Hidden Utilities and TEX

Headers

pdf

Pensieve header: Quantum sl_n.

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References.
[1] Rosso “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$”.
[2] Yamane “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.

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In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

pdf

```
This version: April 2020. Original version: July 1994.
```

In[•]:=
```
MakePDF[]
```

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In[•]:=
```
δij__ := KroneckerDelta[ij];
```

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In[•]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := PPSS@Collect[PPNormal@Normal@PPSeries@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Identity];
```

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In[•]:=
```
$N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝔄[i_] := 𝔄 <> i;
y[i_, j_] := y <> i <> j; b[i_] := b <> i; B[i_] := B <> i;
η[i_, j_] := η <> i <> j; β[i_] := β <> i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

In[•]:=
```
{τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.02 seconds Out[•]=
```
List["x", List[3, 4]]
```

The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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Greek - Latin duality:

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In[•]:=
```
Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]^* = ξ[i, j];
 ξ[i, j]^* = x[i, j]; y[i, j]^* = η[i, j];
 η[i, j]^* = y[i, j]
]
Do[As[i, 1, $N - 1], A[i]^* = 𝒜[i]; 𝒜[i]^* = A[i]; a[i]^* = α[i]; α[i]^* = a[i];
 b[i]^* = β[i]; β[i]^* = b[i]]
(vs_List)^* := (v ↦ v^*) /@ vs;
(u_αβ__)^* := (u^*)_αβ;
F[u_i_] := F[u_i] = ToExpression["F" <> ToString[u]]_i (*May conflict with τ,
  which assumes that generator names are a single letter (plus indices) *)
```

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In[•]:=
```
LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
    Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars^*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ⋃ GreekVars))_;
```

Upper and lower case variables:

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In[•]:=
```
Module[{i, s, p, c, d, χ, f, v},
  U2lRule = Union @@ Table[As[i, 1, $N - 1], {𝒜[i]_s_^p_. → e^(p α[i]_s), B[i]_s_^p_. → e^(-p ħ b[i]_s)}];
  l2URule = Append[Union @@ Table[As[i, 1, $N - 1],
      {e^(d_. + c_. b[i]_s_) → B[i]_s^(-c/ħ) e^d, e^(d_. + c_. α[i]_s_) → 𝒜[i]_s^c e^d}], e^χ_ :> e^Expand[l2U[χ]]];
  U2l[ℰ_] := ℰ /. U2lRule;
  l2U[ℰ_] := ℰ //. l2URule;
  Do[As[i, 1, $N - 1], With[{αi = α[i], 𝒜i = 𝒜[i], bi = b[i], Bi = B[i]},
    D_αi_s_[f_] := ∂_αi_s f + 𝒜i_s ∂_𝒜i_s f;
    D_bi_s_[f_] := ∂_bi_s f - ħ Bi_s ∂_Bi_s f;
   ]];
  D_v_[f_] := ∂_v f;
 ];
```

In[•]:=
```
{U2lRule, l2URule}
```

Time: 0.02 seconds Out[•]=

$$\left\{\left\{\text{B1}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b1}_{\text{s\$}}},\ \text{B2}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b2}_{\text{s\$}}},\ \text{B3}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b3}_{\text{s\$}}},\ \text{B4}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b4}_{\text{s\$}}},\ \text{B5}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b5}_{\text{s\$}}},\right.\right.$$
$$\text{B6}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b6}_{\text{s\$}}},\ \text{B7}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b7}_{\text{s\$}}},\ \text{B8}_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{-\text{p\$}\,\hbar\,\text{b8}_{\text{s\$}}},\ \mathcal{A}1_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 1_{\text{s\$}}},\ \mathcal{A}2_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 2_{\text{s\$}}},$$
$$\left.\mathcal{A}3_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 3_{\text{s\$}}},\ \mathcal{A}4_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 4_{\text{s\$}}},\ \mathcal{A}5_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 5_{\text{s\$}}},\ \mathcal{A}6_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 6_{\text{s\$}}},\ \mathcal{A}7_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 7_{\text{s\$}}},\ \mathcal{A}8_{\text{s\$_}}^{\text{p\$_}\cdot}\to e^{\text{p\$}\,\alpha 8_{\text{s\$}}}\right\},$$
$$\left\{e^{\text{d\$_}.+\text{c\$_}.\,\text{b1}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B1}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\text{b2}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B2}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\text{b3}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B3}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\right.$$
$$e^{\text{d\$_}.+\text{c\$_}.\,\text{b4}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B4}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\text{b5}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B5}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\text{b6}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B6}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},$$
$$e^{\text{d\$_}.+\text{c\$_}.\,\text{b7}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B7}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\text{b8}_{\text{s\$_}}}\to e^{\text{d\$}}\,\text{B8}_{\text{s\$}}^{-\frac{\text{c\$}}{\hbar}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 1_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}1_{\text{s\$}}^{\text{c\$}},$$
$$e^{\text{d\$_}.+\text{c\$_}.\,\alpha 2_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}2_{\text{s\$}}^{\text{c\$}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 3_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}3_{\text{s\$}}^{\text{c\$}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 4_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}4_{\text{s\$}}^{\text{c\$}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 5_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}5_{\text{s\$}}^{\text{c\$}},$$
$$\left.\left.e^{\text{d\$_}.+\text{c\$_}.\,\alpha 6_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}6_{\text{s\$}}^{\text{c\$}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 7_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}7_{\text{s\$}}^{\text{c\$}},\ e^{\text{d\$_}.+\text{c\$_}.\,\alpha 8_{\text{s\$_}}}\to e^{\text{d\$}}\,\mathcal{A}8_{\text{s\$}}^{\text{c\$}},\ e^{\mathcal{X}_{-}}:\to e^{\text{Expand}[\text{l2U}[\mathcal{X}]]}\right\}\right\}$$

In[•]:=
```
{D_{b3_i} /@ {b3_i^5, B3_i^7}, D_{α3_i} /@ {α3_i^5, A3_i^7}}
```

Time: 0.02 seconds Out[•]=

$$\left\{\left\{5\,\text{b3}_{\text{i}}^{4},\ -7\,\hbar\,\text{B3}_{\text{i}}^{7}\right\},\ \left\{5\,\alpha 3_{\text{i}}^{4},\ 7\,\mathcal{A}3_{\text{i}}^{7}\right\}\right\}$$

Canonical Form:

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In[•]:=
```
CF[ℰ_] := PP_CF@Module[{s1},
   s1 = If[$k == ∞,
     Expand[ℰ] /. e^{p_} :→ e^{Factor@p},
     Expand[ℰ] /. e^{p_} :→ e^{Factor@p} /. q^{n_.} :→ Sum_{k=0}^{$k} (ħ ϵ γ n)^k/k! /. ϵ^{k_.} /; k > $k :→ 0
    ]
   (*Expand@Collect[s1,Cases[s1,VarPattern,∞],CCF]/.CCF→Factor*)
  ]
```

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Weights: (Need confirmation!!!)

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In[•]:=
```
Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, -(i - j)};
  Wt[ξ[i, j]] = {0, -(i - j)}; Wt[η[i, j]] = {1, (i - j)};
 ];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[α[i]] = {0, 0}; Wt[β[i]] = {1, 0};
 ];
Wt[u_] := Wt[u];
```

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In[•]:=
```
(* q~e^{ħ γ ϵ}; Ai[n]~e^{ħ ϵ ai}; *)
```

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```
In[•]:= $n = 5; $k = ∞; $E := {$n, $k}
```

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```
In[•]:= $EPBWGens := Flatten@{
          Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
          Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
        }
        $EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
        x_ ≼ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $EPBWRule];
        $PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[•]:= $PBWGens*
```

Time: 0.02 seconds Out[•]=

$\{\xi 12, \xi 13, \xi 14, \xi 15, \xi 23, \xi 24, \xi 25, \xi 34, \xi 35, \xi 45, \alpha 1, \alpha 2, \alpha 3, \alpha 4\}$

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```
In[•]:= UCF[ℰ_] := PP_UCF@Module[{F, k},
          Expand@Collect[
            ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
              U_i_[l___, AB_[n_], r___] :↦
                Sum[As[k, 0, $k], (-2)^k ħ^k ϵ^k n^k / k! U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
            }],
            U_[___], F
          ] /. F → CF]
```

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```
In[•]:= m_s__[0] = 0;
        m_s__[x_Plus] := UCF[m_s /@ x];
        m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
        m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
        m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
        m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
          UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
        m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≼ y,
          c U_k[xx, x, y, yy],
          ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
        ];
        m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :↦ i, ∞];
```

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```
In[•]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := PP_NCM@ Module[{i, is = Supp[x] ⋂ Supp[y], σ, z, k},
    z = x; Do[z = m_{i→σ@i}[z], {i, is}];
    z = Expand[y z];
    If[$k < ∞, z = z /. ϵ^(k_ ·) /; k > $k :→ 0];
    Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := PP_UB@UCF[x ⊠ y - y ⊠ x];
```

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```
In[•]:= Δ_{i_→j_,k_}[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[f_, fs___] :→ m_{k,k1→k}@ m_{j,j1→j}[Δ_{j,k}[f] Δ_{i→j1,k1}[U_i[fs]]]
     }];
Δ_{i_→j_,k_,l__}[ℰ_] := ℰ // Δ_{i→j,k} // Δ_{k→k,l}
```

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```
In[•]:= S_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
      U_i[f_, fs___] :→ m_{i1,i→i}[S_i[f] S_{i1}[U_{i1}[fs]]]
     }];
S̄_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
      U_i[f_, fs___] :→ m_{i1,i→i}[S̄_i[f] S̄_{i1}[U_{i1}[fs]]]
     }];
```

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The 𝔸-algebra

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```
In[•]:= RR_k_[ai_, aj_] /; τ[ai] == τ[aj] == "a" := U_k[aj, ai];
RR_k_[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := U_k[Aj[n], ai];
RR_k_[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := U_k[ai, Aj[n]];
RR_k_[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := U_k[Aj[n], Ai[m]];
RR_k_[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@am;
   U_k[xij, am] + γ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[xij] / 2
  ];
RR_k_[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = ι@xij; {m} = ι@Am; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[xij, Am[n]]
 ]
```

```
In[•]:= RR_0[a[1], x[1, 2]]
```

Time: 0.02 seconds Out[•]=

$$\gamma\, U_0[x12] + U_0[x12, a1]$$

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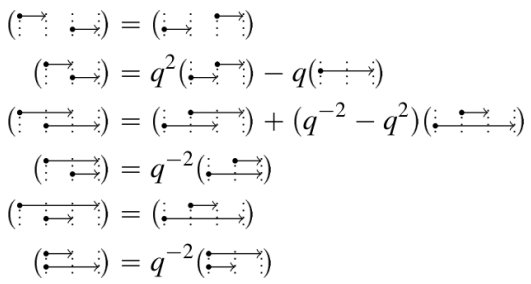
We take the reduction rules RR for xij's from Yamane page 509, rewritten in Monoblog/260928:



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```
In[•]:= RR_k[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
          {i, j} = ι@xij; {m, n} = ι@xmn;
          Which[
           (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[xij, xmn], (* I∨III *)
           (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
           i < m < j ∧ n > j, U_k[xij, xmn] + (q^-2 - q^2) U_k[x[i, n], x[m, j]], (* IV *)
           j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
          ]
         ]
```

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2
```

Time: 0.01 seconds Out[•]=

$$-q\,U_2[x24]\,U_3[x12] + q^2\,U_2[x23, x34]\,U_3[x12]$$

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
```

Time: 0.05 seconds Out[•]=

$$q^2\,U_0[x14] - q^3\,U_0[x12, x24] - q^3\,U_0[x13, x34] + q^4\,U_0[x12, x23, x34]$$

```
In[•]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
          DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
              P = U1[X] U2[Y] U3[Z];
              {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
             ]]
         ]]
```

Time: 14.45 seconds Out[•]=

$$\{\}$$

```
In[•]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
          Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
          DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
              P = U1[X] U2[Y] U3[Z];
              {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
            ]]
         ]]
      » {x12, x13, x23, a1, a2, A1[-1], A1[1], A1[2], A1[2], A2[0], A2[2], A1[0], A1[-1], A2[-1], A2[-2]}
Out[•]=
        {}

In[•]:= Module[{Bas = $PBWGens, P},
          DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
              P = U1[W] U2[X] U3[Y] U4[Z];
              {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
                (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
                (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
            ]]
         ]
Time: 1454.70 seconds Out[•]=
        {}
```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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```
In[•]:= Do[As[i, 1, $n - 1],
          Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
          Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
          Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
         ];
        Do[As[i, 1, $n - 2], As[n, i + 2, $n],
         Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
            q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
           ]]
        ]
```

In[•]:=
```
Block[{$k = ∞, $n = 5}, Column@Module[{Bas = $PBWGens, P},
   Table[As[P, Uᵢ /@ Bas], P → Δ_{i→j,k}[P]]
  ]]
```

Time: 0.06 seconds Out[•]=

$$U_i[x12] \to U_j[A1[-1]]\,U_k[x12] + U_j[x12]\,U_k[A1[1]]$$

$$U_i[x13] \to U_j[A1[-1], A2[-1]]\,U_k[x13] + \left(-\frac{1}{q^2} + q^2\right) U_j[x12, A2[-1]]\,U_k[x23, A1[1]] + U_j[x13]\,U_k[A1[1], A2[1]]$$

$$U_i[x14] \to U_j[A1[-1], A2[-1], A3[-1]]\,U_k[x14] + \left(-\frac{1}{q^2} + q^2\right) U_j[x12, A2[-1], A3[-1]]\,U_k[x24, A1[1]] + \left(-\frac{1}{q^2} + q^2\right) U_j[x13, A3[-1]]\,U_k[x34, A1[1], A2[1]] + U_j[x14]\,U_k[A1[1], A2[1], A3[1]]$$

$$U_i[x15] \to U_j[A1[-1], A2[-1], A3[-1], A4[-1]]\,U_k[x15] + \left(-\frac{1}{q^2} + q^2\right) U_j[x12, A2[-1], A3[-1], A4[-1]]\,U_k[x25, A1[1]] + \left(-\frac{1}{q^2} + q^2\right) U_j[x13, A3[-1], A4[-1]]\,U_k[x35, A1[1], A2[1]] + \left(-\frac{1}{q^2} + q^2\right) U_j[x14, A4[-1]]\,U_k[x45, A1[1], A2[1], A3[1]] + U_j[x15]\,U_k[A1[1], A2[1], A3[1], A4[1]]$$

$$U_i[x23] \to U_j[A2[-1]]\,U_k[x23] + U_j[x23]\,U_k[A2[1]]$$

$$U_i[x24] \to U_j[A2[-1], A3[-1]]\,U_k[x24] + \left(-\frac{1}{q^2} + q^2\right) U_j[x23, A3[-1]]\,U_k[x34, A2[1]] + U_j[x24]\,U_k[A2[1], A3[1]]$$

$$U_i[x25] \to U_j[A2[-1], A3[-1], A4[-1]]\,U_k[x25] + \left(-\frac{1}{q^2} + q^2\right) U_j[x23, A3[-1], A4[-1]]\,U_k[x35, A2[1]] + \left(-\frac{1}{q^2} + q^2\right) U_j[x24, A4[-1]]\,U_k[x45, A2[1], A3[1]] + U_j[x25]\,U_k[A2[1], A3[1], A4[1]]$$

$$U_i[x34] \to U_j[A3[-1]]\,U_k[x34] + U_j[x34]\,U_k[A3[1]]$$

$$U_i[x35] \to U_j[A3[-1], A4[-1]]\,U_k[x35] + \left(-\frac{1}{q^2} + q^2\right) U_j[x34, A4[-1]]\,U_k[x45, A3[1]] + U_j[x35]\,U_k[A3[1], A4[1]]$$

$$U_i[x45] \to U_j[A4[-1]]\,U_k[x45] + U_j[x45]\,U_k[A4[1]]$$

$$U_i[a1] \to U_j[a1]\,U_k[] + U_j[]\,U_k[a1]$$

$$U_i[a2] \to U_j[a2]\,U_k[] + U_j[]\,U_k[a2]$$

$$U_i[a3] \to U_j[a3]\,U_k[] + U_j[]\,U_k[a3]$$

$$U_i[a4] \to U_j[a4]\,U_k[] + U_j[]\,U_k[a4]$$

In[•]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U₁[X];
    X → (P // Δ_{1→1,2} // Δ_{2→2,3}) == (P // Δ_{1→1,3} // Δ_{1→1,2})
   ]]
]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4, A1[-2], A3[-1], A2[0], A3[1], A3[1], A4[0], A1[0], A1[1], A3[-1], A2[0]}

Out[•]=

{}

```
In[•]:= Module[{Bas = $PBWGens, P},
          Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
          DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas],
              P = U1[X] U2[Y];
              {X, Y} → UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
            ]]
        ]
      » {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
         A1[-1], A2[1], A4[2], A1[-1], A1[0], A1[-1], A4[-2], A4[2], A2[2], A2[1]}
Out[•]=
        {}
```

Is Δ a quantization of the co-bracket?

```
In[•]:= $k = 1
Time: 0.00 seconds Out[•]=
        1
```

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U₁ /@ Bas],
   P → UCF[(Δ₁→₁,₂[P] - Δ₁→₂,₁[P]) //.
       {q^n_. :> 1 + n ħ / 2, Uᵢ_[xs___, A_[n_], as___] :> Uᵢ[xs] + n ħ Uᵢ[xs, a @@ ι[A], as] / 2}
     ] /. ħ^n_ /; n > 1 :> 0
   ]
  ]
```

Time: 0.20 seconds Out[•]=

$$U_1[x12] \to -4\in\hbar\, U_1[x12]\, U_2[a1] + 4\in\hbar\, U_1[a1]\, U_2[x12]$$

$$U_1[x13] \to -4\in\hbar\, U_1[x13]\, U_2[a1] - 4\in\hbar\, U_1[x13]\, U_2[a2] - 4\,\gamma\in\hbar\, U_1[x23]\, U_2[x12] + 4\in\hbar\, U_1[a1]\, U_2[x13] + 4\in\hbar\, U_1[a2]\, U_2[x13] + 4\,\gamma\in\hbar\, U_1[x12]\, U_2[x23]$$

$$U_1[x14] \to -4\in\hbar\, U_1[x14]\, U_2[a1] - 4\in\hbar\, U_1[x14]\, U_2[a2] - 4\in\hbar\, U_1[x14]\, U_2[a3] - 4\,\gamma\in\hbar\, U_1[x24]\, U_2[x12] - 4\,\gamma\in\hbar\, U_1[x34]\, U_2[x13] + 4\in\hbar\, U_1[a1]\, U_2[x14] + 4\in\hbar\, U_1[a2]\, U_2[x14] + 4\in\hbar\, U_1[a3]\, U_2[x14] + 4\,\gamma\in\hbar\, U_1[x12]\, U_2[x24] + 4\,\gamma\in\hbar\, U_1[x13]\, U_2[x34]$$

$$U_1[x15] \to -4\in\hbar\, U_1[x15]\, U_2[a1] - 4\in\hbar\, U_1[x15]\, U_2[a2] - 4\in\hbar\, U_1[x15]\, U_2[a3] - 4\in\hbar\, U_1[x15]\, U_2[a4] - 4\,\gamma\in\hbar\, U_1[x25]\, U_2[x12] - 4\,\gamma\in\hbar\, U_1[x35]\, U_2[x13] - 4\,\gamma\in\hbar\, U_1[x45]\, U_2[x14] + 4\in\hbar\, U_1[a1]\, U_2[x15] + 4\in\hbar\, U_1[a2]\, U_2[x15] + 4\in\hbar\, U_1[a3]\, U_2[x15] + 4\in\hbar\, U_1[a4]\, U_2[x15] + 4\,\gamma\in\hbar\, U_1[x12]\, U_2[x25] + 4\,\gamma\in\hbar\, U_1[x13]\, U_2[x35] + 4\,\gamma\in\hbar\, U_1[x14]\, U_2[x45]$$

$$U_1[x23] \to -4\in\hbar\, U_1[x23]\, U_2[a2] + 4\in\hbar\, U_1[a2]\, U_2[x23]$$

$$U_1[x24] \to -4\in\hbar\, U_1[x24]\, U_2[a2] - 4\in\hbar\, U_1[x24]\, U_2[a3] - 4\,\gamma\in\hbar\, U_1[x34]\, U_2[x23] + 4\in\hbar\, U_1[a2]\, U_2[x24] + 4\in\hbar\, U_1[a3]\, U_2[x24] + 4\,\gamma\in\hbar\, U_1[x23]\, U_2[x34]$$

$$U_1[x25] \to -4\in\hbar\, U_1[x25]\, U_2[a2] - 4\in\hbar\, U_1[x25]\, U_2[a3] - 4\in\hbar\, U_1[x25]\, U_2[a4] - 4\,\gamma\in\hbar\, U_1[x35]\, U_2[x23] - 4\,\gamma\in\hbar\, U_1[x45]\, U_2[x24] + 4\in\hbar\, U_1[a2]\, U_2[x25] + 4\in\hbar\, U_1[a3]\, U_2[x25] + 4\in\hbar\, U_1[a4]\, U_2[x25] + 4\,\gamma\in\hbar\, U_1[x23]\, U_2[x35] + 4\,\gamma\in\hbar\, U_1[x24]\, U_2[x45]$$

$$U_1[x34] \to -4\in\hbar\, U_1[x34]\, U_2[a3] + 4\in\hbar\, U_1[a3]\, U_2[x34]$$

$$U_1[x35] \to -4\in\hbar\, U_1[x35]\, U_2[a3] - 4\in\hbar\, U_1[x35]\, U_2[a4] - 4\,\gamma\in\hbar\, U_1[x45]\, U_2[x34] + 4\in\hbar\, U_1[a3]\, U_2[x35] + 4\in\hbar\, U_1[a4]\, U_2[x35] + 4\,\gamma\in\hbar\, U_1[x34]\, U_2[x45]$$

$$U_1[x45] \to -4\in\hbar\, U_1[x45]\, U_2[a4] + 4\in\hbar\, U_1[a4]\, U_2[x45]$$

$$U_1[a1] \to 0$$

$$U_1[a2] \to 0$$

$$U_1[a3] \to 0$$

$$U_1[a4] \to 0$$

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The Antipode

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In[•]:=
```
Do[As[i, 1, $N - 1],
  S_j_[a[i]] = -U_j[a[i]];
  S_j_[A[i][n_]] = U_j[A[i][-n]];
  S_j_[x[i, i + 1]] = -q^-2 U_j[x[i, i + 1]]; (*?*)
  S̄_j_[a[i]] = -U_j[a[i]];
  S̄_j_[A[i][n_]] = U_j[A[i][-n]];
  S̄_j_[x[i, i + 1]] = -q^2 U_j[x[i, i + 1]]; (*?*)
 ];
Do[As[i, 1, $N - 2], As[n, i + 2, $N],
 S_j_[x[i, n]] = m_{j1,j→j}[
   q S_j[x[i, n - 1]] S_j1[x[n - 1, n]] - q^-1 S_j[x[n - 1, n]] S_j1[x[i, n - 1]]];
 S̄_j_[x[i, n]] = m_{j1,j→j}[
   q S̄_j[x[i, n - 1]] S̄_j1[x[n - 1, n]] - q^-1 S̄_j[x[n - 1, n]] S̄_j1[x[i, n - 1]]]
]
```

In[•]:=
```
Block[{$k = ∞, $n = 5}, Column@Module[{Bas = $PBWGens, P},
   Table[As[P, U_i /@ Bas], P → {S_i[P], S̄_i[P]}]
  ]]
```

Time: 0.08 seconds Out[•]=

$$U_i[x12] \to \left\{-\frac{U_i[x12]}{q^2}, -q^2 U_i[x12]\right\}$$

$$U_i[x13] \to \{(-1+2\gamma\in\hbar)\,U_i[x13] + 4\gamma\in\hbar\,U_i[x12, x23],\ (-1-6\gamma\in\hbar)\,U_i[x13] + 4\gamma\in\hbar\,U_i[x12, x23]\}$$

$$U_i[x14] \to \{(-1+2\gamma\in\hbar)\,U_i[x14] + 4\gamma\in\hbar\,U_i[x12, x24] + 4\gamma\in\hbar\,U_i[x13, x34],\ (-1-10\gamma\in\hbar)\,U_i[x14] + 4\gamma\in\hbar\,U_i[x12, x24] + 4\gamma\in\hbar\,U_i[x13, x34]\}$$

$$U_i[x15] \to \{(-1+2\gamma\in\hbar)\,U_i[x15] + 4\gamma\in\hbar\,U_i[x12, x25] + 4\gamma\in\hbar\,U_i[x13, x35] + 4\gamma\in\hbar\,U_i[x14, x45],\ (-1-14\gamma\in\hbar)\,U_i[x15] + 4\gamma\in\hbar\,U_i[x12, x25] + 4\gamma\in\hbar\,U_i[x13, x35] + 4\gamma\in\hbar\,U_i[x14, x45]\}$$

$$U_i[x23] \to \left\{-\frac{U_i[x23]}{q^2}, -q^2 U_i[x23]\right\}$$

$$U_i[x24] \to \{(-1+2\gamma\in\hbar)\,U_i[x24] + 4\gamma\in\hbar\,U_i[x23, x34],\ (-1-6\gamma\in\hbar)\,U_i[x24] + 4\gamma\in\hbar\,U_i[x23, x34]\}$$

$$U_i[x25] \to \{(-1+2\gamma\in\hbar)\,U_i[x25] + 4\gamma\in\hbar\,U_i[x23, x35] + 4\gamma\in\hbar\,U_i[x24, x45],\ (-1-10\gamma\in\hbar)\,U_i[x25] + 4\gamma\in\hbar\,U_i[x23, x35] + 4\gamma\in\hbar\,U_i[x24, x45]\}$$

$$U_i[x34] \to \left\{-\frac{U_i[x34]}{q^2}, -q^2 U_i[x34]\right\}$$

$$U_i[x35] \to \{(-1+2\gamma\in\hbar)\,U_i[x35] + 4\gamma\in\hbar\,U_i[x34, x45],\ (-1-6\gamma\in\hbar)\,U_i[x35] + 4\gamma\in\hbar\,U_i[x34, x45]\}$$

$$U_i[x45] \to \left\{-\frac{U_i[x45]}{q^2}, -q^2 U_i[x45]\right\}$$

$$U_i[a1] \to \{-U_i[a1], -U_i[a1]\}$$
$$U_i[a2] \to \{-U_i[a2], -U_i[a2]\}$$
$$U_i[a3] \to \{-U_i[a3], -U_i[a3]\}$$
$$U_i[a4] \to \{-U_i[a4], -U_i[a4]\}$$

```
In[•]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas],
     P = U₁[X];
     X → (P // Δ₁→2,3 // S₂ // m₂,3→1)
    ]]
 ]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
A1[1], A1[0], A3[1], A1[1], A2[-1], A4[1], A3[-2], A4[-2], A1[-2], A3[-2]}

Out[•]=

$$\{x12 \to 0,\ x13 \to 0,\ x14 \to 0,\ x15 \to 0,\ x23 \to 0,\ x24 \to 0,\ x25 \to 0,\ x34 \to 0,\ x35 \to 0,\ x45 \to 0,\ a1 \to 0,\ a2 \to 0,\ a3 \to 0,\ a4 \to 0,\ A1[1] \to U_1[],\ A1[0] \to U_1[],\ A3[1] \to U_1[],\ A1[1] \to U_1[],\ A2[-1] \to U_1[],\ A4[1] \to U_1[],\ A3[-2] \to U_1[],\ A4[-2] \to U_1[],\ A1[-2] \to U_1[],\ A3[-2] \to U_1[]\}$$

```
In[•]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas],
     P = U₁[X];
     X → UCF[((P // S₁ // S̄₁) - P)]
    ]]
 ]]
```

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
A3[-2], A4[-1], A4[0], A2[-2], A4[-2], A4[2], A3[1], A2[1], A2[2], A2[0]}

Out[•]=

$$\begin{aligned}
&\{x13 \to \left(4\gamma\in\hbar - 4q^5\gamma\in\hbar - 12\gamma^2\in^2\hbar^2\right)U_1[x13] + \left(-4\gamma\in\hbar + 4q^6\gamma\in\hbar + 8\gamma^2\in^2\hbar^2\right)U_1[x12, x23],\\
&x14 \to \left(8\gamma\in\hbar - 8q^3\gamma\in\hbar - 20\gamma^2\in^2\hbar^2 - 48q^3\gamma^2\in^2\hbar^2\right)U_1[x14] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 24q^4\gamma^2\in^2\hbar^2\right)U_1[x12, x24] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 24q^4\gamma^2\in^2\hbar^2\right)U_1[x13, x34] -\\
&\quad 32q^4\gamma^2\in^2\hbar^2 U_1[x12, x23, x34], x15 \to\\
&\left(12\gamma\in\hbar - 4q\gamma\in\hbar - 8q^3\gamma\in\hbar - 28\gamma^2\in^2\hbar^2 - 48q\gamma^2\in^2\hbar^2 - 80q^3\gamma^2\in^2\hbar^2 - 144q\gamma^3\in^3\hbar^3\right)U_1[x15] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 40q^4\gamma^2\in^2\hbar^2 + 96q\gamma^3\in^3\hbar^3\right)\\
&\quad U_1[x12, x25] +\\
&\quad\left(-4\gamma\in\hbar + 4q^2\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 48q^2\gamma^2\in^2\hbar^2 + 32q^3\gamma^2\in^2\hbar^2 + 144q^2\gamma^3\in^3\hbar^3\right)U_1[x13, x35] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 40q^4\gamma^2\in^2\hbar^2 + 96q\gamma^3\in^3\hbar^3\right)\\
&\quad U_1[x14, x45] + \left(-16q^2\gamma^2\in^2\hbar^2 - 16q^4\gamma^2\in^2\hbar^2 - 96q^2\gamma^3\in^3\hbar^3\right)U_1[x12, x23, x35] +\\
&\quad\left(-32q^4\gamma^2\in^2\hbar^2 - 64q\gamma^3\in^3\hbar^3\right)U_1[x12, x24, x45] + \left(-16q^2\gamma^2\in^2\hbar^2 - 16q^4\gamma^2\in^2\hbar^2 - 96q^2\gamma^3\in^3\hbar^3\right)\\
&\quad U_1[x13, x34, x45] + 64q^2\gamma^3\in^3\hbar^3 U_1[x12, x23, x34, x45],\\
&x24 \to \left(4\gamma\in\hbar - 4q^5\gamma\in\hbar - 12\gamma^2\in^2\hbar^2\right)U_1[x24] + \left(-4\gamma\in\hbar + 4q^6\gamma\in\hbar + 8\gamma^2\in^2\hbar^2\right)U_1[x23, x34],\\
&x25 \to \left(8\gamma\in\hbar - 8q^3\gamma\in\hbar - 20\gamma^2\in^2\hbar^2 - 48q^3\gamma^2\in^2\hbar^2\right)U_1[x25] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 24q^4\gamma^2\in^2\hbar^2\right)U_1[x23, x35] +\\
&\quad\left(-4\gamma\in\hbar + 4q^4\gamma\in\hbar + 8\gamma^2\in^2\hbar^2 + 16q^3\gamma^2\in^2\hbar^2 + 24q^4\gamma^2\in^2\hbar^2\right)U_1[x24, x45] -\\
&\quad 32q^4\gamma^2\in^2\hbar^2 U_1[x23, x34, x45],\\
&x35 \to \left(4\gamma\in\hbar - 4q^5\gamma\in\hbar - 12\gamma^2\in^2\hbar^2\right)U_1[x35] + \left(-4\gamma\in\hbar + 4q^6\gamma\in\hbar + 8\gamma^2\in^2\hbar^2\right)U_1[x34, x45]\}
\end{aligned}$$

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The Generating Functions Engine

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Canonical Forms:

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```
In[•]:= LogReduce[ℰ_] :=
          ℰ /. c_ * Log[a_] :> Log@Factor[a^c] //. Log[a_] + Log[b_] :> Log@Factor[a b];
        CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
        CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

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𝔼 operations:

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```
In[•]:= ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
        ℰ_𝔼[k_Integer] := ℰ[[k + 1]]; 𝔼_[ℰs___][k_Integer] := {ℰs}[[k + 1]];
        𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
        𝔼d1_→r1_[ℰ1s___] ≡ 𝔼d2_→r2_[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
        𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
        𝔼d1_→r1_[ℰ1s___] 𝔼d2_→r2_[ℰ2s___] ^:= 𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[ℰ1s] 𝔼[ℰ2s]);
```

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```
In[•]:= 𝔼d1_→r1_[ℰ1s___] // 𝔼d2_→r2_[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
          lvs = Cases[{ℰ1s}, (Alternatives @@ LatinVars)i_ /; MemberQ[is, i], ∞] ⋃
            Cases[{ℰ2s}, (Alternatives @@ GreekVars)i_ /; MemberQ[is, i], ∞]*;
          llvs = lvs /. u_i_ :> u$@i;
          𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Zip llvs⋃llvs* [{(ℱ /@ llvs*).(ℱ /@ llvs), Times[
                𝔼[ℰ1s] /. Thread[lvs → llvs],
                𝔼[ℰ2s] /. Thread[lvs* → llvs*]
              ]}])
        ]
```

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Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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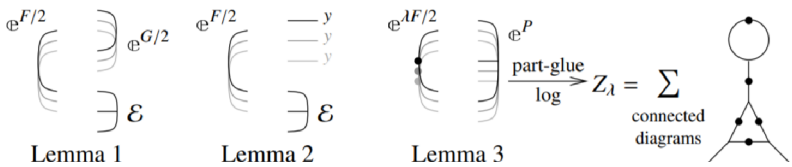

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight *n* variables in perturbations with the weight 0 variables in the coefficients of *Q*.

dvs stands for "Diagonal Variables". In $\mathfrak{gl}_n$ they are the x_{ii}'s and the y_{ii}'s. They have weights {1,0} and {0,0}.

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```
In[•]:= Zip_vs_[{ℱ_, ℰ_}] :=
 PP_Zip@Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
    {ℱ, ℰ} // Zip1_vs(*//Zip2_Complement[vs,dvs]*) // Zip2_dvs //
      Zip3_Complement[vs,dvs] // Zip3_dvs
   ] // Last
```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

pdf

```
In[•]:= Zip1_{} = Identity;
Zip1_vs_@{ℱ_, 𝔼[Q_, P___]} := PP_Zip1@Module[{ℐ, F, G, u, v},
   (*Echo[{vs,ℱ,Q,P};*)
   ℐ = IdentityMatrix@Length@vs;
   F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_F[u],F[v] ℱ, 0], {u, vs}, {v, vs}];
   G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_u,v Q, 0], {u, vs}, {v, vs}];
   (*Echo@MatrixForm[ℐ-G.F];*)
   {CF[(F /@ vs).(F.Inverse[ℐ - G.F]).(F /@ vs) / 2],
    𝔼[CF[Q - PowerExpand@Log[Det[ℐ - G.F]] / 2 - vs.G.vs / 2], P]}
  ]
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

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```
In[•]:= Zip2_{} = Identity;
Zip2_vs_@{ℱ_, 𝔼[Q_, P___]} := PP_Zip2@Module[{F, Y, u, v},
   (*EchoLabel["Zip2"][{vs,ℱ,Q,P};*)
   F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_F[u],F[v] ℱ], 0], {u, vs}, {v, vs}];
   Y = Table[∂_v Q, {v, vs}] /. (*AlsoUpper@*)Table[v → 0, {v, vs}];
   CF /@ ({ℱ, 𝔼[Q - Y.vs + Y.F.Y / 2, P]} /. (*AlsoUpper@*)Thread[vs → vs + F.Y])
  ]
```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power m of λ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda=\sum Z[m]\,\lambda^m$.

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In[•]:=

```
Zip3vs_@{ℱ_, ℰ_, 𝔼} := PPZip3@Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3",vs,ℱ,ℰ}];*)
  Do[Z[0, kk] = ℰ[[kk + 1]], {kk, 0, ℰ@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂F[u],F[v] ℱ, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ ℰ@$, ++kk,
    Z[m + 1, kk] = PPZip3[m,kk]@CF@Sum[
        If[F[u, v] === 0, 0, F[u, v]/(2 (m + 1)) (Z[m, kk, u, v] +
            Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
        {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
      ℱ - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, ℰ@$}]
     } /. (*AlsoUpper@*)Table[v → 0, {v, vs}])
 ]
```

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“Define” Code

pdf

In[•]:=

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[opnisp,$E_, PPBoot@Block[{i, j, k}, opisp,$E = ℰ; opnis,$E]];
    SD[opisp, op{is},$E]; SD[opsis__, op{sis}];
   ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
   }] ]]
```

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The Generating Functions for the Ⱥ operations

```
In[•]:= BeginProfile[]
```

Time: 0.08 seconds Out[•]=

```
ProfileRoot
```

```
In[•]:= $k = 1;
```

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Ad, TT, Seepage

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```
In[•]:= PotQ[0] = True;
PotQ[c_. Uᵢ_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
In[•]:= Ad[x_][y_] /; PotQ[x] ∧ (Head[y] =!= Plus) := If[y === 0, 0, PP_AdPoty[e^(UB[y,x]/y) y]];
Ad[x_][y_Plus] /; PotQ[x] := PP_AdPotPlus[Ad[x] /@ y];
```

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```
In[•]:= Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PP_AdNil@Module[{k = 1, s = y, t = y},
   While[(t = UB[t, x]) =!= 0, s += t / k!; ++k];
   UCF[s]
  ]
```

```
In[•]:= $k = 2
```

Time: 0.00 seconds Out[•]=

```
2
```

```
In[•]:= Ad[ϵ ħ Uⱼ[x12] Uₖ[]][Uⱼ[x23] Uₖ[a4]]
```

Time: 0.03 seconds Out[•]=

$$\left(-\epsilon\,\hbar-\gamma\,\epsilon^2\,\hbar^2\right)\,U_j[x13]\,U_k[a4]+U_j[x23]\,U_k[a4]+2\,\gamma\,\epsilon^2\,\hbar^2\,U_j[x12,x23]\,U_k[a4]$$

```
In[•]:= PotQ[7 ħ Uⱼ[x12] Uₖ[a3]]
```

Time: 0.02 seconds Out[•]=

```
PotQ[7 ħ Uⱼ[x12] Uₖ[a3]]
```

```
In[•]:= Ad[7 ħ Uⱼ[x12] Uₖ[a3]][Uⱼ[x23] Uₖ[a4]]
```

Time: 0.09 seconds Out[•]=

$$U_j[x23]\,U_k[a4]+\left(-7\,\hbar-7\,\gamma\,\epsilon\,\hbar^2-\frac{7}{2}\,\gamma^2\,\epsilon^2\,\hbar^3\right)U_j[x13]\,U_k[a3,a4]+$$
$$\left(14\,\gamma\,\epsilon\,\hbar^2+14\,\gamma^2\,\epsilon^2\,\hbar^3\right)U_j[x12,x23]\,U_k[a3,a4]-98\,\gamma^2\,\epsilon^2\,\hbar^4\,U_j[x12,x13]\,U_k[a3,a3,a4]+$$
$$98\,\gamma^2\,\epsilon^2\,\hbar^4\,U_j[x12,x12,x23]\,U_k[a3,a3,a4]-\frac{686}{3}\,\gamma^2\,\epsilon^2\,\hbar^5\,U_j[x12,x12,x13]\,U_k[a3,a3,a3,a4]$$

```
In[•]:= Ad[π ⅈ Uⱼ[a1]][Uⱼ[x23] Uₖ[a4]]
```

Time: 0.03 seconds Out[•]=

$$e^{\frac{i\pi\gamma}{2}}\ U_j[x23]\ U_k[a4]$$

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```
In[•]:= TTλ_[] = 0; (* Translated Tangent *)
        TTλ_[xs___, x_] := PP_TT@UCF[Ad[x][TTλ[xs]]] + ∂λ x];
        TTλ_[xs_List] := TTλ @@ xs
```

```
In[•]:= TTλ[λ ξ12 U₀[x12] U₁[], λ ξ23 U₀[x23] U₁[]]
```

Time: 0.09 seconds Out[•]=

$$\xi12\ U_0[x12]\ U_1[] + \left(\lambda\ \xi12\ \xi23 + \gamma\in\lambda\ \xi12\ \xi23\ \hbar + \frac{1}{2}\gamma^2\in^2\lambda\ \xi12\ \xi23\ \hbar^2\right) U_0[x13]\ U_1[] +$$
$$\xi23\ U_0[x23]\ U_1[] + \left(-2\gamma\in\lambda\ \xi12\ \xi23\ \hbar - 2\gamma^2\in^2\lambda\ \xi12\ \xi23\ \hbar^2\right) U_0[x12, x23]\ U_1[] -$$
$$2\gamma^2\in^2\lambda^2\ \xi12\ \xi23^2\ \hbar^2\ U_0[x13, x23]\ U_1[] + 2\gamma^2\in^2\lambda^2\ \xi12\ \xi23^2\ \hbar^2\ U_0[x12, x23, x23]\ U_1[] +$$
$$\frac{2}{3}\gamma^2\in^2\lambda^3\ \xi12\ \xi23^3\ \hbar^2\ U_0[x13, x23, x23]\ U_1[]$$

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```
In[•]:= 𝕆i_[poly_] := PP𝕆@Total[
            CoefficientRules[poly, #i & /@ $PBWGens] /.
              (p_ → c_) :↦ c Ui @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
            ];
        𝕆i_,is__[poly_] := 𝕆i[𝕆is[poly]];
        ℙi_[ℰ_] := ℰ /. Ui[xs___] :↦ Times @@ (#i & /@ {xs});
        ℙi_,is__[poly_] := ℙi[ℙis[poly]];
```

```
In[•]:= 𝕆ᵢ[x12ᵢ² x13ᵢ + 7 x23ᵢ¹⁷]
```

Time: 0.01 seconds Out[•]=

$$U_i[x12, x12, x13] + 7\ U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$$

```
In[•]:= 𝕆ᵢ,ⱼ[x12ᵢ² x13ᵢ x24ⱼ + 7 x23ᵢ¹⁷ x25ⱼ³]
```

Time: 0.05 seconds Out[•]=

$$U_i[x12, x12, x13]\ U_j[x24] + 7\ U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]\ U_j[x25, x25, x25]$$

```
In[•]:= ℙᵢ,ⱼ@𝕆ᵢ,ⱼ[x12ᵢ² x13ᵢ x24ⱼ + 7 x23ᵢ¹⁷ x25ⱼ³]
```

Time: 0.03 seconds Out[•]=

$$x12_i^2\ x13_i\ x24_j + 7\ x23_i^{17}\ x25_j^3$$

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```
In[•]:= Seepage_i[{}, c_] = c;
        Seepage_i[ξs_List, P_] := PP_Seepage@Module[{lx},
            lx = $PBWGens[[Length@ξs]];
            CF@Collect[P, lx_i, ℙ_i[Ad[Last[ξs]] U_i[lx]][𝕆_i[Seepage_i[Most@ξs, #]]]]
              &]
           ];
        Seepage_i_,is_[ξs_List, ξss__List, P_] := Seepage_i[ξs, Seepage_is[ξss, P]]

In[•]:= $k = 1

Time: 0.00 seconds Out[•]=
        1

In[•]:= $PBWGens*

Time: 0.02 seconds Out[•]=
        {ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

In[•]:= Seepage_i[$PBWGens*, x35_i]

Time: 0.14 seconds Out[•]=
        e^(α2γ/2 − α3γ/2 − α4γ/2) x35_i + 2 e^(α2γ/2 − 3α4γ/2) γ ∈ ξ45 ℏ x35_i x45_i

In[•]:= Seepage_i,j[#_i & /@ $PBWGens*, #_j & /@ $PBWGens*, x35_i x45_j]

Time: 0.27 seconds Out[•]=
        e^(γα2_i/2 − γα3_i/2 + γα3_j/2 − γα4_i/2 − γ α4_j) x35_i x45_j + 2 e^(γα2_i/2 + γα3_j/2 − 3γα4_i/2 − γ α4_j) γ ∈ ℏ x35_i x45_i x45_j ξ45_i
```

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```
In[•]:= Invert_is__[ℰ_] := PP_Invert@Module[{A, B, U1, j}, (* ℰ=A U[]+∈ B *)
            U1 = Times @@ (U_#[] & /@ {is});
            A = Coefficient[ℰ, U1] /. ∈ → 0;
            B = UCF[(ℰ - A U1)];
            UCF[
              A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply @@ ConstantArray[B / A, j]])
            ]

In[•]:= $k = 3;
        Invert_i,j[7 U_i[] U_j[] + ∈ U_i[x12] U_j[x23]]

Time: 0.05 seconds Out[•]=
        1/7 U_i[] U_j[] − 1/49 ∈ U_i[x12] U_j[x23] +
          1/343 ∈^2 U_i[x12, x12] U_j[x23, x23] − (∈^3 U_i[x12, x12, x12] U_j[x23, x23, x23])/2401

In[•]:= {$k = 1, $n = 3}

Time: 0.00 seconds Out[•]=
        {1, 3}
```

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In[•]:=
```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
 PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinits = inits, rfs = fs, s, p},
   While[rfs =!= {},
    {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
    {{s}} = DSolve[reqns[[p]] ∧ rinits[[p]], rfs[[p]], λ];
    AppendTo[ss, s = s /. F_Function :↦ ReplacePart[F, 2 → CF[F[[2]]]]];
    reqns = Delete[reqns, p] /. s /. e_ == 0 :↦ CF[e] == 0;
    rinits = Delete[rinits, p];
    rfs = Delete[rfs, p];
   ];
   ss
  ]
```

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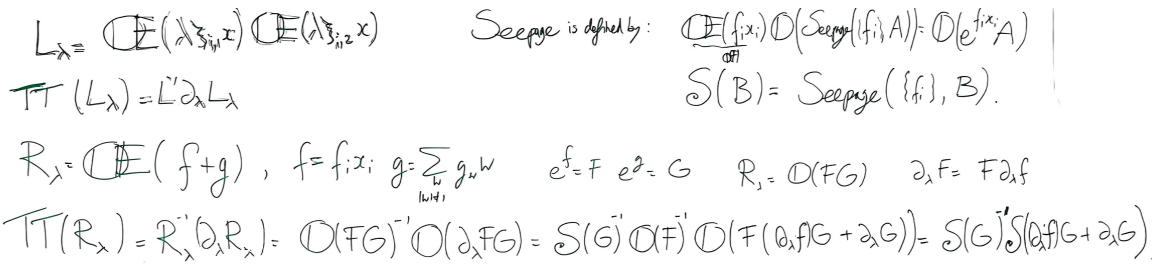

pdf

In[•]:=

```
Define[amt_{i,j→k} = PP"amt"[$k,$n]@Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ); ES[s_][ℰ_] := ℰ;
    TTL = ES["TTL"]@PP"TTL"@TT_λ@
        Join[l1 = Table[If[τ[g] === "x", λ, 1] (g*)_i U_k[g], {g, $PBWGens}], l1 /. i → j];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], U_k[___], ∞];
    ff = Table[
       Sum[ϵ^p f_{p,u_k}[λ], {p, 0, $k}],
       {u, $PBWGens}
      ] /. f_{p_,u_k}[λ] /; τ[u] == "a" :↦ If[p == 0, (u*)_i + (u*)_j, 0];
    fs = Flatten[Table[
       f_{p,ℙ_k[m]},
       {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
      ϵ^p f_{p,ℙ_k[u]}[λ] ℙ_k[u],
      {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
     ];
    G = Series[e^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[𝒪_k[Seepage_k[ff, UCF@B]]];
    𝒮G = ES["𝒮G"]@PP"𝒮G"@𝒮[G];
    TTR = ES["TTR"]@PP"TTR"@UCF[Invert_k[𝒮G] ⊠ 𝒮[(∂_λ (ff. (#_k & /@ $PBWGens)) G + ∂_λ G)]];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
       Coefficient[Coefficient[diff, m], ϵ, p] == 0,
       {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼_{{i,j}→{k}} @@ Table[As[p, 0, $k], Plus[
       If[p == 0, Sum[As[m, 1, $n - 1], (α[m]_i + α[m]_j) a[m]_k], 0],
       Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :↦ u (f_{p,u} /. ss)[λ] /. λ | ħ | γ → 1]
     ]
   ]
]
```

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In[•]:=

```
Define[aΔt_{i→j,k} = PP"aΔt"[$k,$n]@Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
    TTL = TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g^*)_i Δ_{i→j,k}[U_i[g]]], {g, $PBWGens}];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃
      Union@Cases[Coefficient[TTL, ϵ, p], c_. U_j[x___] U_k[y___] :> U_j[x] U_k[y], ∞];
    ffj = Table[
       Sum[ϵ^p f_{p,u_j}[λ], {p, 0, $k}],
       {u, $PBWGens}
      ] /. f_{p_,u_j}[λ] /; τ[u] == "a" :> If[p == 0, (u^*)_i, 0];
    ffk = ffj /. j → k;
    fs = Flatten[Table[
       f_{p,𝕡_{j,k}[m]},
       {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
      ϵ^p f_{p,𝕡_{j,k}[u]}[λ] 𝕡_{j,k}[u],
      {p, 0, $k}, {u, Complement[mons[p], U_j[] (U_k /@ $PBWGens) ⋃ U_k[] (U_j /@ $PBWGens)]}
     ];
    G = Series[ℯ^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[𝒪_{j,k}[Seepage_{j,k}[ffj, ffk, UCF@B]]];
    𝒮G = PP"𝒮G"@𝒮[G];
    BB = PP"BB"[∂_λ (ffj. (#_j & /@ $PBWGens) + ffk. (#_k & /@ $PBWGens)) G + ∂_λ G];
    𝒮BB = PP"𝒮BB"@𝒮[BB];
    TTR = PP"TTR"@UCF[Invert_{j,k}[𝒮G] ⊠ 𝒮BB];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
       Coefficient[Coefficient[diff, m], ϵ, p] == 0,
       {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼_{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
       If[p == 0, Sum[As[m, 1, $n - 1], α[m]_i (a[m]_j + a[m]_k)], 0],
       Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :> u (f_{p,u} /. ss)[λ] /. λ | ħ | γ → 1]
     ]
   ]]
```

In[•]:=
```
{$k = 1, $n = 2}; amt_{i,j→k}
(amt_{1,2→2} // amt_{2,3→1}) ≡ (amt_{2,3→2} // amt_{1,2→1})
```

Time: 0.27 seconds Out[•]=

$$\mathbb{E}_{\{i,j\}\to\{k\}}\left[a1_k\ (\alpha 1_i+\alpha 1_j)+x12_k\ \left(\xi 12_i+e^{\alpha 1_i}\ \xi 12_j\right),\ 0\right]$$

Out[•]=

True

In[•]:=
```
{$k = 1, $n = 3}; amt_{i,j→k}
(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})
```

Time: 3.58 seconds Out[•]=

$$\begin{aligned}&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a1_k\ (\alpha 1_i+\alpha 1_j)+a2_k\ (\alpha 2_i+\alpha 2_j)+x12_k\ \left(\xi 12_i+e^{\frac{1}{2}\ (2\,\alpha 1_i-\alpha 2_i)}\ \xi 12_j\right)+\\&\quad x13_k\ \left(\xi 13_i+e^{\frac{1}{2}\ (\alpha 1_i+\alpha 2_i)}\ \xi 13_j-e^{\frac{1}{2}\ (2\,\alpha 1_i-\alpha 2_i)}\ \xi 12_j\ \xi 23_i\right)+x23_k\ \left(\xi 23_i+e^{\frac{1}{2}\ (-\alpha 1_i+2\,\alpha 2_i)}\ \xi 23_j\right),\\&\quad -2\,e^{\frac{1}{2}\ (2\,\alpha 1_i-\alpha 2_i)}\ x12_k\ x13_k\ \xi 12_j\ \xi 13_i-e^{\frac{1}{2}\ (2\,\alpha 1_i-\alpha 2_i)}\ x13_k\ \xi 12_j\ \xi 23_i+\\&\quad 2\,e^{\frac{1}{2}\ (2\,\alpha 1_i-\alpha 2_i)}\ x12_k\ x23_k\ \xi 12_j\ \xi 23_i-2\,e^{\frac{1}{2}\ (\alpha 1_i+\alpha 2_i)}\ x13_k\ x23_k\ \xi 13_j\ \xi 23_i\Big]\end{aligned}$$

Out[•]=

True

In[•]:=
```
{$k = 2, $n = 3}; amt_{i,j→k}
(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})
```

Time: 44.66 seconds Out[•]=

$$\begin{aligned}
&\mathbb{E}_{\{i,j\}\to\{k\}}\Big[\mathrm{a1}_k\,(\alpha 1_i+\alpha 1_j)+\mathrm{a2}_k\,(\alpha 2_i+\alpha 2_j)+\mathrm{x12}_k\left(\xi 12_i+e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\right)+\\
&\quad \mathrm{x13}_k\left(\xi 13_i+e^{\frac{1}{2}(\alpha 1_i+\alpha 2_i)}\,\xi 13_j-e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 23_i\right)+\mathrm{x23}_k\left(\xi 23_i+e^{\frac{1}{2}(-\alpha 1_i+2\alpha 2_i)}\,\xi 23_j\right),\\
&\quad -2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x12}_k\,\mathrm{x13}_k\,\xi 12_j\,\xi 13_i-e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x13}_k\,\xi 12_j\,\xi 23_i+\\
&\quad 2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x12}_k\,\mathrm{x23}_k\,\xi 12_j\,\xi 23_i-2\,e^{\frac{1}{2}(\alpha 1_i+\alpha 2_i)}\,\mathrm{x13}_k\,\mathrm{x23}_k\,\xi 13_j\,\xi 23_i,\\
&\quad -\frac{1}{2}\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x13}_k\,\xi 12_j\,\xi 23_i+2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x12}_k\,\mathrm{x23}_k\,\xi 12_j\,\xi 23_i+2\,e^{2\alpha 1_i-\alpha 2_i}\,\mathrm{x12}_k^2\,\mathrm{x23}_k\,\xi 12_j^2\,\xi 23_i+\\
&\quad 2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x12}_k\,\mathrm{x23}_k^2\,\xi 12_j\,\xi 23_i^2+e^{2\alpha 1_i-\alpha 2_i}\,\mathrm{x13}_k^2\,\xi 12_j^2\,\xi 23_i^2-\frac{2}{9}\,e^{\frac{3}{2}(2\alpha 1_i-\alpha 2_i)}\,\mathrm{x13}_k^3\,\xi 12_j^3\,\xi 23_i^3+\\
&\quad \mathrm{x12}_k\,\mathrm{x13}_k\left(2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 13_i-2\,e^{2\alpha 1_i-\alpha 2_i}\,\xi 12_j^2\,\xi 23_i\right)+\\
&\quad \mathrm{x12}_k^2\,\mathrm{x13}_k\left(2\,e^{2\alpha 1_i-\alpha 2_i}\,\xi 12_j^2\,\xi 13_i-\frac{2}{3}\,e^{\frac{3}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j^3\,\xi 23_i\right)+\\
&\quad \mathrm{x12}_k\,\mathrm{x13}_k\,\mathrm{x23}_k\left(-4\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 13_i\,\xi 23_i-4\,e^{\frac{3\alpha 1_i}{2}}\,\xi 12_j\,\xi 13_j\,\xi 23_i\right)+\\
&\quad \mathrm{x13}_k\,\mathrm{x23}_k\left(2\,e^{\frac{1}{2}(\alpha 1_i+\alpha 2_i)}\,\xi 13_j\,\xi 23_i-2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 23_i^2\right)+\\
&\quad \mathrm{x12}_k\,\mathrm{x13}_k^2\left(2\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 13_i^2+\frac{2}{3}\,e^{\frac{3}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j^3\,\xi 23_i^2\right)+\\
&\quad \mathrm{x13}_k\,\mathrm{x23}_k^2\left(2\,e^{\frac{1}{2}(\alpha 1_i+\alpha 2_i)}\,\xi 13_j\,\xi 23_i^2-\frac{2}{3}\,e^{\frac{1}{2}(2\alpha 1_i-\alpha 2_i)}\,\xi 12_j\,\xi 23_i^3\right)+\\
&\quad \mathrm{x13}_k^2\,\mathrm{x23}_k\left(2\,e^{\alpha 1_i+\alpha 2_i}\,\xi 13_j^2\,\xi 23_i+\frac{2}{3}\,e^{2\alpha 1_i-\alpha 2_i}\,\xi 12_j^2\,\xi 23_i^3\right)\Big]
\end{aligned}$$

Out[•]=
```
True
```

In[•]:=
```
{$k = 1, $n = 2}; aΔt_{i→j,k}
(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})
```

Time: 0.23 seconds Out[•]=

$$\mathbb{E}_{\{i\}\to\{j,k\}}\left[\left(\mathrm{a1}_j+\mathrm{a1}_k\right)\alpha 1_i+\mathrm{x12}_j\,\xi 12_i+\mathrm{x12}_k\,\xi 12_i,\ -2\,\mathrm{a1}_k\,\mathrm{x12}_j\,\xi 12_i+2\,\mathrm{a1}_j\,\mathrm{x12}_k\,\xi 12_i\right]$$

Out[•]=
```
True
```

In[•]:=
```
{$k = 1, $n = 3}; aΔt_{i→j,k}
(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})
```

Time: 0.08 seconds Out[•]=

$$\begin{aligned}
&\mathbb{E}_{\{i\}\to\{j,k\}}\Big[\\
&\quad \left(\mathrm{a1}_j+\mathrm{a1}_k\right)\alpha 1_i+\left(\mathrm{a2}_j+\mathrm{a2}_k\right)\alpha 2_i+\mathrm{x12}_j\,\xi 12_i+\mathrm{x12}_k\,\xi 12_i+\mathrm{x13}_j\,\xi 13_i+\mathrm{x13}_k\,\xi 13_i+\mathrm{x23}_j\,\xi 23_i+\mathrm{x23}_k\,\xi 23_i,\\
&\quad -2\,\mathrm{a1}_k\,\mathrm{x12}_j\,\xi 12_i+2\,\mathrm{a1}_j\,\mathrm{x12}_k\,\xi 12_i-2\,\mathrm{a1}_k\,\mathrm{x13}_j\,\xi 13_i-2\,\mathrm{a2}_k\,\mathrm{x13}_j\,\xi 13_i+2\,\mathrm{a1}_j\,\mathrm{x13}_k\,\xi 13_i+\\
&\quad 2\,\mathrm{a2}_j\,\mathrm{x13}_k\,\xi 13_i+\mathrm{x12}_k\,\mathrm{x13}_j\,\xi 12_i\,\xi 13_i-\mathrm{x12}_j\,\mathrm{x13}_k\,\xi 12_i\,\xi 13_i-2\,\mathrm{a2}_k\,\mathrm{x23}_j\,\xi 23_i+2\,\mathrm{a2}_j\,\mathrm{x23}_k\,\xi 23_i-\\
&\quad \mathrm{x12}_k\,\mathrm{x23}_j\,\xi 12_i\,\xi 23_i+\mathrm{x13}_k\,\mathrm{x23}_j\,\xi 13_i\,\xi 23_i-\mathrm{x13}_j\,\mathrm{x23}_k\,\xi 13_i\,\xi 23_i+\mathrm{x12}_j\,\mathrm{x23}_k\left(4\,\xi 13_i+\xi 12_i\,\xi 23_i\right)\Big]
\end{aligned}$$

Out[•]=
```
True
```

In[•]:=
```
{$k = 1, $n = 2}; (amt1,2→1 // aΔt1→1,2) ≡ (aΔt2→3,4 // aΔt1→1,2 // amt1,3→1 // amt2,4→2)
```

Time: 0.11 seconds Out[•]=

True

In[•]:=
```
Block[{$k = 1, $n = 3}, Echo@{amt1,2→1, aΔt1→1,2};
 (amt1,2→1 // aΔt1→1,2) ≡ (aΔt2→3,4 // aΔt1→1,2 // amt1,3→1 // amt2,4→2)]
```

» $\left\{\mathbb{E}_{\{1,2\}\to\{1\}}\left[a1_1\ (\alpha1_1+\alpha1_2)+a2_1\ (\alpha2_1+\alpha2_2)+x12_1\left(\xi12_1+e^{\frac{1}{2}(2\alpha1_1-\alpha2_1)}\ \xi12_2\right)+\right.\right.$
$x13_1\left(\xi13_1+e^{\frac{1}{2}(\alpha1_1+\alpha2_1)}\ \xi13_2-e^{\frac{1}{2}(2\alpha1_1-\alpha2_1)}\ \xi12_2\ \xi23_1\right)+x23_1\left(\xi23_1+e^{\frac{1}{2}(-\alpha1_1+2\alpha2_1)}\ \xi23_2\right),$
$-2\,e^{\frac{1}{2}(2\alpha1_1-\alpha2_1)}\ x12_1\ x13_1\ \xi12_2\ \xi13_1-e^{\frac{1}{2}(2\alpha1_1-\alpha2_1)}\ x13_1\ \xi12_2\ \xi23_1+$
$\left.2\,e^{\frac{1}{2}(2\alpha1_1-\alpha2_1)}\ x12_1\ x23_1\ \xi12_2\ \xi23_1-2\,e^{\frac{1}{2}(\alpha1_1+\alpha2_1)}\ x13_1\ x23_1\ \xi13_2\ \xi23_1\right],\ \mathbb{E}_{\{1\}\to\{1,2\}}[$
$(a1_1+a1_2)\ \alpha1_1+(a2_1+a2_2)\ \alpha2_1+x12_1\ \xi12_1+x12_2\ \xi12_1+x13_1\ \xi13_1+x13_2\ \xi13_1+x23_1\ \xi23_1+x23_2\ \xi23_1,$
$-2\,a1_2\ x12_1\ \xi12_1+2\,a1_1\ x12_2\ \xi12_1-2\,a1_2\ x13_1\ \xi13_1-2\,a2_2\ x13_1\ \xi13_1+2\,a1_1\ x13_2\ \xi13_1+$
$2\,a2_1\ x13_2\ \xi13_1+x12_2\ x13_1\ \xi12_1\ \xi13_1-x12_1\ x13_2\ \xi12_1\ \xi13_1-2\,a2_2\ x23_1\ \xi23_1+2\,a2_1\ x23_2\ \xi23_1-$
$x12_2\ x23_1\ \xi12_1\ \xi23_1+x13_2\ x23_1\ \xi13_1\ \xi23_1-x13_1\ x23_2\ \xi13_1\ \xi23_1+x12_1\ x23_2\ (4\ \xi13_1+\xi12_1\ \xi23_1)\ ]\Big\}$

Out[•]=

True

pdf

In[•]:=

```
Define[aSt_i = Module[{(*TTL, *)},
   ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
   TTL = TT_λ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g^*)_i S_i[U_i[g]]], {g, $PBWGens}];
   mons[-1] = {};
   mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], U_i[x___] :> U_i[x], ∞];
   ff = Table[
      Sum[ϵ^p f_{p,u_i}[λ], {p, 0, $k}],
      {u, $PBWGens}
     ] /. f_{p_,u_i}[λ] /; τ[u] == "a" :> If[p == 0, -(u^*)_i, 0];
   fs = Flatten[Table[
      f_{p,ℙ_i[m]},
      {p, 0, $k}, {m, mons[p]}
     ]];
   gg = Sum[
     ϵ^p f_{p,ℙ_i[u]}[λ] ℙ_i[u],
     {p, 0, $k}, {u, Complement[mons[p], (U_i /@ $PBWGens)]}
    ];
   G = Series[ⅇ^gg, {ϵ, 0, $k}];
   𝒮[B_] := UCF[𝒪_i[Seepage_i[ff, UCF@B]]];
   𝒮G = 𝒮[G];
   TTR = UCF[Invert_i[𝒮G] ⊠ 𝒮[(∂_λ (ff.(#_i & /@ $PBWGens)) G + ∂_λ G)]];
   diff = UCF[TTL - TTR];
   eqns = Flatten[Table[
      Coefficient[Coefficient[diff, m], ϵ, p] == 0,
      {p, 0, $k}, {m, mons[p]}
     ]];
   inits = (#[0] == 0) & /@ fs;
   ss = TriangularDSolve[eqns, inits, fs, λ, CF];
   𝔼_{i}→{i} @@ Table[As[p, 0, $k], Plus[
      If[p == 0, Sum[As[m, 1, $n - 1], -α[m]_i (a[m]_i)], 0],
      Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :> u (f_{p,u} /. ss)[λ] /. λ | ħ | γ → 1]
    ]
  ]]
```

pdf

In[•]:=

```
Define[aSt_i = Module[{(*TTL, *)},
   TTL = TT_λ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g^*)_i S_i[U_i[g]]], {g, $PBWGens}];
   mons[-1] = {};
   mons[p_] :=
    mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], c_. U_i[x___] :> U_i[x], ∞];
   ff = Table[
      Sum[ϵ^p f_{p,u_i}[λ], {p, 0, $k}],
      {u, $PBWGens}
     ] /. f_{p_,u_i}[λ] /; τ[u] == "a" :> If[p == 0, -(u^*)_i, 0];
   fs = Flatten[Table[
      f_{p,ℙ_i[m]},
      {p, 0, $k}, {m, mons[p]}
     ]];
   gg = Sum[
     ϵ^p f_{p,ℙ_i[u]}[λ] ℙ_i[u],
     {p, 0, $k}, {u, Complement[mons[p], (U_i /@ $PBWGens)]}
    ];
   G = Series[ℯ^gg, {ϵ, 0, $k}];
   𝒮[B_] := UCF[𝒪_i[Seepage_i[ff, UCF@B]]];
   𝒮G = 𝒮[G];
   TTR = UCF[Invert_i[𝒮G] ⊠ 𝒮[(∂_λ (ff.(#_i & /@ $PBWGens)) G + ∂_λ G)]];
   diff = UCF[TTL - TTR];
   eqns = Flatten[Table[
      Coefficient[Coefficient[diff, m], ϵ, p] == 0,
      {p, 0, $k}, {m, mons[p]}
     ]];
   inits = (#[0] == 0) & /@ fs;
   ss = TriangularDSolve[eqns, inits, fs, λ, CF];
   𝔼_{{i}→{i}} @@ Table[As[p, 0, $k], Plus[
      If[p == 0, Sum[As[m, 1, $n - 1], -α[m]_i a[m]_i], 0],
      Total[Cases[fs, f_{p,_}]] /. f_{p_,u_} :> u (f_{p,u} /. ss)[λ] /. λ | ħ | γ → 1]
    ]
  ]]
```

In[•]:=
```
$k = 1; $n = 3; aSt_i
aΔt_{1→2,3} // aSt_2 // amt_{2,3→1}
```

Time: 32.98 seconds Out[•]=

$$\mathbb{E}_{\{i\}\to\{i\}}\Big[-a1_i\,\alpha1_i - a2_i\,\alpha2_i - e^{\frac{1}{2}(-2\alpha1_i+\alpha2_i)}\,x12_i\,\xi12_i - e^{\frac{1}{2}(\alpha1_i-2\alpha2_i)}\,x23_i\,\xi23_i + x13_i\left(-e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi13_i - e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi12_i\,\xi23_i\right),\ 2\,e^{\frac{1}{2}(-2\alpha1_i+\alpha2_i)}\,x12_i\,\xi12_i + 2\,e^{-\frac{3\alpha1_i}{2}}\,x12_i\,x13_i\,\xi12_i\,\xi13_i + 2\,e^{\frac{1}{2}(\alpha1_i-2\alpha2_i)}\,x23_i\,\xi23_i + 2\,e^{-\frac{3\alpha2_i}{2}}\,x13_i\,x23_i\,\xi13_i\,\xi23_i + 2\,e^{-\alpha1_i-\alpha2_i}\,x13_i^2\,\xi12_i\,\xi13_i\,\xi23_i + x12_i\,x23_i\left(4\,e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi13_i + 2\,e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi12_i\,\xi23_i\right) + x13_i\left(2\,e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi13_i + 3\,e^{\frac{1}{2}(-\alpha1_i-\alpha2_i)}\,\xi12_i\,\xi23_i\right)\Big]$$

Out[•]=

$$\mathbb{E}_{\{1\}\to\{1\}}[0,\,0]$$

pdf

In[•]:=
```
(*Splitter 𝔹⊗𝔸 *)
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{i}→{j,k,l,m}[
  Sum[η[r, s]_i y[r, s]_j + ξ[r, s]_i x[r, s]_m, {r, 1, $n - 1}, {s, r + 1, $n}] +
   Sum[α[r]_i a[r]_l + β[r]_i b[r]_k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
 ]
```

pdf

$P:\mathbb{A}\otimes\mathbb{B}\to\mathbb{Q}$ and $R\in\mathbb{B}\otimes\mathbb{A}$:

pdf

In[•]:=
```
P_{i_,j_} := 𝔼_{i,j}→{} @@ PadRight[
   {Sum[η[r, s]_j ξ[r, s]_i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]_j α[r]_i, {r, 1, $n - 1}]},
   $k + 1]
R_{i_,j_} := 𝔼_{}→{i,j} @@ PadRight[
   {Sum[y[r, s]_i x[r, s]_j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]_i a[r]_j, {r, 1, $n - 1}]},
   $k + 1]
```

In[•]:=
```
{R_{1,2} // P_{2,3}, R_{1,2} // P_{5,1}}
```

Time: 0.12 seconds Out[•]=

$$\{\mathbb{E}_{\{3\}\to\{1\}}[b1_1\,\beta1_3 + b2_1\,\beta2_3 + b3_1\,\beta3_3 + b4_1\,\beta4_3 + y12_1\,\eta12_3 + y13_1\,\eta13_3 + y14_1\,\eta14_3 + y15_1\,\eta15_3 + y23_1\,\eta23_3 + y24_1\,\eta24_3 + y25_1\,\eta25_3 + y34_1\,\eta34_3 + y35_1\,\eta35_3 + y45_1\,\eta45_3,\ 0],\ \mathbb{E}_{\{5\}\to\{2\}}[a1_2\,\alpha1_5 + a2_2\,\alpha2_5 + a3_2\,\alpha3_5 + a4_2\,\alpha4_5 + x12_2\,\xi12_5 + x13_2\,\xi13_5 + x14_2\,\xi14_5 + x15_2\,\xi15_5 + x23_2\,\xi23_5 + x24_2\,\xi24_5 + x25_2\,\xi25_5 + x34_2\,\xi34_5 + x35_2\,\xi35_5 + x45_2\,\xi45_5,\ 0]\}$$

pdf

In[•]:=
```
Define[bmt_{j,k→i} = Module[{i0, i1, j1, k1}, (R_{i0,i1} // aΔt_{i1→j1,k1} // P_{j1,j} // P_{k1,k}) /. i0 → i],
 bΔt_{k→i,j} = Module[{i1, j1, k1, k2}, ((R_{i,i1} R_{j,j1} // amt_{i1,j1→k1}) // P_{k1,k2}) /. k2 → k],
 bSt_i = Module[{i1, i2}, (R_{i,i1} // aSt_{i1} // P_{i1,i2}) /. i2 → i],
 ‾bSt_i = Module[{i1, i2}, (R_{i,i1} // ‾aSt_{i1} // P_{i1,i2}) /. i2 → i]]
Define[‾R_{i,j} = R_{i,j} // bSt_i]
```

```
In[•]:= Block[{$k = 1, $n = 2}, bmt_{1,2→3}]
```

Time: 0.22 seconds Out[•]=

$$\mathbb{E}_{\{1,2\}\to\{3\}}\left[b1_3\,\beta1_1 + b1_3\,\beta1_2 + y12_3\,\eta12_1 + y12_3\,\eta12_2,\ -2\,y12_3\,\beta1_2\,\eta12_1 + 2\,y12_3\,\beta1_1\,\eta12_2\right]$$

Testing 𝔸 and 𝔹 dually paired Hopf algebras

```
In[•]:= $k = 1; $n = 3;
```

co-associativity

```
In[•]:= {(aΔt_{1→1,2} // aΔt_{2→2,3}) ≡ (aΔt_{1→2,3} // aΔt_{2→1,2}), (bΔt_{1→1,2} // bΔt_{2→2,3}) ≡ (bΔt_{1→2,3} // bΔt_{2→1,2})}
```

Time: 0.22 seconds Out[•]=

```
{True, True}
```

algebra morphism

```
In[•]:= {(aΔt_{i→1,2} aΔt_{j→3,4} // amt_{1,3→i} // amt_{2,4→j}) ≡ (amt_{i,j→k} // aΔt_{k→i,j}),
         (bΔt_{i→1,2} bΔt_{j→3,4} // bmt_{1,3→i} // bmt_{2,4→j}) ≡ (bmt_{i,j→k} // bΔt_{k→i,j})}
```

Time: 0.44 seconds Out[•]=

```
{True, True}
```

associativity

```
In[•]:= {(amt_{1,2→k} // amt_{k,3→k}) ≡ (amt_{2,3→k} // amt_{1,k→k}), (bmt_{1,2→k} // bmt_{k,3→k}) ≡ (bmt_{2,3→k} // bmt_{1,k→k})}
```

Time: 0.12 seconds Out[•]=

```
{True, True}
```

antipode

```
In[•]:= {aΔt_{i→1,2} // aSt_1 // amt_{1,2→1}, aΔt_{i→1,2} // aSt_2 // amt_{1,2→1},
         bΔt_{i→1,2} // bSt_1 // bmt_{1,2→1}, bΔt_{i→1,2} // bSt_2 // bmt_{1,2→1}}
```

Time: 0.89 seconds Out[•]=

$$\{\mathbb{E}_{\{i\}\to\{1\}}[0,0], \mathbb{E}_{\{i\}\to\{1\}}[0,0], \mathbb{E}_{\{i\}\to\{1\}}[0,0], \mathbb{E}_{\{i\}\to\{1\}}[0,0]\}$$

inverse antipode

```
In[•]:= {aSt_i // conj[aSt_i], bSt_i // conj[bSt_i]}
```

Time: 3.53 seconds Out[•]=

$$\{\mathbb{E}_{\{i\}\to\{i\}}\left[a1_i\,\alpha1_i + a2_i\,\alpha2_i + x12_i\,\xi12_i + x13_i\,\xi13_i + x23_i\,\xi23_i,\ 0\right],$$
$$\mathbb{E}_{\{i\}\to\{i\}}\left[b1_i\,\beta1_i + b2_i\,\beta2_i + y12_i\,\eta12_i + y13_i\,\eta13_i + y23_i\,\eta23_i,\ 0\right]\}$$

Pairing

```
In[•]:= {(amt_{1,2→3} // P_{3,-3}) ≡ (bΔt_{-3→-1,-2} // P_{1,-1} // P_{2,-2}),
         (bmt_{1,2→3} // P_{-3,3}) ≡ (aΔt_{-3→-1,-2} // P_{-1,1} // P_{-2,2}),
         (aSt_1 // P_{1,-1}) ≡ (bSt_{-1} // P_{1,-1}), (conj[aSt_1] // P_{1,-1}) ≡ (conj[bSt_{-1}] // P_{1,-1})}
```

Time: 0.23 seconds Out[•]=

```
{True, True, True, True}
```

Testing Double

pdf

```
In[•]:= Define[dm_{i,j→k} = Module[{i1, i2, i3, i4},
    ((sϒ_{i→i4,i4,i1,i1} // aΔt_{i1→i1,i2} // aΔt_{i2→i2,i3} // aSt_{i3}) (sϒ_{j→-i1,-i1,-i4,-i4} // bΔt_{-i1→-i1,-i2} //
        bΔt_{-i2→-i2,-i3})) // (P_{i1,-i3} P_{i3,-i1} amt_{i2,-i4→k} bmt_{i4,-i2→k})]]
Define[dS_i = sϒ_{i→1,1,2,2} // (bSt_1 aSt_2) // dm_{2,1→i},
 dS_i = sϒ_{i→1,1,2,2} // (bSt_1 aSt_2) // dm_{2,1→i},
 dΔ_{i→j,k} = (bΔt_{i→1,3} aΔt_{i→4,2}) // (dm_{3,4→k} dm_{1,2→j})]
(*Notice we take co-opposite on the 𝔸-side! to make the double*)
```

```
In[•]:= {$k = 1, $n = 3, ħ = 1};
        dm_{i,j→k} // l2U // CF // Short
```

Time: 33.12 seconds Out[•]=

$$\mathbb{E}_{\{i,j\}\to\{k\}}\left[a1_k\,\alpha1_i + a1_k\,\alpha1_j + a2_k\,\alpha2_i + a2_k\,\alpha2_j + \text{<<29>>} + \frac{\sqrt{B2_k}\;\eta13_j\,\xi12_i\,\xi23_i}{B1_k\,\sqrt{\mathcal{A}1_i}\;\sqrt{\mathcal{A}2_i}} - \frac{x13_k\,\mathcal{A}1_i\,\xi12_j\,\xi23_i}{\sqrt{\mathcal{A}2_i}} + \frac{x23_k\,\mathcal{A}2_i\,\xi23_j}{\sqrt{\mathcal{A}1_i}},\; -2\,y12_k\,\beta1_j\,\eta12_i + \text{<<344>>} + \frac{\text{<<1>>}}{\text{<<1>>}}\right]$$

associativity

```
In[•]:= (dm_{1,2→k} // dm_{k,3→k}) ≡ (dm_{2,3→k} // dm_{1,k→k})
```

Time: 5.34 seconds Out[•]=

True

co-associativity

```
In[•]:= (dΔ_{1→1,2} // dΔ_{2→2,3}) ≡ (dΔ_{1→2,3} // dΔ_{2→1,2})
```

Time: 1.22 seconds Out[•]=

True

algebra morphism

```
In[•]:= (dΔ_{1→1,2} dΔ_{2→3,4} // (dm_{1,3→1} dm_{2,4→2})) ≡ (dm_{1,2→3} // dΔ_{3→1,2})
```

Time: 61.20 seconds Out[•]=

True

antipode

```
In[•]:= {dΔ_{i→1,2} // dS_1 // dm_{1,2→1}, dΔ_{i→1,2} // dS_2 // dm_{1,2→1}, dS_1 // dS_1}
```

Time: 341.30 seconds Out[•]=

$$\{\mathbb{E}_{\{i\}\to\{1\}}[0,0],\ \mathbb{E}_{\{i\}\to\{1\}}[0,0],\ \mathbb{E}_{\{1\}\to\{1\}}[a1_1\,\alpha1_1 + a2_1\,\alpha2_1 + b1_1\,\beta1_1 + b2_1\,\beta2_1 + y12_1\,\eta12_1 + y13_1\,\eta13_1 + y23_1\,\eta23_1 + x12_1\,\xi12_1 + x13_1\,\xi13_1 + x23_1\,\xi23_1, 0]\}$$

quasi-triangular axioms

```
In[•]:= (R1,3 // dΔ1→1,2) ≡ (R1,3 R2,4 // dm3,4→3)
        (R1,3 // dΔ3→2,3) ≡ (R1,3 R0,2 // dm1,0→1)
        (dΔi→k,j R1,2 // dmj,1→1 // dmk,2→2) ≡ (R1,2 dΔi→j,k // dm1,j→1 // dm2,k→2)
```

Time: 14.95 seconds Out[•]=

True

Out[•]=

True

Out[•]=

True

```
In[•]:= (R1,2 // aSt2) ≡ (R̄1,2)
```

Time: 0.16 seconds Out[•]=

True

Braid-like Reidemeister moves: R2,R3

```
In[•]:= {R1,2 R̄3,4 // dm1,3→1 // dm2,4→2, R̄1,2 R3,4 // dm1,3→1 // dm2,4→2,
          (R1,2 R4,3 R5,6 // dm1,4→1 // dm2,5→5 // dm3,6→3) ≡ (R1,6 R2,3 R4,5 // dm1,4→1 // dm2,5→5 // dm3,6→3)}
```

Time: 15.94 seconds Out[•]=

$\{\mathbb{E}_{\{\}\to\{1,2\}}[0, 0], \mathbb{E}_{\{\}\to\{1,2\}}[0, 0], \text{True}\}$

```
In[•]:= Block[{$k = 2, $n = 3, ħ = 1}, dmi,j→k // l2U // CF]
```

Time: 18271.69 seconds Out[•]=

```
In[•]:= PrintProfile[]
```

Time: 0.06 seconds Out[•]=

```
ProfileRoot is root. Profiled time: 16908.9
  (      6)     53.466/ 16629.843 above Boot
  (  24675)    279.082/   279.082 above CF
CF: called 974127 times, time in 15807.8/15807.8
  (  16977)    152.488/   152.488 under Boot
  (  24675)    279.082/   279.082 under ProfileRoot
  (    483)  15170.329/ 15170.329 under Seepage
  (  11573)      8.895/     8.895 under TriangularDSolve
  ( 920419)    196.959/   196.959 under UCF
NCM: called 5266 times, time in 817.677/1204.65
  (      4)    659.641/   713.328 under TTR
  (      3)      1.636/     3.265 under Boot
  (      3)     37.519/    40.437 under Invert
  (   5256)    118.881/   447.616 under UB
  ( 506363)    201.465/   386.969 above UCF
UCF: called 510163 times, time in 206.803/403.762
  (      8)      0.063/     0.391 under TTR
  (      4)      1.110/     2.078 under 𝒮BB
  (      8)      0.095/     0.220 under 𝒮G
  (    960)      1.328/     3.669 under AdNil
  (     69)      0.079/     0.157 under Boot
  (     14)      0.267/     0.516 under Invert
```

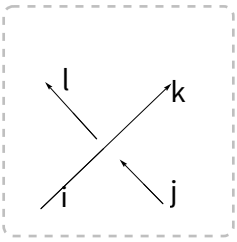
```
  ( 506363)    201.465/   386.969 under NCM
  (     45)      0.172/     4.406 under TT
  (   2628)      2.083/     4.919 under UB
  (      1)         0/     0.015 under amt[1, 3]
  (      1)      0.015/     0.047 under amt[2, 3]
  (     31)      0.016/     0.031 under a∆t[1, 3]
  (     31)      0.110/     0.344 under a∆t[2, 3]
  ( 920419)    196.959/   196.959 above CF
Boot: called 19 times, time in 54.106/49403.5
  (     13)      0.640/ 32773.671 under Boot
  (      6)     53.466/ 16629.843 under ProfileRoot
  (     13)      0.640/ 32773.671 above Boot
  (  16977)    152.488/   152.488 above CF
  (      3)         0/     0.313 above Invert
  (      3)      1.636/     3.265 above NCM
  (      6)      0.016/    19.202 above Seepage
  (      3)      0.346/     0.501 above TriangularDSolve
  (      3)         0/     0.296 above TT
  (     69)      0.079/     0.157 above UCF
  (      6)      0.109/     0.109 above 𝕆
  (      1)      0.063/     4.734 above amt[1, 3]
  (      1)      0.250/    47.031 above amt[2, 3]
  (      1)      0.267/    31.500 above a∆t[1, 3]
  (      1)      2.125/ 16316.141 above a∆t[2, 3]
𝕆: called 1027 times, time in 11.739/11.739
  (      2)      0.109/     0.109 under TTR
  (      4)      3.281/     3.281 under 𝒮BB
  (      6)      0.329/     0.329 under 𝒮G
  (      6)      0.109/     0.109 under Boot
  (   1009)      7.911/     7.911 under Seepage
TriangularDSolve: called 7 times, time in 3.652/12.547
  (      3)      0.346/     0.501 under Boot
  (      1)      0.282/     0.328 under amt[1, 3]
  (      1)      0.591/     1.203 under amt[2, 3]
  (      1)      0.235/     0.515 under a∆t[1, 3]
  (      1)      2.198/    10.000 under a∆t[2, 3]
  (  11573)      8.895/     8.895 above CF
Seepage: called 483 times, time in 2.917/22420.
  (      2)      0.015/    31.172 under TTR
  (      4)      0.841/ 15424.063 under 𝒮BB
  (      6)      0.062/   151.811 under 𝒮G
  (      6)      0.016/    19.202 under Boot
  (    465)      1.983/  6793.770 under Seepage
  (    933)      0.344/   110.047 above AdNil
  (     62)      0.267/   335.013 above AdPotPlus
  (     14)         0/     0.031 above AdPoty
  (    483)  15170.329/ 15170.329 above CF
```

```
  (    465)      1.983/  6793.770 above Seepage
  (   1009)      7.911/     7.911 above 𝕆
a△t[2, 3]: called 1 times, time in 2.125/16316.1
  (      1)      2.125/ 16316.141 under Boot
  (      1)      0.062/     0.062 above BB
  (      1)      0.015/   748.453 above TTR
  (      1)      0.062/ 15402.703 above 𝒮BB
  (      1)         0/   147.563 above 𝒮G
  (      1)      2.198/    10.000 above TriangularDSolve
  (      1)         0/     4.891 above TT
  (     31)      0.110/     0.344 above UCF
UB: called 2628 times, time in 0.464/452.999
  (   1182)      0.278/   110.346 under AdNil
  (   1446)      0.186/   342.653 under AdPoty
  (   5256)    118.881/   447.616 above NCM
  (   2628)      2.083/     4.919 above UCF
AdNil: called 960 times, time in 0.344/114.359
  (    933)      0.344/   110.047 under Seepage
  (     27)         0/     4.312 under TT
  (   1182)      0.278/   110.346 above UB
  (    960)      1.328/     3.669 above UCF
AdPotPlus: called 74 times, time in 0.298/343.122
  (     62)      0.267/   335.013 under Seepage
  (     12)      0.031/     8.109 under TT
  (   1432)      0.202/   342.824 above AdPoty
a△t[1, 3]: called 1 times, time in 0.267/31.5
  (      1)      0.267/    31.500 under Boot
  (      1)      0.078/     0.078 above BB
  (      1)         0/     1.703 above TTR
  (      1)      0.016/    26.797 above 𝒮BB
  (      1)         0/     1.109 above 𝒮G
  (      1)      0.235/     0.515 above TriangularDSolve
  (      1)         0/     1.000 above TT
  (     31)      0.016/     0.031 above UCF
amt[2, 3]: called 1 times, time in 0.25/47.031
  (      1)      0.250/    47.031 under Boot
  (      1)         0/     9.125 above TTL
  (      1)      0.016/    32.953 above TTR
  (      1)         0/     3.453 above 𝒮G
  (      1)      0.591/     1.203 above TriangularDSolve
  (      1)      0.015/     0.047 above UCF
AdPoty: called 1446 times, time in 0.202/342.855
  (   1432)      0.202/   342.824 under AdPotPlus
  (     14)         0/     0.031 under Seepage
  (   1446)      0.186/   342.653 above UB
BB: called 2 times, time in 0.14/0.14
  (      1)      0.078/     0.078 under a△t[1, 3]
```

```
  (      1)      0.062/     0.062 under a∆t[2, 3]
𝒮BB: called 2 times, time in 0.078/15429.5
  (      1)      0.016/    26.797 under a∆t[1, 3]
  (      1)      0.062/ 15402.703 under a∆t[2, 3]
  (      4)      0.841/ 15424.063 above Seepage
  (      4)      1.110/     2.078 above UCF
  (      4)      3.281/     3.281 above 𝕆
amt[1, 3]: called 1 times, time in 0.063/4.734
  (      1)      0.063/     4.734 under Boot
  (      1)         0/     1.515 above TTL
  (      1)         0/     2.578 above TTR
  (      1)         0/     0.235 above 𝒮G
  (      1)      0.282/     0.328 above TriangularDSolve
  (      1)         0/     0.015 above UCF
TTR: called 4 times, time in 0.031/785.687
  (      1)         0/     2.578 under amt[1, 3]
  (      1)      0.016/    32.953 under amt[2, 3]
  (      1)         0/     1.703 under a∆t[1, 3]
  (      1)      0.015/   748.453 under a∆t[2, 3]
  (      4)      0.016/    40.656 above Invert
  (      4)    659.641/   713.328 above NCM
  (      2)      0.015/    31.172 above Seepage
  (      8)      0.063/     0.391 above UCF
  (      2)      0.109/     0.109 above 𝕆
Invert: called 7 times, time in 0.016/40.969
  (      4)      0.016/    40.656 under TTR
  (      3)         0/     0.313 under Boot
  (      3)     37.519/    40.437 above NCM
  (     14)      0.267/     0.516 above UCF
TT: called 45 times, time in 0./35.167
  (      2)         0/    10.640 under TTL
  (      3)         0/     0.296 under Boot
  (     38)         0/    18.340 under TT
  (      1)         0/     1.000 under a∆t[1, 3]
  (      1)         0/     4.891 under a∆t[2, 3]
  (     27)         0/     4.312 above AdNil
  (     12)      0.031/     8.109 above AdPotPlus
  (     38)         0/    18.340 above TT
  (     45)      0.172/     4.406 above UCF
𝒮G: called 4 times, time in 0./152.36
  (      1)         0/     0.235 under amt[1, 3]
  (      1)         0/     3.453 under amt[2, 3]
  (      1)         0/     1.109 under a∆t[1, 3]
  (      1)         0/   147.563 under a∆t[2, 3]
  (      6)      0.062/   151.811 above Seepage
  (      8)      0.095/     0.220 above UCF
  (      6)      0.329/     0.329 above 𝕆
```

```
TTL: called 2 times, time in 0./10.64
   (      1)         0/     1.515 under amt[1, 3]
   (      1)         0/     9.125 under amt[2, 3]
   (      2)         0/    10.640 above TT
```

gification



In[•]:= **Block[{$k = 1, $n = 2}, $R_{k,1}$ // $dm_{i,k\to k}$ // $dm_{j,1\to 1}$]**

Time: 0.69 seconds Out[•]=

$$\mathbb{E}_{\{i,j\}\to\{k,l\}}\Big[a1_l\, b1_k + e^{-\alpha1_i+\alpha1_j}\, x12_l\, y12_k + a1_k\,\alpha1_i + a1_l\,\alpha1_j + b1_k\,\beta1_i + b1_l\,\beta1_j + y12_k\,\eta12_i + y12_l\,\eta12_j + x12_k\,\xi12_i - e^{-\alpha1_i+\alpha1_j}\, x12_l\,\xi12_i + e^{b1_k-\alpha1_i+\alpha1_j}\, x12_l\,\xi12_i + x12_l\,\xi12_j,$$
$$2\,e^{-\alpha1_i+\alpha1_j}\, x12_l\, y12_k\,\beta1_i - 2\, a1_l\, y12_k\,\eta12_i + 4\, a1_l\, x12_k\,\xi12_i - 2\,e^{-\alpha1_i+\alpha1_j}\, x12_l\,\xi12_i + 2\,e^{b1_k-\alpha1_i+\alpha1_j}\, x12_l\,\xi12_i - 2\,e^{-\alpha1_i+\alpha1_j}\, a1_k\, x12_l\,\xi12_i - 2\,e^{b1_k-\alpha1_i+\alpha1_j}\, a1_k\, x12_l\,\xi12_i - 2\,e^{-\alpha1_i+\alpha1_j}\, a1_l\, x12_l\,\xi12_i + 2\,e^{b1_k-\alpha1_i+\alpha1_j}\, a1_l\, x12_l\,\xi12_i + 2\,e^{-\alpha1_i+\alpha1_j}\, x12_k\, x12_l\, y12_k\,\xi12_i + 2\,e^{b1_k-2\,\alpha1_i+2\,\alpha1_j}\, x12_l^2\, y12_k\,\xi12_i - 2\,e^{b1_k-\alpha1_i+\alpha1_j}\, x12_l\, y12_k\,\eta12_i\,\xi12_i - 2\,e^{-\alpha1_i+\alpha1_j}\, x12_k\, x12_l\,\xi12_i^2 + 2\,e^{b1_k-\alpha1_i+\alpha1_j}\, x12_k\, x12_l\,\xi12_i^2 + 2\,e^{2\,(b1_k-\alpha1_i+\alpha1_j)}\, x12_l^2\,\xi12_i^2 - 2\,e^{b1_k-2\,\alpha1_i+2\,\alpha1_j}\, x12_l^2\,\xi12_i^2\Big]$$

In[•]:= **Block[{$k = 1, $n = 3}, $R_{k,1}$ // $dm_{i,k\to k}$ // $dm_{j,1\to 1}$]**

Time: 49.28 seconds Out[•]=

$$\mathbb{E}_{\{i,j\}\to\{k,l\}}\Big[a1_l\, b1_k + a2_l\, b2_k + e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\, y12_k + e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y13_k + e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\, y23_k + a1_k\,\alpha1_i + a1_l\,\alpha1_j + a2_k\,\alpha2_i + a2_l\,\alpha2_j + b1_k\,\beta1_i + b1_l\,\beta1_j + b2_k\,\beta2_i + b2_l\,\beta2_j + y12_k\,\eta12_i + y12_l\,\eta12_j + y13_k\,\eta13_i + y13_l\,\eta13_j + y23_k\,\eta23_i + y23_l\,\eta23_j + x12_k\,\xi12_i - e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\,\xi12_i + e^{\frac{1}{2}\,(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\,\xi12_i - e^{\frac{1}{2}\,(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y23_k\,\xi12_i + x12_l\,\xi12_j + x13_k\,\xi13_i - e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\,\xi13_i + e^{\frac{1}{2}\,(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\,\xi13_i + x13_l\,\xi13_j + x23_k\,\xi23_i - e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\,\xi23_i + e^{\frac{1}{2}\,(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\,\xi23_i + e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y12_k\,\xi23_i - e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\,\xi12_i\,\xi23_i + e^{\frac{1}{2}\,(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\,\xi12_i\,\xi23_i + x23_l\,\xi23_j - e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x13_l\, y12_k\,\xi23_j + e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x13_l\,\xi12_i\,\xi23_j - e^{\frac{1}{2}\,(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x13_l\,\xi12_i\,\xi23_j,$$
$$2\,e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\, y12_k\,\beta1_i + 2\,e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y13_k\,\beta1_i + 2\,e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y13_k\,\beta2_i + 2\,e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\, y23_k\,\beta2_i - 2\, a1_l\, y12_k\,\eta12_i + 4\,e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\, y13_k\,\eta12_i - e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y12_k\, y13_k\,\eta12_i + e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\, y12_k\, y23_k\,\eta12_i - 2\, a1_l\, y13_k\,\eta13_i - 2\, a2_l\, y13_k\,\eta13_i + e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\, y12_k\, y13_k\,\eta13_i - e^{\frac{1}{2}\,(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\, x23_l\, y13_k\, y23_k\,\eta13_i - 2\, a2_l\, y23_k\,\eta23_i - e^{\frac{1}{2}\,(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\, x12_l\, y12_k\, y23_k\,\eta23_i + e^{\frac{1}{2}\,(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\, x13_l\, y13_k\, y23_k\,\eta23_i +$$

$$4\,a1_l\,x12_k\,\xi12_i - 2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,\xi12_i + 2\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_k\,x12_l\,\xi12_i - 2\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_k\,x12_l\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_l\,x12_l\,\xi12_i + 2\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_l\,x12_l\,\xi12_i +$$
$$2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x12_l\,y12_k\,\xi12_i + 2\,e^{\frac{1}{2}(2b1_k-b2_k-4\alpha1_i+4\alpha1_j+2\alpha2_i-2\alpha2_j)}\,x12_l^2\,y12_k\,\xi12_i +$$
$$e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,y13_k\,\xi12_i - e^{\frac{1}{2}(2b1_k-b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y13_k\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y23_k\,\xi12_i + 2\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_k\,x13_l\,y23_k\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_l\,x13_l\,y23_k\,\xi12_i - e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x12_k\,x23_l\,y23_k\,\xi12_i +$$
$$e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,y23_k\,\xi12_i - 2\,e^{\frac{1}{2}(2b1_k-b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y12_k\,y23_k\,\xi12_i -$$
$$e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x13_l^2\,y13_k\,y23_k\,\xi12_i + e^{\frac{1}{2}(2b1_k-b2_k-3\alpha2_i+3\alpha2_j)}\,x13_l\,x23_l\,y23_k^2\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y23_k\,\beta2_i\,\xi12_i - 2\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,y12_k\,\eta12_i\,\xi12_i -$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y13_k\,\eta12_i\,\xi12_i + e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,y23_k\,\eta12_i\,\xi12_i -$$
$$e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,y13_k\,\eta13_i\,\xi12_i + 2\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y13_k\,y23_k\,\eta13_i\,\xi12_i +$$
$$e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,y23_k\,\eta23_i\,\xi12_i - e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y23_k^2\,\eta23_i\,\xi12_i -$$
$$2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x12_l\,\xi12_i^2 + 2\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x12_l\,\xi12_i^2 -$$
$$2\,e^{\frac{1}{2}(2b1_k-b2_k-4\alpha1_i+4\alpha1_j+2\alpha2_i-2\alpha2_j)}\,x12_l^2\,\xi12_i^2 + 2\,e^{2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j}\,x12_l^2\,\xi12_i^2 -$$
$$e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,y23_k\,\xi12_i^2 - 3\,e^{\frac{1}{2}(4b1_k-2b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y23_k\,\xi12_i^2 +$$
$$2\,e^{\frac{1}{2}(2b1_k-b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y23_k\,\xi12_i^2 + e^{2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,y23_k^2\,\xi12_i^2 + 4\,a1_l\,x13_k\,\xi13_i +$$
$$4\,a2_l\,x13_k\,\xi13_i - 6\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,\xi13_i + 4\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,\xi13_i +$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,\xi13_i - 2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_k\,x13_l\,\xi13_i -$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_k\,x13_l\,\xi13_i - 2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_l\,x13_l\,\xi13_i +$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_l\,x13_l\,\xi13_i - 2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_k\,x13_l\,\xi13_i -$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_k\,x13_l\,\xi13_i - 2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_l\,x13_l\,\xi13_i +$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_l\,x13_l\,\xi13_i + 4\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_k\,\xi13_i -$$
$$4\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x12_k\,x23_l\,\xi13_i + 4\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi13_i -$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi13_i + e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x13_k\,y12_k\,\xi13_i +$$
$$4\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,y12_k\,\xi13_i - 2\,e^{-\frac{3}{2}(\alpha1_i-\alpha1_j)}\,x12_l\,x13_l\,y12_k\,\xi13_i +$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y12_k\,\xi13_i + e^{\frac{1}{2}(b1_k+b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,y12_k\,\xi13_i +$$
$$2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,y13_k\,\xi13_i + 2\,e^{\frac{1}{2}(b1_k+b2_k-2\alpha1_i+2\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x13_l^2\,y13_k\,\xi13_i -$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,y23_k\,\xi13_i + e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x13_k\,x23_l\,y23_k\,\xi13_i +$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-3\alpha2_i+3\alpha2_j)}\,x13_l\,x23_l\,y23_k\,\xi13_i - e^{\frac{1}{2}(b1_k+b2_k-3\alpha2_i+3\alpha2_j)}\,x13_l\,x23_l\,y23_k\,\xi13_i -$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-2\alpha1_i+2\alpha1_j-2\alpha2_i+2\alpha2_j)}\,x13_l^2\,y12_k\,y23_k\,\xi13_i - e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,\eta12_i\,\xi13_i -$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y13_k\,\eta13_i\,\xi13_i - e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y23_k\,\eta23_i\,\xi13_i -$$
$$2\,e^{\frac{1}{2}(-2\alpha1_i+2\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x13_k\,\xi12_i\,\xi13_i - 4\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,\xi12_i\,\xi13_i +$$
$$4\,e^{\frac{1}{2}(2b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,\xi12_i\,\xi13_i + 2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,\xi12_i\,\xi13_i +$$
$$2\,e^{-\frac{3}{2}(\alpha1_i-\alpha1_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_i + 2\,e^{\frac{3}{2}(b1_k-\alpha1_i+\alpha1_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_i +$$
$$4\,e^{\frac{1}{2}(4b1_k-2b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_i - 6\,e^{\frac{1}{2}(2b1_k-b2_k-3\alpha1_i+3\alpha1_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_i -$$

$$2\,e^{\frac{1}{2}(b1_k+b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_i-e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,y23_k\,\xi12_i\,\xi13_i-$$
$$4\,e^{2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,y23_k\,\xi12_i\,\xi13_i-3\,e^{\frac{1}{2}(3\,b1_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y23_k\,\xi12_i\,\xi13_i+$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y23_k\,\xi12_i\,\xi13_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,\xi13_i^2+$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,\xi13_i^2+2\,e^{b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,\xi13_i^2-$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,\xi13_i^2-2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x13_l\,y12_k\,\xi13_j+$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_j-2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x13_l\,\xi12_i\,\xi13_j+$$
$$4\,a2_l\,x23_k\,\xi23_i-2\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_l\,\xi23_i+2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_l\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,a2_k\,x23_l\,\xi23_i-2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,a2_k\,x23_l\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,a2_l\,x23_l\,\xi23_i+2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,a2_l\,x23_l\,\xi23_i+$$
$$3\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,\xi23_i+2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_k\,x13_l\,y12_k\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_l\,x13_l\,y12_k\,\xi23_i-e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_k\,y12_k\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,y12_k\,\xi23_i-e^{\frac{1}{2}(-b1_k+2\,b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,y12_k\,\xi23_i+$$
$$e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,y13_k\,\xi23_i+2\,e^{-\frac{3}{2}(\alpha2_i-\alpha2_j)}\,x13_l\,x23_l\,y13_k\,\xi23_i+$$
$$e^{\frac{1}{2}(-b1_k+2\,b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,y13_k\,\xi23_i+2\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_k\,x23_l\,y23_k\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+2\,\alpha1_i-2\,\alpha1_j-4\,\alpha2_i+4\,\alpha2_j)}\,x23_l^2\,y23_k\,\xi23_i+2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,\beta1_i\,\xi23_i+$$
$$e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_l\,y12_k\,\eta12_i\,\xi23_i-e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_l\,y13_k\,\eta13_i\,\xi23_i+$$
$$e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,y13_k\,\eta13_i\,\xi23_i-2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_l\,y23_k\,\eta23_i\,\xi23_i-$$
$$e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y12_k\,y23_k\,\eta23_i\,\xi23_i-5\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,\xi12_i\,\xi23_i+$$
$$5\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_k\,x13_l\,\xi12_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_k\,x13_l\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_l\,x13_l\,\xi12_i\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a1_l\,x13_l\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_k\,x13_l\,\xi12_i\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_k\,x13_l\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_l\,x13_l\,\xi12_i\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,a2_l\,x13_l\,\xi12_i\,\xi23_i+2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_k\,\xi12_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x12_k\,x23_l\,\xi12_i\,\xi23_i+2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_i+2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,y12_k\,\xi12_i\,\xi23_i+$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x12_l\,x13_l\,y12_k\,\xi12_i\,\xi23_i-e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y13_k\,\xi12_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,y23_k\,\xi12_i\,\xi23_i-e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,y23_k\,\xi12_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y12_k\,y23_k\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}$$
$$x13_l\,y12_k\,\eta12_i\,\xi12_i\,\xi23_i-e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y13_k\,\eta13_i\,\xi12_i\,\xi23_i+$$
$$e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,y23_k\,\eta23_i\,\xi12_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,\xi12_i^2\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x12_k\,x13_l\,\xi12_i^2\,\xi23_i+4\,e^{\frac{1}{2}(4\,b1_k-2\,b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x12_l\,x13_l\,\xi12_i^2\,\xi23_i-$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x12_l\,x13_l\,\xi12_i^2\,\xi23_i-3\,e^{2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,y23_k\,\xi12_i^2\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y23_k\,\xi12_i^2\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,\xi13_i\,\xi23_i+$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,\xi13_i\,\xi23_i+2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_k\,x23_l\,\xi13_i\,\xi23_i+$$
$$2\,e^{-\frac{3}{2}(\alpha2_i-\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_i+2\,e^{\frac{3}{2}(b2_k-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_i-$$

$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_i+2\,e^{\frac{1}{2}(b1_k+b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_i+e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,y12_k\,\xi13_i\,\xi23_i-$$
$$2\,e^{-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_i+4\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_i+$$
$$e^{\frac{1}{2}(b1_k+b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_i-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_k\,x13_l\,\xi12_i\,\xi13_i\,\xi23_i+$$
$$2\,e^{-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_i+4\,e^{2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_i+$$
$$2\,e^{\frac{1}{2}(3\,b1_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_i-6\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_i-$$
$$2\,e^{\frac{1}{2}(b1_k+b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_i-2\,e^{\frac{1}{2}(\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_k\,x23_l\,\xi23_i^2+$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x23_k\,x23_l\,\xi23_i^2+2\,e^{-b1_k+2\,b2_k+\alpha1_i-\alpha1_j-2\,\alpha2_i+2\,\alpha2_j}\,x23_l^2\,\xi23_i^2-$$
$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k+2\,\alpha1_i-2\,\alpha1_j-4\,\alpha2_i+4\,\alpha2_j)}\,x23_l^2\,\xi23_i^2+e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,y12_k\,\xi23_i^2+$$
$$e^{\frac{1}{2}(-b1_k+2\,b2_k-3\,\alpha2_i+3\,\alpha2_j)}\,x13_l\,x23_l\,y12_k\,\xi23_i^2+2\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_k\,\xi12_i\,\xi23_i^2+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,y12_k\,\xi12_i\,\xi23_i^2+2\,e^{2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j}\,x13_l^2\,\xi12_i^2\,\xi23_i^2-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j-2\,\alpha2_i+2\,\alpha2_j)}\,x13_l^2\,\xi12_i^2\,\xi23_i^2-e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y12_k\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_l\,y12_k\,\xi23_j-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,y13_k\,\xi23_j-$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y12_k\,\beta1_i\,\xi23_j-e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y12_k\,y13_k\,\eta13_i\,\xi23_j+$$
$$e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y12_k\,y23_k\,\eta23_i\,\xi23_j+3\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,\xi12_i\,\xi23_j-$$
$$3\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,\xi12_i\,\xi23_j+2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_k\,x13_l\,\xi12_i\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_k\,x13_l\,\xi12_i\,\xi23_j+2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_l\,x13_l\,\xi12_i\,\xi23_j-$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,a1_l\,x13_l\,\xi12_i\,\xi23_j-2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_l\,x23_l\,\xi12_i\,\xi23_j-2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x13_l\,y12_k\,\xi12_i\,\xi23_j-$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-4\,\alpha1_i+4\,\alpha1_j+2\,\alpha2_i-2\,\alpha2_j)}\,x12_l\,x13_l\,y12_k\,\xi12_i\,\xi23_j+e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y13_k\,\xi12_i\,\xi23_j+$$
$$e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,y23_k\,\xi12_i\,\xi23_j+2\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y12_k\,y23_k\,\xi12_i\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y12_k\,\eta12_i\,\xi12_i\,\xi23_j+e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}$$
$$x13_l\,y13_k\,\eta13_i\,\xi12_i\,\xi23_j-e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,y23_k\,\eta23_i\,\xi12_i\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x13_l\,\xi12_i^2\,\xi23_j-2\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x12_k\,x13_l\,\xi12_i^2\,\xi23_j+$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-4\,\alpha1_i+4\,\alpha1_j+2\,\alpha2_i-2\,\alpha2_j)}\,x12_l\,x13_l\,\xi12_i^2\,\xi23_j-4\,e^{2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j}\,x12_l\,x13_l\,\xi12_i^2\,\xi23_j+$$
$$3\,e^{\frac{1}{2}(4\,b1_k-2\,b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y23_k\,\xi12_i^2\,\xi23_j-2\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y23_k\,\xi12_i^2\,\xi23_j-$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,x23_k\,\xi13_i\,\xi23_j-2\,e^{\frac{1}{2}(-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_j+$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_j-2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,\xi13_i\,\xi23_j-$$
$$e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_k\,x13_l\,y12_k\,\xi13_i\,\xi23_j+2\,e^{-\frac{3}{2}(\alpha1_i-\alpha1_j)}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_j-$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_j-e^{\frac{1}{2}(b1_k+b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,y12_k\,\xi13_i\,\xi23_j+$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_k\,x13_l\,\xi12_i\,\xi13_i\,\xi23_j-2\,e^{-\frac{3}{2}(\alpha1_i-\alpha1_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_j-$$
$$2\,e^{\frac{3}{2}(b1_k-\alpha1_i+\alpha1_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_j-4\,e^{\frac{1}{2}(4\,b1_k-2\,b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_j+$$
$$6\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_j+2\,e^{\frac{1}{2}(b1_k+b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_l^2\,\xi12_i\,\xi13_i\,\xi23_j+$$
$$e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,x23_k\,y12_k\,\xi23_i\,\xi23_j+e^{\frac{1}{2}(-b1_k+2\,b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,y12_k\,\xi23_i\,\xi23_j-$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j)}\,x13_l\,x23_k\,\xi12_i\,\xi23_i\,\xi23_j+2\,e^{\frac{1}{2}(b1_k+b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_l\,x23_l\,\xi12_i\,\xi23_i\,\xi23_j-$$

$$2\,e^{\frac{1}{2}(-b1_k+2\,b2_k-\alpha1_i+\alpha1_j-\alpha2_i+\alpha2_j)}\,x13_1\,x23_1\,\xi12_i\,\xi23_i\,\xi23_j -$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_1^2\,y12_k\,\xi12_i\,\xi23_i\,\xi23_j - 4\,e^{\frac{1}{2}(4\,b1_k-2\,b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_1^2\,\xi12_i^2\,\xi23_i\,\xi23_j +$$
$$4\,e^{\frac{1}{2}(2\,b1_k-b2_k-3\,\alpha1_i+3\,\alpha1_j)}\,x13_1^2\,\xi12_i^2\,\xi23_i\,\xi23_j + 2\,e^{\frac{1}{2}(2\,b1_k-b2_k-4\,\alpha1_i+4\,\alpha1_j+2\,\alpha2_i-2\,\alpha2_j)}\,x13_1^2\,y12_k\,\xi12_i\,\xi23_j^2 -$$
$$2\,e^{\frac{1}{2}(2\,b1_k-b2_k-4\,\alpha1_i+4\,\alpha1_j+2\,\alpha2_i-2\,\alpha2_j)}\,x13_1^2\,\xi12_i^2\,\xi23_j^2 + 2\,e^{2\,b1_k-b2_k-2\,\alpha1_i+2\,\alpha1_j+\alpha2_i-\alpha2_j}\,x13_1^2\,\xi12_i^2\,\xi23_j^2\Big]$$

Scratch

In[•]:=
```
Block[{$k = 1, $n = 3}, dS1]
```

Time: 61.30 seconds Out[•]=

$$\mathbb{E}_{\{1\}\to\{1\}}\Big[-a1_1\,\alpha1_1 - a2_1\,\alpha2_1 - b1_1\,\beta1_1 - b2_1\,\beta2_1 - e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1 -$$
$$e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1 - e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1 - e^{\frac{1}{2}(b1_1-2\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta23_1 -$$
$$e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\xi12_1 + \eta12_1\,\xi12_1 - e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta12_1\,\xi12_1 - e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1\,\xi12_1 -$$
$$e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\xi13_1 + \eta13_1\,\xi13_1 - e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1\,\xi13_1 - e^{\frac{1}{2}(\alpha1_1-2\,\alpha2_1)}\,x23_1\,\xi23_1 +$$
$$e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1\,\xi23_1 + \eta23_1\,\xi23_1 - e^{\frac{1}{2}(b1_1-2\,b2_1)}\,\eta23_1\,\xi23_1 -$$
$$e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\xi12_1\,\xi23_1 - e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1\,\xi12_1\,\xi23_1 + e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta13_1\,\xi12_1\,\xi23_1,$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1 + 2\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1 + 3\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1 +$$
$$2\,e^{-\frac{3}{2}(b1_1-\alpha1_1)}\,y12_1\,y13_1\,\eta12_1\,\eta13_1 + 2\,e^{-b1_1-b2_1+\alpha1_1+\alpha2_1}\,y12_1\,y13_1\,y23_1\,\eta13_1^2 +$$
$$2\,e^{\frac{1}{2}(b1_1-2\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta23_1 + 4\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta12_1\,\eta23_1 +$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta12_1\,\eta23_1 + 2\,e^{-\frac{3}{2}(b2_1-\alpha2_1)}\,y13_1\,y23_1\,\eta13_1\,\eta23_1 -$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\xi12_1 + 4\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\beta1_1\,\xi12_1 - 2\,a1_1\,\eta12_1\,\xi12_1 -$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,a1_1\,\eta12_1\,\xi12_1 - 2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,x12_1\,y12_1\,\eta12_1\,\xi12_1 - 2\,\beta1_1\,\eta12_1\,\xi12_1 +$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\beta1_1\,\eta12_1\,\xi12_1 - 2\,e^{\frac{1}{2}(-4\,b1_1+2\,b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1^2\,\xi12_1 -$$
$$e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,x12_1\,y13_1\,\eta13_1\,\xi12_1 - 2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,a1_1\,y23_1\,\eta13_1\,\xi12_1 -$$
$$e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,x12_1\,y12_1\,y23_1\,\eta13_1\,\xi12_1 + 2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\beta1_1\,\eta13_1\,\xi12_1 +$$
$$e^{\frac{1}{2}(-3\,b1_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta12_1\,\eta13_1\,\xi12_1 - 3\,e^{\frac{1}{2}(-3\,b1_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta12_1\,\eta13_1\,\xi12_1 +$$
$$e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+3\,\alpha2_1)}\,y13_1\,y23_1\,\eta13_1^2\,\xi12_1 - e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+3\,\alpha2_1)}\,y12_1\,y23_1^2\,\eta13_1^2\,\xi12_1 +$$
$$e^{\frac{1}{2}(b1_1-2\,b2_1-3\,\alpha1_1+3\,\alpha2_1)}\,x12_1\,y23_1\,\eta23_1\,\xi12_1 + 3\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta12_1\,\eta23_1\,\xi12_1 +$$
$$e^{\frac{1}{2}(-3\,b2_1-2\,\alpha1_1+4\,\alpha2_1)}\,y23_1^2\,\eta13_1\,\eta23_1\,\xi12_1 + 2\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta12_1\,\xi12_1^2 -$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta12_1\,\xi12_1^2 + 2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta12_1^2\,\xi12_1^2 - 2\,e^{-2\,b1_1+b2_1}\,\eta12_1^2\,\xi12_1^2 -$$
$$e^{\frac{1}{2}(-b1_1-b2_1-3\,\alpha1_1+3\,\alpha2_1)}\,x12_1\,y23_1\,\eta13_1\,\xi12_1^2 - 3\,e^{\frac{1}{2}(-3\,b1_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta12_1\,\eta13_1\,\xi12_1^2 +$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta12_1\,\eta13_1\,\xi12_1^2 - e^{-b1_1-b2_1-\alpha1_1+2\,\alpha2_1}\,y23_1^2\,\eta13_1^2\,\xi12_1^2 - 6\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\xi13_1 +$$
$$4\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x12_1\,x23_1\,\xi13_1 + 4\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\beta1_1\,\xi13_1 + 4\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\beta2_1\,\xi13_1 +$$
$$4\,e^{\frac{1}{2}(-2\,b1_1+b2_1+\alpha1_1-2\,\alpha2_1)}\,x23_1\,\eta12_1\,\xi13_1 - e^{\frac{1}{2}(-2\,b1_1+b2_1+\alpha1_1-2\,\alpha2_1)}\,x13_1\,y12_1\,\eta12_1\,\xi13_1 -$$
$$4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1\,\xi13_1 + 4\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta13_1\,\xi13_1 - 2\,a1_1\,\eta13_1\,\xi13_1 - 2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,a1_1\,\eta13_1\,\xi13_1 -$$
$$2\,a2_1\,\eta13_1\,\xi13_1 - 2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,a2_1\,\eta13_1\,\xi13_1 - 4\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,x12_1\,y12_1\,\eta13_1\,\xi13_1 -$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y13_1\,\eta13_1\,\xi13_1 + 4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x23_1\,y23_1\,\eta13_1\,\xi13_1 -$$

$$2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y12_1\,y23_1\,\eta13_1\,\xi13_1-2\,\beta1_1\,\eta13_1\,\xi13_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\beta1_1\,\eta13_1\,\xi13_1-$$
$$2\,\beta2_1\,\eta13_1\,\xi13_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\beta2_1\,\eta13_1\,\xi13_1-e^{\frac{1}{2}(-3\,b1_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta13_1\,\xi13_1+$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta13_1\,\xi13_1-4\,e^{\frac{1}{2}(-4\,b1_1+2\,b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta13_1\,\xi13_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1^2\,\xi13_1-4\,e^{\frac{1}{2}(-3\,b1_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1^2\,\xi13_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1^2\,\xi13_1-4\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta23_1\,\xi13_1-$$
$$e^{\frac{1}{2}(b1_1-2\,b2_1-2\,\alpha1_1+\alpha2_1)}\,x13_1\,y23_1\,\eta23_1\,\xi13_1+4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta12_1\,\eta23_1\,\xi13_1-4\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta12_1\,\eta23_1\,\xi13_1+$$
$$e^{\frac{1}{2}(-3\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1\,\eta23_1\,\xi13_1-4\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1\,\eta23_1\,\xi13_1+$$
$$2\,e^{-\frac{3\,\alpha1_1}{2}}\,x12_1\,x13_1\,\xi12_1\,\xi13_1+2\,e^{\frac{1}{2}(-2\,b1_1+b2_1-\alpha1_1-\alpha2_1)}\,x13_1\,\eta12_1\,\xi12_1\,\xi13_1+$$
$$2\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta13_1\,\xi12_1\,\xi13_1-4\,e^{\frac{1}{2}(-2\,b1_1+b2_1-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta13_1\,\xi12_1\,\xi13_1+$$
$$e^{\frac{1}{2}(-b1_1-b2_1-2\,\alpha1_1+\alpha2_1)}\,x13_1\,y23_1\,\eta13_1\,\xi12_1\,\xi13_1+4\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta12_1\,\eta13_1\,\xi12_1\,\xi13_1-$$
$$4\,e^{-2\,b1_1+b2_1}\,\eta12_1\,\eta13_1\,\xi12_1\,\xi13_1-4\,e^{\frac{1}{2}(-3\,b1_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1^2\,\xi12_1\,\xi13_1-$$
$$e^{\frac{1}{2}(-2\,b1_1-2\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1^2\,\xi12_1\,\xi13_1+2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1^2\,\xi12_1\,\xi13_1+$$
$$2\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\eta13_1\,\xi13_1^2-2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1-\alpha2_1)}\,x13_1\,\eta13_1\,\xi13_1^2+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1^2\,\xi13_1^2-$$
$$2\,e^{-b1_1-b2_1}\,\eta13_1^2\,\xi13_1^2-2\,e^{\frac{1}{2}(\alpha1_1-2\,\alpha2_1)}\,x23_1\,\xi23_1+4\,e^{\frac{1}{2}(\alpha1_1-2\,\alpha2_1)}\,x23_1\,\beta2_1\,\xi23_1+$$
$$e^{\frac{1}{2}(-2\,b1_1+b2_1+3\,\alpha1_1-3\,\alpha2_1)}\,x23_1\,y12_1\,\eta12_1\,\xi23_1-e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,a2_1\,y12_1\,\eta13_1\,\xi23_1-e^{\frac{1}{2}(-b1_1-b2_1+2\,\alpha1_1-\alpha2_1)}\,x23_1\,y13_1\,\eta13_1\,\xi23_1-$$
$$e^{\frac{1}{2}(-b1_1-b2_1+2\,\alpha1_1-\alpha2_1)}\,x23_1\,y12_1\,y23_1\,\eta13_1\,\xi23_1-2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\beta2_1\,\eta13_1\,\xi23_1-$$
$$2\,e^{-\frac{3}{2}(b1_1-\alpha1_1)}\,y12_1\,y13_1\,\eta13_1^2\,\xi23_1-2\,a2_1\,\eta23_1\,\xi23_1-2\,e^{\frac{1}{2}(b1_1-2\,b2_1)}\,a2_1\,\eta23_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(b1_1-2\,b2_1)}\,x23_1\,y23_1\,\eta23_1\,\xi23_1-2\,\beta2_1\,\eta23_1\,\xi23_1+2\,e^{\frac{1}{2}(b1_1-2\,b2_1)}\,\beta2_1\,\eta23_1\,\xi23_1+$$
$$e^{\frac{1}{2}(-b1_1-b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta23_1\,\xi23_1-2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta23_1\,\xi23_1-$$
$$e^{\frac{1}{2}(-3\,b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1\,\eta23_1\,\xi23_1-2\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1\,\eta23_1\,\xi23_1-$$
$$e^{\frac{1}{2}(-3\,b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1\,\eta23_1\,\xi23_1-2\,e^{\frac{1}{2}(2\,b1_1-4\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta23_1^2\,\xi23_1-$$
$$5\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\xi12_1\,\xi23_1+2\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x12_1\,x23_1\,\xi12_1\,\xi23_1+4\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\beta1_1\,\xi12_1\,\xi23_1+$$
$$4\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\beta2_1\,\xi12_1\,\xi23_1+2\,e^{\frac{1}{2}(-2\,b1_1+b2_1+\alpha1_1-2\,\alpha2_1)}\,x23_1\,\eta12_1\,\xi12_1\,\xi23_1-$$
$$e^{\frac{1}{2}(-2\,b1_1+b2_1+\alpha1_1-2\,\alpha2_1)}\,x13_1\,y12_1\,\eta12_1\,\xi12_1\,\xi23_1-3\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1\,\xi12_1\,\xi23_1+$$
$$3\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta13_1\,\xi12_1\,\xi23_1-2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,a1_1\,\eta13_1\,\xi12_1\,\xi23_1+2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,a1_1\,\eta13_1\,\xi12_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,a2_1\,\eta13_1\,\xi12_1\,\xi23_1-2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,a2_1\,\eta13_1\,\xi12_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y13_1\,\eta13_1\,\xi12_1\,\xi23_1-2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y12_1\,y23_1\,\eta13_1\,\xi12_1\,\xi23_1+$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\beta1_1\,\eta13_1\,\xi12_1\,\xi23_1-2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\beta1_1\,\eta13_1\,\xi12_1\,\xi23_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\beta2_1\,\eta13_1\,\xi12_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\beta2_1\,\eta13_1\,\xi12_1\,\xi23_1-e^{\frac{1}{2}(-3\,b1_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta13_1\,\xi12_1\,\xi23_1+$$
$$2\,e^{\frac{1}{2}(-4\,b1_1+2\,b2_1+2\,\alpha1_1-\alpha2_1)}\,y12_1\,\eta12_1\,\eta13_1\,\xi12_1\,\xi23_1-e^{\frac{1}{2}(-3\,b1_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1^2\,\xi12_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+\alpha1_1+\alpha2_1)}\,y13_1\,\eta13_1^2\,\xi12_1\,\xi23_1+e^{\frac{1}{2}(-3\,b1_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1^2\,\xi12_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-2\,b1_1-2\,b2_1+\alpha1_1+\alpha2_1)}\,y12_1\,y23_1\,\eta13_1^2\,\xi12_1\,\xi23_1-2\,e^{\frac{1}{2}(-2\,\alpha1_1+\alpha2_1)}\,x12_1\,\eta23_1\,\xi12_1\,\xi23_1-$$
$$e^{\frac{1}{2}(b1_1-2\,b2_1-2\,\alpha1_1+\alpha2_1)}\,x13_1\,y23_1\,\eta23_1\,\xi12_1\,\xi23_1+4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta12_1\,\eta23_1\,\xi12_1\,\xi23_1-$$
$$4\,e^{\frac{1}{2}(-2\,b1_1+b2_1)}\,\eta12_1\,\eta23_1\,\xi12_1\,\xi23_1-e^{\frac{1}{2}(-3\,b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1\,\eta23_1\,\xi12_1\,\xi23_1-$$
$$3\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1+2\,\alpha2_1)}\,y23_1\,\eta13_1\,\eta23_1\,\xi12_1\,\xi23_1+2\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\eta12_1\,\xi12_1^2\,\xi23_1-$$

$$2\,e^{\frac{1}{2}(-b1_1-b2_1-2\alpha1_1+\alpha2_1)}\,x12_1\,\eta13_1\,\xi12_1^2\,\xi23_1-e^{\frac{1}{2}(-b1_1-b2_1-2\alpha1_1+\alpha2_1)}\,x13_1\,y23_1\,\eta13_1\,\xi12_1^2\,\xi23_1-$$
$$2\,e^{-\frac{3b1_1}{2}}\,\eta12_1\,\eta13_1\,\xi12_1^2\,\xi23_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta12_1\,\eta13_1\,\xi12_1^2\,\xi23_1-2\,e^{\frac{1}{2}(-2b1_1+b2_1)}\,\eta12_1\,\eta13_1\,\xi12_1^2\,\xi23_1+$$
$$2\,e^{-2b1_1+b2_1}\,\eta12_1\,\eta13_1\,\xi12_1^2\,\xi23_1+e^{\frac{1}{2}(-3b1_1-\alpha1_1+2\alpha2_1)}\,y23_1\,\eta13_1^2\,\xi12_1^2\,\xi23_1-$$
$$3\,e^{\frac{1}{2}(-2b1_1-2b2_1-\alpha1_1+2\alpha2_1)}\,y23_1\,\eta13_1^2\,\xi12_1^2\,\xi23_1+2\,e^{-\frac{3\alpha2_1}{2}}\,x13_1\,x23_1\,\xi13_1\,\xi23_1+$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1+\alpha1_1-2\alpha2_1)}\,x23_1\,\eta13_1\,\xi13_1\,\xi23_1-4\,e^{\frac{1}{2}(-2b1_1+b2_1+\alpha1_1-2\alpha2_1)}\,x23_1\,\eta13_1\,\xi13_1\,\xi23_1-$$
$$e^{\frac{1}{2}(-2b1_1+b2_1+\alpha1_1-2\alpha2_1)}\,x13_1\,y12_1\,\eta13_1\,\xi13_1\,\xi23_1-e^{\frac{1}{2}(-3b1_1+2\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1^2\,\xi13_1\,\xi23_1+$$
$$4\,e^{\frac{1}{2}(-4b1_1+2b2_1+2\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1^2\,\xi13_1\,\xi23_1-2\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\eta23_1\,\xi13_1\,\xi23_1-$$
$$4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1\,\eta23_1\,\xi13_1\,\xi23_1+4\,e^{\frac{1}{2}(-2b1_1+b2_1)}\,\eta13_1\,\eta23_1\,\xi13_1\,\xi23_1+2\,e^{-\alpha1_1-\alpha2_1}\,x13_1^2\,\xi12_1\,\xi13_1\,\xi23_1+$$
$$2\,e^{\frac{1}{2}(-\alpha1_1-\alpha2_1)}\,x13_1\,\eta13_1\,\xi12_1\,\xi13_1\,\xi23_1-4\,e^{\frac{1}{2}(-2b1_1+b2_1-\alpha1_1-\alpha2_1)}\,x13_1\,\eta13_1\,\xi12_1\,\xi13_1\,\xi23_1-$$
$$2\,e^{-\frac{3b1_1}{2}}\,\eta13_1^2\,\xi12_1\,\xi13_1\,\xi23_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,\eta13_1^2\,\xi12_1\,\xi13_1\,\xi23_1-2\,e^{-b1_1-b2_1}\,\eta13_1^2\,\xi12_1\,\xi13_1\,\xi23_1-$$
$$2\,e^{\frac{1}{2}(-2b1_1+b2_1)}\,\eta13_1^2\,\xi12_1\,\xi13_1\,\xi23_1+4\,e^{-2b1_1+b2_1}\,\eta13_1^2\,\xi12_1\,\xi13_1\,\xi23_1+$$
$$e^{\frac{1}{2}(-2b1_1+b2_1+3\alpha1_1-3\alpha2_1)}\,x23_1\,y12_1\,\eta13_1\,\xi23_1^2+2\,e^{\frac{1}{2}(\alpha1_1-2\alpha2_1)}\,x23_1\,\eta23_1\,\xi23_1^2-$$
$$2\,e^{\frac{1}{2}(b1_1-2b2_1+\alpha1_1-2\alpha2_1)}\,x23_1\,\eta23_1\,\xi23_1^2+e^{\frac{1}{2}(-b1_1-b2_1+2\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1\,\eta23_1\,\xi23_1^2+$$
$$2\,e^{\frac{1}{2}(b1_1-2b2_1)}\,\eta23_1^2\,\xi23_1^2-2\,e^{b1_1-2b2_1}\,\eta23_1^2\,\xi23_1^2+e^{\frac{1}{2}(-2b1_1+b2_1+\alpha1_1-2\alpha2_1)}\,x13_1\,y12_1\,\eta13_1\,\xi12_1\,\xi23_1^2+$$
$$e^{\frac{1}{2}(-3b1_1+2\alpha1_1-\alpha2_1)}\,y12_1\,\eta13_1^2\,\xi12_1\,\xi23_1^2-2\,e^{\frac{1}{2}(b1_1-2b2_1-\alpha1_1-\alpha2_1)}\,x13_1\,\eta23_1\,\xi12_1\,\xi23_1^2-$$
$$2\,e^{-\frac{3b2_1}{2}}\,\eta13_1\,\eta23_1\,\xi12_1\,\xi23_1^2+2\,e^{\frac{1}{2}(-2b1_1+b2_1)}\,\eta13_1\,\eta23_1\,\xi12_1\,\xi23_1^2-$$
$$2\,e^{\frac{1}{2}(-b1_1-b2_1-\alpha1_1-\alpha2_1)}\,x13_1\,\eta13_1\,\xi12_1^2\,\xi23_1^2+2\,e^{-\frac{3b1_1}{2}}\,\eta13_1^2\,\xi12_1^2\,\xi23_1^2-2\,e^{-b1_1-b2_1}\,\eta13_1^2\,\xi12_1^2\,\xi23_1^2\Big]$$

In[•]:=
```
{$k = 1, $n = 3}
```

Time: 0.02 seconds Out[•]=

$$\{1,\,3\}$$

In[•]:=
```
Block[{$k = 1, $n = 3}, R1,2 // dS2 // dm2,1→1]
```

Time: 8988.86 seconds Out[•]=

$$\mathbb{E}_{\{\}\to\{1\}}\Big[\text{Log}\big[e^{-b1_1-b2_1}\big]-a1_1\,b1_1-a2_1\,b2_1-e^{\frac{1}{2}(-2b1_1+b2_1)}\,x12_1\,y12_1-$$
$$e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y13_1-e^{\frac{1}{2}(b1_1-2b2_1)}\,x23_1\,y23_1-e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y12_1\,y23_1,$$
$$8\,a1_1+8\,a2_1-2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y13_1+4\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x12_1\,x23_1\,y13_1+$$
$$2\,e^{-\frac{3b1_1}{2}}\,x12_1\,x13_1\,y12_1\,y13_1-e^{\frac{1}{2}(-b1_1-b2_1)}\,x13_1\,y12_1\,y23_1+2\,e^{\frac{1}{2}(-b1_1-b2_1)}\,x12_1\,x23_1\,y12_1\,y23_1+$$
$$2\,e^{-\frac{3b2_1}{2}}\,x13_1\,x23_1\,y13_1\,y23_1+2\,e^{-b1_1-b2_1}\,x13_1^2\,y12_1\,y13_1\,y23_1\Big]$$

In[•]:=
```
R1,2 // dS2 // dm2,1→1
```

Time: 21.28 seconds Out[•]=

$Aborted