 0.00 seconds Out[ ]=

Ad[0][0]

In[ ]:=
```
$k
```
Time: 0.00 seconds Out[ ]=

1

pdf

In[ ]:=
```
O_i_[poly_] := Total[
    CoefficientRules[poly, #_i & /@ $PBWGens] /.
     (p_ → c_) :> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
   ];
O_i_,is__[poly_] := O_i[O_is[poly]];
P_i_[ℰ_] := ℰ /. U_i[xs___] :> Times @@ (#_i & /@ {xs});
P_i_,is__[poly_] := P_i[P_is[poly]];
```

In[ ]:=
```
O_i[x12_i^2 x13_i + 7 x23_i^17]
```
Time: 0.06 seconds Out[ ]=

U_i[x12, x12, x13] +
 7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]

In[ ]:= $\mathbb{O}_{i,j}\left[x12_i^2\, x13_i\, x24_j + 7\, x23_i^{17}\, x25_j^3\right]$

Time: 0.03 seconds Out[ ]=

```
Ui[x12, x12, x13] Uj[x24] +
 7 Ui[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]
  Uj[x25, x25, x25]
```

In[ ]:= $\mathbb{P}_{i,j}$@$\mathbb{O}_{i,j}\left[x12_i^2\, x13_i\, x24_j + 7\, x23_i^{17}\, x25_j^3\right]$

Time: 0.06 seconds Out[ ]=

$x12_i^2\, x13_i\, x24_j + 7\, x23_i^{17}\, x25_j^3$

pdf

```
In[ ]:= Seepage_i_ [{}, c_] = c;
Seepage_i_ [ξs_List, P_] := Module[{lx},
    lx = $PBWGens[[Length@ξs]];
    CF@Collect[P, lx_i, ℙ_i[Ad[Last[ξs] U_i[lx]][𝕆_i[Seepage_i[Most@ξs, #]]]]
       &]
   ];
Seepage_i_,is_ [ξs_List, ξss__List, P_] := Seepage_i[ξs, Seepage_is[ξss, P]]
```

In[ ]:= $k = 1

Time: 0.00 seconds Out[ ]=

1

In[ ]:= $PBWGens*

Time: 0.01 seconds Out[ ]=

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

In[ ]:= Seepage_i[$PBWGens*, x35_i]

Time: 0.16 seconds Out[ ]=

$e^{\frac{1}{2}(\alpha 2-\alpha 3-\alpha 4)\gamma}\, x35_i + 2\, e^{\frac{1}{2}(\alpha 2-3\,\alpha 4)\gamma}\,\gamma\,\epsilon\,\xi 45\,\hbar\, x35_i\, x45_i$

In[ ]:= Seepage_{i,j}[#_i & /@ $PBWGens*, #_j & /@ $PBWGens*, x35_i x45_j]

Time: 0.25 seconds Out[ ]=

$e^{\frac{1}{2}\gamma(\alpha 2_i-\alpha 3_i+\alpha 3_j-\alpha 4_i-2\,\alpha 4_j)}\, x35_i\, x45_j + 2\, e^{-\frac{1}{2}\gamma(-\alpha 2_i-\alpha 3_j+3\,\alpha 4_i+2\,\alpha 4_j)}\,\gamma\,\epsilon\,\hbar\, x35_i\, x45_i\, x45_j\, \xi 45_i$

pdf

```
In[ ]:= Invert_is__[ℰ_] := Module[{A, B, U1, j}, (* ℰ=A U[]+ϵ B *)
   U1 = Times @@ (U_#[] & /@ {is});
   A = Coefficient[ℰ, U1] /. ϵ → 0;
   B = UCF[(ℰ - A U1)];
   UCF[
    A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
  ]
```

In[∘]:=
```
$k = 3;
Invert_{i,j}[7 U_i[] U_j[] + ∈ U_i[x12] U_j[x23]]
```

Time: 0.09 seconds Out[∘]=

$$\frac{1}{7}U_i[]\,U_j[] - \frac{1}{49}\in U_i[x12]\,U_j[x23] + \frac{1}{343}\in^2 U_i[x12, x12]\,U_j[x23, x23] - \frac{\in^3 U_i[x12, x12, x12]\,U_j[x23, x23, x23]}{2401}$$

```
{$k = 1, $n = 3}
```

Time: 0.01 seconds Out[∘]=

{2, 3}

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In[∘]:=
```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
 PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinits = inits, rfs = fs, s, p},
   While[rfs =!= {},
    {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
    {{s}} = DSolve[reqns⟦p⟧ ∧ rinits⟦p⟧, rfs⟦p⟧, λ];
    AppendTo[ss, s = s /. F_Function :→ ReplacePart[F, 2 → CF[F⟦2⟧]]];
    reqns = Delete[reqns, p] /. s /. e_ == 0 :→ CF[e] == 0;
    rinits = Delete[rinits, p];
    rfs = Delete[rfs, p];
   ];
   ss
  ]
```

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“Define” Code

pdf

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_{is__} = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
    ReleaseHold[Hold[
       SD[op_{nisp,$E_}, PP_Boot@Block[{i, j, k}, op_{isp,$E} = ℰ; op_{nis,$E}]];
       SD[op_{isp}, op_{{is},$E}]; SD[op_{sis__}, op_{{sis}}];
      ] /. {SD → SetDelayed,
       isp → {is} /. {i → i_, j → j_, k → k_},
       nis → {is} /. {i → ii, j → jj, k → kk},
       nisp → {is} /. {i → ii_, j → jj_, k → kk_}
      }] ]]
```

pdf

$$L_\lambda = \mathbb{OE}(\lambda\xi_{i,1}x)\,\mathbb{OE}(\lambda\xi_{i,2}x)$$

Seepage is defined by: $\underbrace{\mathbb{OE}(f_i x_i)}_{\text{OH}}\,\mathbb{O}(\text{Seepage}(\{f_i\},A)) = \mathbb{O}(e^{f_i x_i}A)$

$$\mathbb{T}\mathbb{T}(L_\lambda) = L^{-1}\partial_\lambda L_\lambda \qquad \mathcal{S}(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \mathbb{OE}(f+g),\quad f = f_i x_i \quad g = \sum_{|w|\neq 1} g_w w \qquad e^f = F \quad e^g = G \qquad R_\lambda = \mathbb{O}(FG) \qquad \partial_\lambda F = F\partial_\lambda f$$

$$\mathbb{T}\mathbb{T}(R_\lambda) = R_\lambda^{-1}\partial_\lambda R_\lambda = \mathbb{O}(FG)^{-1}\mathbb{O}(\partial_\lambda FG) = \mathcal{S}(G)^{-1}\mathbb{O}(F)^{-1}\mathbb{O}(F(\partial_\lambda f)G + \partial_\lambda G)) = \mathcal{S}(G)^{-1}\mathcal{S}((\partial_\lambda f)G + \partial_\lambda G)$$

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In[∘]:=

```
Define[amt_{i,j→k} = PP_{"amt"[$k,$n]}@Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ); ES[s_][ℰ_] := ℰ;
    TTL = ES["TTL"]@PP_{"TTL"}@TT_λ@
        Join[l1 = Table[If[τ[g] === "x", λ, 1] (g*)_i U_k[g], {g, $PBWGens}], l1 /. i → j];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], U_k[___], ∞];
    ff = Table[
        Sum[ϵ^p f_{p,u_k}[λ], {p, 0, $k}],
        {u, $PBWGens}
      ] /. f_{p_,u_k}[λ] /; τ[u] == "a" :> If[p == 0, (u*)_i + (u*)_j, 0];
    fs = Flatten[Table[
        f_{p,ℙ_k[m]},
        {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
       ϵ^p f_{p,ℙ_k[u]}[λ] ℙ_k[u],
       {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
      ];
    G = Series[e^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[SS@𝕆_k[Seepage_k[ff, SS@UCF@B]]];
    𝒮G = ES["𝒮G"]@PP_{"𝒮G"}@𝒮[G];
    TTR = ES["TTR"]@PP_{"TTR"}@UCF[Invert_k[𝒮G] ⊠ 𝒮[(∂_λ (ff.(#_k & /@ $PBWGens)) G + ∂_λ G)]];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
        Coefficient[Coefficient[diff, m], ϵ, p] == 0,
        {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼_{{i,j}→{k}} @@ Table[As[p, 0, $k], Plus[
        If[p == 0, Sum[As[m, 1, $n - 1], (α[m]_i + α[m]_j) a[m]_k], 0],
        Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :> u (f_{p,u} /. ss)[λ] /. λ | ℏ | γ → 1]
      ]
   ]
]
```

pdf

In[ ]:=
```
Define[aΔt_{i→j,k} = PP_{"aΔt"[$k,$n]}@Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
    TTL = ES["TTL"]@
      TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g^*)_i Δ_{i→j,k}[U_i[g]]], {g, $PBWGens}];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃
      Union@Cases[Coefficient[TTL, ϵ, p], c_. U_j[x___] U_k[y___] ⧴ U_j[x] U_k[y], ∞];
    ffj = Table[
       Sum[ϵ^p f_{p,u_j}[λ], {p, 0, $k}],
       {u, $PBWGens}
      ] /. f_{p_,u_j}[λ] /; τ[u] == "a" ⧴ If[p == 0, (u^*)_i, 0];
    ffk = ffj /. j → k;
    fs = Flatten[Table[
       f_{p,ℙ_{j,k}[m]},
       {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
      ϵ^p f_{p,ℙ_{j,k}[u]}[λ] ℙ_{j,k}[u],
      {p, 0, $k}, {u, Complement[mons[p], U_j[] (U_k /@ $PBWGens) ⋃ U_k[] (U_j /@ $PBWGens)]}
     ];
    G = Series[ⅇ^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[SS@𝕆_{j,k}[Seepage_{j,k}[ffj, ffk, SS@UCF@B]]];
    𝒮G = ES["𝒮G"]@PP_{"𝒮G"}@𝒮[G];
    TTR = ES["TTR"]@PP_{"TTR"}@UCF[
        Invert_{j,k}[𝒮G] ⊠ 𝒮[(∂_λ (ffj.(#_j & /@ $PBWGens) + ffk.(#_k & /@ $PBWGens)) G + ∂_λ G)]];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
       Coefficient[Coefficient[diff, m], ϵ, p] == 0,
       {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼_{{i}→{j,k}} @@ Table[As[p, 0, $k], Plus[
       If[p == 0, Sum[As[m, 1, $n - 1], α[m]_i (a[m]_j + a[m]_k)], 0],
        Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} ⧴ u (f_{p,u} /. ss)[λ] /. λ | ℏ | γ → 1]
     ]
   ]]
```

In[ ]:=
```
$k = 1; $n = 2;
```

In[ ]:=
```
$PBWGens
```

Time: 0.03 seconds Out[ ]=
```
{x12, a1}
```

In[ ]:= $S_i[U_i[x12]]$

Time: 1.16 seconds Out[ ]=

$Aborted

In[ ]:= `Table[UCF[If[τ[g] === "x", λ, 1] (g*)_i S_i[g]], {g, $PBWGens}]`

Time: 0.03 seconds Out[ ]=

$$\{-\lambda\,(1-2\,\gamma\,\epsilon\,\hbar)\,\xi 12_i\,U_i[x12],\ (-\lambda\,\xi 13_i+2\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 13_i)\,U_i[x13]+4\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 13_i\,U_i[x12,\,x23],\ -\lambda\,(1-2\,\gamma\,\epsilon\,\hbar)\,\xi 23_i\,U_i[x23],\ -\alpha 1_i\,U_i[a1],\ -\alpha 2_i\,U_i[a2]\}$$

In[ ]:= `TTL = TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)_i S_i[U_i[g]]], {g, $PBWGens}]`

Time: 0.16 seconds Out[ ]=

$$\left(-e^{\frac{1}{2}\gamma\,(2\,\alpha 1_i-\alpha 2_i)}\,\xi 12_i+2\,e^{\frac{1}{2}\gamma\,(2\,\alpha 1_i-\alpha 2_i)}\,\gamma\,\epsilon\,\hbar\,\xi 12_i\right)U_i[x12]+$$
$$\Big(-e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\xi 13_i+2\,e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\gamma\,\epsilon\,\hbar\,\xi 13_i+$$
$$e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\lambda\,\xi 12_i\,\xi 23_i-3\,e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 12_i\,\xi 23_i\Big)\,U_i[x13]+$$
$$\left(-e^{-\frac{1}{2}\gamma\,(\alpha 1_i-2\,\alpha 2_i)}\,\xi 23_i+2\,e^{-\frac{1}{2}\gamma\,(\alpha 1_i-2\,\alpha 2_i)}\,\gamma\,\epsilon\,\hbar\,\xi 23_i\right)U_i[x23]-2\,e^{\frac{3\,\gamma\,\alpha 1_i}{2}}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 12_i\,\xi 13_i\,U_i[x12,\,x13]+$$
$$\left(4\,e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\gamma\,\epsilon\,\hbar\,\xi 13_i-2\,e^{\frac{1}{2}\gamma\,(\alpha 1_i+\alpha 2_i)}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 12_i\,\xi 23_i\right)U_i[x12,\,x23]+$$
$$2\,e^{\gamma\,(\alpha 1_i+\alpha 2_i)}\,\gamma\,\epsilon\,\lambda^2\,\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_i\,U_i[x13,\,x13]-2\,e^{\frac{3\,\gamma\,\alpha 2_i}{2}}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 13_i\,\xi 23_i\,U_i[x13,\,x23]$$

```
Define[aSt_i = Module[{(*TTL, *)},
  TTL = TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)_i S_i[U_i[g]]], {g, $PBWGens}] ×
    mons[-1] = {};
  mons[p_] :=
   mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], c_. U_i[x___] ⧴ U_i[x], ∞];
  ff = Table[
     Sum[ϵ^p f_{p,u_i}[λ], {p, 0, $k}],
     {u, $PBWGens}
    ] /. f_{p_,u_i}[λ] /; τ[u] == "a" ⧴ If[p == 0, (u*)_i, 0];
  fs = Flatten[Table[
     f_{p,ℙ_i[m]},
     {p, 0, $k}, {m, mons[p]}
    ]];
  gg = Sum[
    ϵ^p f_{p,ℙ_i[u]}[λ] ℙ_i[u],
    {p, 0, $k}, {u, Complement[mons[p], (U_i /@ $PBWGens)]}
   ];
  G = Series[𝕖^gg, {ϵ, 0, $k}];
  𝒮[B_] := UCF[SS@𝕆_i[Seepage_i[ff, SS@UCF@B]]];
  𝒮G = 𝒮[G];
  TTR = UCF[Invert_i[𝒮G] ⊠ 𝒮[(∂_λ (ff.(#_i & /@ $PBWGens)) G + ∂_λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
     Coefficient[Coefficient[diff, m], ϵ, p] == 0,
     {p, 0, $k}, {m, mons[p]}
    ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  𝔼_{{i}→{j,k}} @@ Table[As[p, 0, $k], Plus[
     If[p == 0, Sum[As[m, 1, $n - 1], α[m]_i (a[m]_j + a[m]_k)], 0],
     Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} ⧴ u (f_{p,u} /. ss)[λ] /. λ | ℏ | γ → 1]
   ]
```

pdf

In[•]:=
```
Define[aΔt_{i→j,k} = PP_"aΔt"[$k,$n] @Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
    TTL = ES["TTL"]@
      TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g^*)_i Δ_{i→j,k}[U_i[g]]], {g, $PBWGens}];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃
      Union@Cases[Coefficient[TTL, ϵ, p], c_. U_j[x___] U_k[y___] :→ U_j[x] U_k[y], ∞];
    ffj = Table[
        Sum[ϵ^p f_{p,u_j}[λ], {p, 0, $k}],
        {u, $PBWGens}
      ] /. f_{p_,u_j}[λ] /; τ[u] == "a" :→ If[p == 0, (u^*)_i, 0];
    ffk = ffj /. j → k;
    fs = Flatten[Table[
        f_{p,ℙ_{j,k}[m]},
        {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
      ϵ^p f_{p,ℙ_{j,k}[u]}[λ] ℙ_{j,k}[u],
      {p, 0, $k}, {u, Complement[mons[p], U_j[] (U_k /@ $PBWGens) ⋃ U_k[] (U_j /@ $PBWGens)]}
     ];
    G = Series[e^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[SS@𝕆_{j,k}[Seepage_{j,k}[ffj, ffk, SS@UCF@B]]];
    𝒮G = ES["𝒮G"]@PP_"𝒮G"@𝒮[G];
    TTR = ES["TTR"]@PP_"TTR"@UCF[
        Invert_{j,k}[𝒮G] ⊠ 𝒮[(∂_λ (ffj.(#_j & /@ $PBWGens) + ffk.(#_k & /@ $PBWGens)) G + ∂_λ G)]];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
        Coefficient[Coefficient[diff, m], ϵ, p] == 0,
        {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼_{{i}→{j,k}} @@ Table[As[p, 0, $k], Plus[
        If[p == 0, Sum[As[m, 1, $n - 1], α[m]_i (a[m]_j + a[m]_k)], 0],
        Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :→ u (f_{p,u} /. ss)[λ] /. λ | ℏ | γ → 1]
     ]
  ]]
```

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The Generating Functions Engine

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Canonical Forms:

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In[∘]:=
```
LogReduce[ℰ_] :=
  ℰ /. c_ * Log[a_] :> Log@Factor[a^c] //. Log[a_] + Log[b_] :> Log@Factor[a b];
CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

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𝔼 operations:

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In[∘]:=
```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼_{d1_→r1_}[ℰ1s___] ≡ 𝔼_{d2_→r2_}[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼_{d1_→r1_}[ℰ1s___] 𝔼_{d2_→r2_}[ℰ2s___] ^:= 𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
```

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In[∘]:=
```
𝔼_{d1_→r1_}[ℰ1s___] // 𝔼_{d2_→r2_}[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
  lvs = Cases[{ℰ1s}, (Alternatives @@ LatinVars)_{i_} /; MemberQ[is, i], ∞] ⋃
    Cases[{ℰ2s}, (Alternatives @@ GreekVars)_{i_} /; MemberQ[is, i], ∞]^*;
  llvs = lvs /. u_{i_} :> u_{$@i};
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ (Zip_{llvs⋃llvs*}[{(llvs)^*.(llvs), Times[
      𝔼[ℰ1s] /. Thread[lvs → llvs],
      𝔼[ℰ2s] /. Thread[lvs^* → llvs^*]
     ]}])
 ]
```

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Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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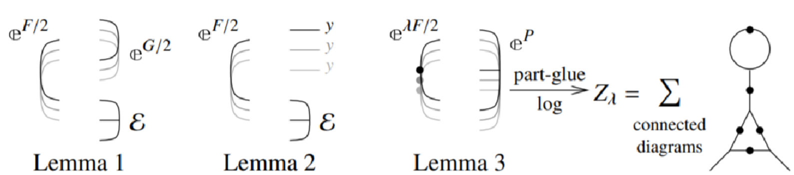

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q.

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii}’s and the y_{ii}’s. They have weights {1,0} and {0,0}.

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```
In[∘]:= Zip_vs_[{𝓕_, 𝓔_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
    {𝓕, 𝓔} // Zip1_vs (*//Zip2_Complement[vs,dvs]*) // Zip2_dvs //
      Zip3_Complement[vs,dvs] // Zip3_dvs
   ] // Last
```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon \mathcal{E}\right\rangle_B$$

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```
In[∘]:= Zip1_{} = Identity;
Zip1_vs_@{𝓕_, 𝔼[Q_, P___]} := Module[{𝓘, F, G, u, v},
  𝓘 = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}𝓕, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[𝓘 - G.F]).(vs) / 2],
   𝔼[CF[Q - PowerExpand@Log[Det[𝓘 - G.F]] / 2 - vs.G.vs / 2], P]}
 ]
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F\colon \mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

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```
In[∘]:= Zip2_{} = Identity;
Zip2_vs_@{𝓕_, 𝔼[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_{u,v}𝓕], 0], {u, vs}, {v, vs}];
  Y = Table[∂_v Q, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {𝓕, 𝔼[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
 ]
```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F\colon \mathbb{e}^P]_B$ satisfies
and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda = \sum Z[m]\,\lambda^m$.

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In[∘]:=
```
Zip3_vs_@{ℱ_, ℰ_𝔼} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = ℰ[[kk + 1]], {kk, 0, ℰ@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v}ℱ, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂_u Z[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂_v Z[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ ℰ@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
        If[F[u, v] === 0, 0, F[u, v]/(2 (m + 1)) (Z[m, kk, u, v] +
            Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
        {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
  ]];
  CF /@ ({
      ℱ - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, ℰ@$}]
    } /. Table[v → 0, {v, vs}])
]
```

In[∘]:= `{$k = 1, $n = 2}; amt_{i,j→k}`

Time: 0.19 seconds Out[∘]=

$$\mathbb{E}_{\{i,j\}\to\{k\}}\left[a1_k\,(\alpha1_i + \alpha1_j) + x12_k\left(\xi12_i + e^{\alpha1_i}\,\xi12_j\right),\ 0\right]$$

In[∘]:= `(amt_{1,2→2} // amt_{2,3→1}) ≡ (amt_{2,3→2} // amt_{1,2→1})`

Time: 0.08 seconds Out[∘]=

True

In[∘]:= `{$k = 1, $n = 3}; amt_{i,j→k}`
`(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})`

Time: 0.09 seconds Out[∘]=

$$\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a1_k\,(\alpha1_i + \alpha1_j) + a2_k\,(\alpha2_i + \alpha2_j) + x12_k\left(\xi12_i + e^{\frac{1}{2}(2\alpha1_i - \alpha2_i)}\,\xi12_j\right) +$$
$$x13_k\left(\xi13_i + e^{\frac{1}{2}(\alpha1_i + \alpha2_i)}\,\xi13_j - e^{\frac{1}{2}(2\alpha1_i - \alpha2_i)}\,\xi12_j\,\xi23_i\right) + x23_k\left(\xi23_i + e^{\frac{1}{2}(-\alpha1_i + 2\alpha2_i)}\,\xi23_j\right),$$
$$-2\,e^{\frac{1}{2}(2\alpha1_i - \alpha2_i)}\,x12_k\,x13_k\,\xi12_j\,\xi13_i - e^{\frac{1}{2}(2\alpha1_i - \alpha2_i)}\,x13_k\,\xi12_j\,\xi23_i +$$
$$2\,e^{\frac{1}{2}(2\alpha1_i - \alpha2_i)}\,x12_k\,x23_k\,\xi12_j\,\xi23_i - 2\,e^{\frac{1}{2}(\alpha1_i + \alpha2_i)}\,x13_k\,x23_k\,\xi13_j\,\xi23_i\Big]$$

Out[∘]=

True

In[ ]:=
```
{$k = 2, $n = 3}; amt_{i,j→k}
(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})
```

Time: 83.27 seconds Out[ ]=

$$\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a1_k\,(\alpha1_i+\alpha1_j)+a2_k\,(\alpha2_i+\alpha2_j)+x12_k\left(\xi12_i+e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\right)+$$
$$x13_k\left(\xi13_i+e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j-e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i\right)+x23_k\left(\xi23_i+e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i)}\,\xi23_j\right),$$
$$-2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x12_k\,x13_k\,\xi12_j\,\xi13_i-e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x13_k\,\xi12_j\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x12_k\,x23_k\,\xi12_j\,\xi23_i-2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,x13_k\,x23_k\,\xi13_j\,\xi23_i,$$
$$-\frac{1}{2}\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x13_k\,\xi12_j\,\xi23_i+2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x12_k\,x23_k\,\xi12_j\,\xi23_i+2\,e^{2\alpha1_i-\alpha2_i}\,x12_k^2\,x23_k\,\xi12_j^2\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,x12_k\,x23_k^2\,\xi12_j\,\xi23_i^2+e^{2\alpha1_i-\alpha2_i}\,x13_k^2\,\xi12_j^2\,\xi23_i^2-\frac{2}{9}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,x13_k^3\,\xi12_j^3\,\xi23_i^3+$$
$$x12_k\,x13_k\left(2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i-2\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi23_i\right)+$$
$$x12_k^2\,x13_k\left(2\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi13_i-\frac{2}{3}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j^3\,\xi23_i\right)+$$
$$x12_k\,x13_k\,x23_k\left(-4\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i\,\xi23_i-4\,e^{\frac{3\alpha1_i}{2}}\,\xi12_j\,\xi13_j\,\xi23_i\right)+$$
$$x13_k\,x23_k\left(2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j\,\xi23_i-2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i^2\right)+$$
$$x12_k\,x13_k^2\left(2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i^2+\frac{2}{3}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j^3\,\xi23_i^2\right)+$$
$$x13_k\,x23_k^2\left(2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j\,\xi23_i^2-\frac{2}{3}\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i^3\right)+$$
$$x13_k^2\,x23_k\left(2\,e^{\alpha1_i+\alpha2_i}\,\xi13_j^2\,\xi23_i+\frac{2}{3}\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi23_i^3\right)\Big]$$

Out[ ]=

True

In[ ]:=
```
{$k = 1, $n = 2}; aΔt_{i→j,k}
(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})
```

Time: 0.06 seconds Out[ ]=

$$\mathbb{E}_{\{i\}\to\{j,k\}}\left[(a1_j+a1_k)\,\alpha1_i+x12_j\,\xi12_i+x12_k\,\xi12_i,\ -2\,a1_k\,x12_j\,\xi12_i+2\,a1_j\,x12_k\,\xi12_i\right]$$

Out[ ]=

True

In[ ]:=
```
{$k = 1, $n = 3}; aΔt_{i→j,k}
(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})
```

Time: 0.08 seconds Out[ ]=

$$\mathbb{E}_{\{i\}\to\{j,k\}}\Big[$$
$$(a1_j+a1_k)\,\alpha1_i+(a2_j+a2_k)\,\alpha2_i+x12_j\,\xi12_i+x12_k\,\xi12_i+x13_j\,\xi13_i+x13_k\,\xi13_i+x23_j\,\xi23_i+x23_k\,\xi23_i,$$
$$-2\,a1_k\,x12_j\,\xi12_i+2\,a1_j\,x12_k\,\xi12_i-2\,a1_k\,x13_j\,\xi13_i-2\,a2_k\,x13_j\,\xi13_i+2\,a1_j\,x13_k\,\xi13_i+$$
$$2\,a2_j\,x13_k\,\xi13_i+x12_k\,x13_j\,\xi12_i\,\xi13_i-x12_j\,x13_k\,\xi12_i\,\xi13_i-2\,a2_k\,x23_j\,\xi23_i+2\,a2_j\,x23_k\,\xi23_i-$$
$$x12_k\,x23_j\,\xi12_i\,\xi23_i+x13_k\,x23_j\,\xi13_i\,\xi23_i-x13_j\,x23_k\,\xi13_i\,\xi23_i+x12_j\,x23_k\,(4\,\xi13_i+\xi12_i\,\xi23_i)\Big]$$

Out[ ]=

True

In[∘]:= {$k = 2, $n = 3}; $a\Delta t_{i\to j,k}$
($a\Delta t_{1\to 1,3}$ // $a\Delta t_{1\to 1,2}$) ≡ ($a\Delta t_{1\to 1,2}$ // $a\Delta t_{2\to 2,3}$)

» TTL $e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\xi 12_i\, U_j[x12]\, U_k[\,] + \ll 125\gg +$
$\left(2\, e^{\frac{1}{2}\gamma(\alpha 1_i-5\alpha 2_i)}\,\gamma^2\,\epsilon^2\,\lambda^2\,\hbar^2\,\xi 13_i\,\xi 23_i^2 + \frac{2}{3}\,e^{\ll 1\gg}\,\gamma^2\,\epsilon^2 \ll 1\gg \hbar^2\,\xi 12_i\,\xi 23_i^3\right) \ll 1\gg \ll 1\gg$

In[∘]:= {$k = 1, $n = 2}; ($amt_{1,2\to 1}$ // $a\Delta t_{1\to 1,2}$) ≡ ($a\Delta t_{2\to 3,4}$ // $a\Delta t_{1\to 1,2}$ // $amt_{1,3\to 1}$ // $amt_{2,4\to 2}$)

Time: 0.06 seconds Out[∘]=

True

In[∘]:= {$k = 1, $n = 3}; ($amt_{1,2\to 1}$ // $a\Delta t_{1\to 1,2}$) ≡ ($a\Delta t_{2\to 3,4}$ // $a\Delta t_{1\to 1,2}$ // $amt_{1,3\to 1}$ // $amt_{2,4\to 2}$)

Time: 7.16 seconds Out[∘]=

True