Hidden Utilities and $\TeX$

Headers

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Pensieve header: Quantum sl_n following Rosso and Yamane.

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Notes.
1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso’s X_i and Y_j. Rosso’s $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i,\ j] \leftrightarrow$ xij. Likewise Rosso’s $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i,\ j] \leftrightarrow$ yij.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i.

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In[ ]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[ ]:=
```
MakePDF[]
```

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In[ ]:=
```
δ_{ij__} := KroneckerDelta[ij];
```

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In[ ]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
```

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In[ ]:=
```
CF[ℰ_] := If[$k == ∞,
   Expand[ℰ],
   Expand[Normal@Series[ℰ /. q → e^(ℏ ϵ γ), {ϵ, 0, $k}]]
  ];
```

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In[ ]:=
```
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝒜[i_] := 𝒜 <> i;
```

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In[•]:=
```
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

In[•]:=
```
{τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[•]=
```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for *R* at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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In[•]:=
```
(* q=e^(ℏ γ ϵ); Ai[n]=e^(ℏ ϵ ai); Bi[n]=e^(ℏ γ bi) *)
```

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In[•]:=
```
$n = 3; $k = ∞; $E := {$n, $k}
```

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In[•]:=
```
$EPBWGens := Flatten@{
   Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]^* = ξ[i, j]; ξ[i, j]^* = x[i, j]; x[i, j]],
   Table[As[i, 1, $n - 1],
    A[i]^* = 𝒜[i]; 𝒜[i]^* = A[i]; A[i][_]],
   Table[As[i, 1, $n - 1],
    a[i]^* = α[i]; α[i]^* = a[i]; a[i]]
  }
(#_U = U_0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

In[•]:=
```
#^* & /@ $EPBWGens
```

Time: 0.01 seconds Out[•]=
```
{ξ12, ξ13, ξ23, A1[_]^*, A2[_]^*, α1, α2}
```

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In[∘]:=
```
UCF[ℰ_] := Module[{F, k}, Expand@Collect[
    ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
        U_i_[l___, AB_[n_], r___] ↦
         Sum[As[k, 0, $k], (ħ^k ϵ^k)/k! U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
       }],
    U_[___], F
   ] /. F → CF]
```

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In[∘]:=
```
m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_(i_→j_)[ℰ_] := ℰ /. U_i → U_j;
m_(i_,j_→k_)[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_(i_,j_→k_)[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_(i_,j_→k_)[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_(i,j→k);
m_(i_,j_→k_)[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((c U_i[xx] RR_j[x, y] // UCF // m_(i,j→i)) U_j[yy] // UCF // m_(i,j→k)) // UCF
  ];
m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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In[∘]:=
```
Δ_(i_→j_,k_)[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[f_, fs___] ↦ m_(k,k1→k)@ m_(j,j1→j)[Δ_(j,k)[f] Δ_(i→j1,k1)[U_i[fs]]]
     }];
Δ_(i_→j_,k_,l__)[ℰ_] := ℰ // Δ_(i→j,k) // Δ_(k→k,l)
```

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In[∘]:=
```
S_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
      U_i[f_, fs___] ↦ m_(i1,i→i)[S_i[f] S_i1[U_i1[fs]]]
     }];
```

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The 𝔸-algebra

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```
In[∘]:= RRk_ [ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk_ [ai_, Aj_[n_]] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := Uk[Aj[n], ai];
RRk_ [Ai_[m_], Aj_[n_]] /;  τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk_ [am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = ι@xij; {m} = ι@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
   ];
RRk_ [Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@Am; q^(-n (δm,i-δm+1,i-δm,j+δm+1,j)) Uk[xij, Am[n]]
  ]
```

```
In[∘]:= RR0[a[1], x[1, 2]]
```

Time: 0.00 seconds Out[∘]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

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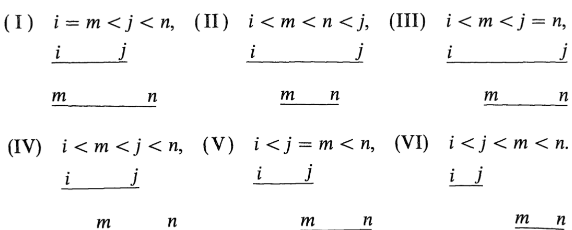
We take the reduction rules RR for xij's from Yamane page 509:



(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$$q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$
$$[e_{ij}, e_{mn}] = 0 \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$
$$[e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{IV})}.$$
$$q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = q e_{in} \qquad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})}.$$

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```
In[∘]:= RRk_ [xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
   {i, j} = ι@xij; {m, n} = ι@xmn;
   Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 Uk[xij, xmn], (* I∨III *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* II∨VI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q^-2 - q^2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q^2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
   ]
  ]
```

In[ ]:=
```
U1[x34] U2[x23] U3[x12] // m1,2→2
```
Time: 0.00 seconds Out[ ]=

$-q\, U_2[x24]\, U_3[x12] + q^2\, U_2[x23, x34]\, U_3[x12]$

In[ ]:=
```
U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
```
Time: 0.02 seconds Out[ ]=

$q^2\, U_0[x14] - q^3\, U_0[x12, x24] - q^3\, U_0[x13, x34] + q^4\, U_0[x12, x23, x34]$

In[ ]:=
```
Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
   ]]
]
```
Time: 0.77 seconds Out[ ]=

{}

In[ ]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
   ]]
]
```
» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}

Out[ ]=

{}

```
Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
   ]]
]
```
» {x12, x13, x23, a1, a2}

Out[ ]=

{}

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The Co-Product, Co-Associativity, and the Hopf Axiom

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```
In[◦]:= Do[As[i, 1, $n - 1],
    Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
    Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
    Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
  ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
      q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
    ]]
]
```

```
In[◦]:= Column@Module[{Bas = $PBWGens, P},
   Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
  ]
```

Time: 0.05 seconds Out[◦]=

$U_i[x12] \to U_j[A1[-1]]\, U_k[x12] + U_j[x12]\, U_k[A1[1]]$

$U_i[x13] \to U_j[A1[-1], A2[-1]]\, U_k[x13] + \left(-\frac{1}{q^2} + q^2\right) U_j[x12, A2[-1]]\, U_k[x23, A1[1]] + U_j[x13]\, U_k[A1[1], A2[1]]$

$U_i[x23] \to U_j[A2[-1]]\, U_k[x23] + U_j[x23]\, U_k[A2[1]]$

$U_i[a1] \to U_j[a1]\, U_k[] + U_j[]\, U_k[a1]$

$U_i[a2] \to U_j[a2]\, U_k[] + U_j[]\, U_k[a2]$

```
In[◦]:= Short[{Δ4,5[a2], Δ7,9[A2[77]], Δ1,2[x12], Δ1,2[x15]}, 8]
```

Time: 0.01 seconds Out[◦]=

$\{U_4[a2]\, U_5[] + U_4[]\, U_5[a2],\ U_7[A2[77]]\, U_9[A2[77]],\ U_1[A1[-1]]\, U_2[x12] + U_1[x12]\, U_2[A1[1]],\ \Delta_{1,2}[x15]\}$

```
In[◦]:= Δ1→1,2[U1[x13, a2] U0[x34]]
```

Time: 0.01 seconds Out[◦]=

$$U_0[x34]\, U_1[A1[-1], A2[-1], a2]\, U_2[x13] + U_0[x34]\, U_1[A1[-1], A2[-1]]\, U_2[x13, a2] + \left(-\frac{1}{q^2} + q^2\right) U_0[x34]\, U_1[x12, A2[-1], a2]\, U_2[x23, A1[1]] + U_0[x34]\, U_1[x13, a2]\, U_2[A1[1], A2[1]] + \left(-\frac{1}{q^2} + q^2\right) U_0[x34]\, U_1[x12, A2[-1]]\, U_2[x23, A1[1], a2] + U_0[x34]\, U_1[x13]\, U_2[A1[1], A2[1], a2]$$

```
In[•]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas],
      P = U1[X];
      X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
    ]]
 ]

» {x12, x13, x23, a1, a2, A2[-1], A1[0], A1[-2], A1[1], A2[-2], A1[2], A2[2], A1[-1], A1[2], A2[1]}

Out[•]=
 {}

In[•]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas],
      P = U1[X] U2[Y];
      {X, Y} → (P // m1,2→1 // Δ1→1,2) == (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
    ]]
 ]

» {x12, x13, x23, a1, a2, A2[2], A2[1], A1[0], A2[1], A2[1], A2[-1], A1[-1], A2[-1], A2[2], A2[-2]}

Out[•]=
 {}
```

Is Δ a quantization of the co-bracket?

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
   Table[As[P, U1 /@ Bas],
    P → UCF[ (Δ1→1,2[P] - Δ2→2,1[P]) //.
        {q^n_. :→ 1 + n ℏ / 2, Ui_[xs___, A_[n_], as___] :→ Ui[xs] + n ℏ Ui[xs, a @@ ι[A], as] / 2}
      ] /. ℏ^n_ /; n > 1 :→ 0
   ]
  ]

Time: 0.08 seconds Out[•]=
 U1[x12] → ℏ U1[x12] U2[a1] - ℏ U1[a1] U2[x12]
 U1[x13] → ℏ U1[x13] U2[a1] - 2 ℏ U1[x23] U2[x12] - ℏ U1[a1] U2[x13] + 2 ℏ U1[x12] U2[x23]
 U1[x23] → ℏ U1[x23] U2[a2] - ℏ U1[a2] U2[x23]
 U1[a1] → 0
 U1[a2] → 0
```

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The Antipode

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In[∘]:=
```
Do[As[i, 1, $n - 1],
  S_j_[a[i]] = -U_j[a[i]];
  S_j_[A[i][n_]] = U_j[A[i][-n]];
  S_j_[x[i, i + 1]] = -q^-2 U_j[x[i, i + 1]] (*?*)
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 S_j_[x[i, n]] = m_{j1,j→j}[
   q S_j[x[i, n - 1]] S_j1[x[n - 1, n]] - q^-1 S_j[x[n - 1, n]] S_j1[x[i, n - 1]]
  ]
]
```

In[∘]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U_1[X];
    X → (P // Δ_{1→2,3} // S_2 // m_{2,3→1})
   ]]
]
```

» {x12, x13, x23, a1, a2, A2[1], A2[-2], A1[-1], A1[2], A1[-2], A1[-1], A1[1], A1[2], A1[-1], A1[-1]}

Out[∘]=
```
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[1] → U_1[],
 A2[-2] → U_1[], A1[-1] → U_1[], A1[2] → U_1[], A1[-2] → U_1[],
 A1[-1] → U_1[], A1[1] → U_1[], A1[2] → U_1[], A1[-1] → U_1[], A1[-1] → U_1[]}
```

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The Generating Functions for the 𝔸 operations

In[∘]:=
```
$k = 1;
```

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NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

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In[∘]:=
```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] ⧴ i, ∞];
```

In[∘]:=
```
Supp[2 ℏ U_1[x45] U_2[x34]]
```

Time: 0.01 seconds Out[∘]=
```
{1, 2}
```

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In[∘]:=
```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{i, is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := UCF[x ⊠ y - y ⊠ x];
```

In[∘]:=
```
(2 ℏ_1 U_1[x45] U_2[x34]) ⊠ (2 ℏ_2 U_1[x45] U_2[x34])
```

Time: 0.03 seconds Out[∘]=

$4\,\hbar_1\,\hbar_2\,U_1[x45, x45]\,U_2[x34, x34]$

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In[∘]:=
```
(*adPower[x_,k_,Env_][c_*y_QU]:=UCF[c adPower[x,k,Env][y]];
adPower[x_,k_,Env_][y_Plus] := adPower[x,k,Env]/@ y;*)
adPower[_, 0, Env_][y_] := y;
(*adPower[x_,k_,Env_][0]=0;*)
adPower[x_, k_, Env_][y_] :=
  (*adPower[x,k,Env][y]=*)UCF@UB[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

In[∘]:=
```
Table[As[k, 0, 5], adPower[U_0[x12], k][U_0[x23]]]
```

Time: 0.06 seconds Out[∘]=

$\{U_0[x23],\ (-1 - \gamma \in \hbar)\,U_0[x13] + 2\,\gamma \in \hbar\,U_0[x12, x23],\ 0,\ 0,\ 0,\ 0\}$

In[∘]:=
```
Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U_i /@ Bas], P → Δ_{i→j,k}[P]]
 ]
```

Time: 0.08 seconds Out[∘]=

$U_i[x12] \to U_j[x12]\,U_k[\,] + \in \hbar\,U_j[x12]\,U_k[a1] + U_j[\,]\,U_k[x12] + \in \hbar\,U_j[a1]\,U_k[x12]$

$U_i[x13] \to U_j[x13]\,U_k[\,] + \in \hbar\,U_j[x13]\,U_k[a1] + \in \hbar\,U_j[x13]\,U_k[a2] +$
$\quad U_j[\,]\,U_k[x13] + \in \hbar\,U_j[a1]\,U_k[x13] + \in \hbar\,U_j[a2]\,U_k[x13] + 4\,\gamma \in \hbar\,U_j[x12]\,U_k[x23]$

$U_i[x23] \to U_j[x23]\,U_k[\,] + \in \hbar\,U_j[x23]\,U_k[a2] + U_j[\,]\,U_k[x23] + \in \hbar\,U_j[a2]\,U_k[x23]$

$U_i[a1] \to U_j[a1]\,U_k[\,] + U_j[\,]\,U_k[a1]$

$U_i[a2] \to U_j[a2]\,U_k[\,] + U_j[\,]\,U_k[a2]$

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In[∘]:=
```
Ad[0] = Identity;
```

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In[∘]:=
```
Ad[c_. U_i_[xjk_] (EUs : U_[] ...)][y_] /; τ[xjk] == "x" := Module[{k = 0, s = 0, t},
  While[(t = adPower[U_i[xjk], k][y]) =!= 0,
   s += t / k!;
   ++k
  ];
  UCF[s ⊠ EUs]
 ]
```

```
In[]:= Ad[ϵ ℏ U_j[x12] U_k[]][U_j[x23] U_k[a4]]
```

Time: 0.01 seconds Out[]=

$(-1 - \gamma \epsilon \hbar)\, U_j[x13]\, U_k[a4] + U_j[x23]\, U_k[a4] + 2\, \gamma \epsilon \hbar\, U_j[x12, x23]\, U_k[a4]$

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```
In[]:= Wt_aj_[x_] := Wt_aj[x] = UB[U_0[aj], U_0[x]] / U_0[x]
```

```
In[]:= Table[Wt_a[j][x23], {j, 1, $n}]
```

Time: 0.00 seconds Out[]=

$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}\right\}$

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```
In[]:= Ad[c_. U_i_[aj_] (EUs : U_[] ...)][y_] /; τ[aj] == "a" :=
  (y /. U_i[gs___] :> U_i[gs] ⅇ^(c Total[Wt_aj /@ {gs}]))
```

```
In[]:= Ad[π ⅈ U_j[a1]][U_j[x23] U_k[a4]]
```

Time: 0.02 seconds Out[]=

$e^{-\frac{1}{2} i \pi \gamma}\, U_j[x23]\, U_k[a4]$

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```
In[]:= TT_λ_[] = 0; (* Translated Tangent *)
TT_λ_[xs___, x_] := UCF[Ad[x][TT_λ[xs]] + ∂_λ x];
TT_λ_[xs_List] := TT_λ @@ xs
```

```
In[]:= TT_λ[λ ξ12 U_0[x12] U_1[], λ ξ23 U_0[x23] U_1[]]
```

Time: 0.03 seconds Out[]=

$\xi 12\, U_0[x12]\, U_1[] + (\xi 12 + \gamma \epsilon\, \xi 12\, \hbar)\, U_0[x13]\, U_1[] + \xi 23\, U_0[x23]\, U_1[] - 2\, \gamma \epsilon\, \xi 12\, \hbar\, U_0[x12, x23]\, U_1[]$

```
In[]:= $PBWGens
```

Time: 0.00 seconds Out[]=

{x12, x13, x23, a1, a2}

```
In[]:= Join[
  Cases[$PBWGens, g_ :> Which[
     τ[g] === "x", λ (g*)_i U_k[g],
     τ[g] === "a", (g*)_i U_k[g]
    ]],
  Cases[$PBWGens, g_ :> Which[
     τ[g] === "x", λ (g*)_j U_k[g],
     τ[g] === "a", (g*)_j U_k[g]
    ]]
 ]
```

Time: 0.06 seconds Out[]=

$\{\lambda\, \xi 12_i\, U_k[x12], \lambda\, \xi 13_i\, U_k[x13], \lambda\, \xi 23_i\, U_k[x23], \alpha 1_i\, U_k[a1], \alpha 2_i\, U_k[a2],$
$\lambda\, \xi 12_j\, U_k[x12], \lambda\, \xi 13_j\, U_k[x13], \lambda\, \xi 23_j\, U_k[x23], \alpha 1_j\, U_k[a1], \alpha 2_j\, U_k[a2]\}$

In[∘]:=
```
$n = 3
```

Time: 0.00 seconds Out[∘]=

3

In[∘]:=
```
TTL = TTλ@Take[Join[
    Cases[$PBWGens, g_ :→ Which[
      τ[g] === "x", λ (g*)_i U_k[g],
      τ[g] === "a", (g*)_i U_k[g]
     ]],
    Cases[$PBWGens, g_ :→ Which[
      τ[g] === "x", λ (g*)_j U_k[g],
      τ[g] === "a", (g*)_j U_k[g]
     ]]
   ], All]
```

Time: 4.72 seconds Out[∘]=

$$
\begin{aligned}
&\left(e^{\gamma\alpha 1_i+\gamma\alpha 1_j-\frac{\gamma\alpha 2_i}{2}-\frac{\gamma\alpha 2_j}{2}}\,\xi 12_i+e^{\gamma\alpha 1_j-\frac{\gamma\alpha 2_j}{2}}\,\xi 12_j\right)U_k[x12]+\\
&\left(e^{\frac{3\gamma\alpha 1_i}{2}+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 12_i+e^{\gamma(\alpha 1_i+\alpha 2_i)+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 12_i+e^{\frac{1}{2}\gamma(2\alpha 1_i+\alpha 1_j-\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_i+\right.\\
&\quad e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_i+e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i)+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 12_j+e^{\frac{\gamma\alpha 1_j}{2}+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_j+\\
&\quad e^{\gamma(\alpha 1_i+\alpha 2_i)+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 13_i+e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i)+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 13_j-\\
&\quad \left.e^{\frac{3\gamma\alpha 2_i}{2}+\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j-\alpha 2_i+\alpha 2_j)}\,\xi 23_i-e^{-\frac{1}{2}\gamma(\alpha 1_i-\alpha 1_j-2\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 23_i\right)U_k[x13]+\\
&\left(e^{-\frac{\gamma\alpha 1_i}{2}-\frac{\gamma\alpha 1_j}{2}+\gamma\alpha 2_i+\gamma\alpha 2_j}\,\xi 23_i+e^{-\frac{\gamma\alpha 1_j}{2}+\gamma\alpha 2_j}\,\xi 23_j\right)U_k[x23]+\\
&\left(2\,e^{\frac{3}{2}\gamma(\alpha 1_i+\alpha 1_j)}\,\gamma\epsilon\hbar\,\xi 12_i+2\,e^{\frac{1}{2}\gamma(2\alpha 1_i+3\alpha 1_j-\alpha 2_i)}\,\gamma\epsilon\hbar\,\xi 12_i+\right.\\
&\quad \left.2\,e^{\frac{3\gamma\alpha 1_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_j-2\,e^{\frac{1}{2}\gamma(\alpha 1_i+3\alpha 1_j+\alpha 2_i)}\,\gamma\epsilon\hbar\,\xi 13_i\right)U_k[x12,x13]+\\
&\left(-2\,e^{\frac{1}{2}\gamma(2\alpha 1_i+\alpha 1_j-\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_i-2\,e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_i-\right.\\
&\quad \left.2\,e^{\frac{\gamma\alpha 1_j}{2}+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 12_j+2\,e^{\frac{1}{2}\gamma(-\alpha 1_i+\alpha 1_j+2\alpha 2_i)+\frac{\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 23_i\right)U_k[x12,x23]+\\
&\left(2\,e^{\gamma\alpha 1_i+\gamma\alpha 1_j-\frac{\gamma\alpha 2_i}{2}+\gamma\alpha 2_j}\,\gamma\epsilon\hbar\,\xi 12_i+2\,e^{\gamma\alpha 1_i+\gamma\alpha 1_j+\gamma\alpha 2_i+\gamma\alpha 2_j}\,\gamma\epsilon\hbar\,\xi 12_i+2\,e^{\gamma\alpha 1_i+\gamma\alpha 1_j-\frac{\gamma\alpha 2_i}{2}+\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i)+\gamma\alpha 2_j}\right.\\
&\quad \left.\gamma\epsilon\hbar\,\xi 12_i+2\,e^{\gamma\alpha 1_j+\gamma\alpha 2_j}\,\gamma\epsilon\hbar\,\xi 12_j-2\,e^{\frac{\gamma\alpha 1_i}{2}+\gamma\alpha 1_j+\frac{\gamma\alpha 2_i}{2}+\gamma\alpha 2_j}\,\gamma\epsilon\hbar\,\xi 13_i-2\,e^{\gamma\alpha 1_j+\frac{3\gamma\alpha 2_i}{2}+\gamma\alpha 2_j}\,\gamma\epsilon\hbar\,\xi 13_i\right)\\
&U_k[x13,x13]+\left(2\,e^{\frac{3\gamma\alpha 2_i}{2}+\frac{3\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 13_i+2\,e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i)+\frac{3\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 13_i+\right.\\
&\quad \left.2\,e^{\frac{3\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 13_j-2\,e^{-\frac{\gamma\alpha 1_i}{2}+\gamma\alpha 2_i+\frac{3\gamma\alpha 2_j}{2}}\,\gamma\epsilon\hbar\,\xi 23_i\right)U_k[x13,x23]
\end{aligned}
$$

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In[∘]:=
```
mons[p_] := Union@Cases[Coefficient[TTL, ϵ, p], U_k[___], ∞];
mons /@ Range[0, $k]
```

Time: 0.02 seconds Out[∘]=
pdf

$\{\{U_k[x12], U_k[x13], U_k[x23]\},$
$\{U_k[x13], U_k[x12, x13], U_k[x12, x23], U_k[x13, x13], U_k[x13, x23]\}\}$

In[∘]:=
```
U_k /@ $PBWGens
```
Time: 0.00 seconds Out[∘]=

$\{U_k[x12], U_k[x13], U_k[x23], U_k[a1], U_k[a2]\}$

pdf

In[∘]:=
```
ff = Table[
   Sum[ϵ^p f_{p,u}[λ], {p, 0, $k}],
   {u, $PBWGens}
  ] /. f_{0,u_}[λ] /; τ[u] == "a" :> (u*)_i + (u*)_j
```
Time: 0.01 seconds Out[∘]=

pdf

$\{f_{0,x12}[\lambda] + \epsilon f_{1,x12}[\lambda], f_{0,x13}[\lambda] + \epsilon f_{1,x13}[\lambda], f_{0,x23}[\lambda] + \epsilon f_{1,x23}[\lambda], \alpha 1_i + \alpha 1_j + \epsilon f_{1,a1}[\lambda], \alpha 2_i + \alpha 2_j + \epsilon f_{1,a2}[\lambda]\}$

In[∘]:=
```
fs = Flatten[Table[
   f_{k,Times@@m},
   {k, 0, $k}, {m, mons[k]}
  ]]
```
Time: 0.02 seconds Out[∘]=

$\{f_{0,x12}, f_{0,x13}, f_{0,x23}, f_{1,x13}, f_{1,x12\,x13}, f_{1,x12\,x23}, f_{1,x13^2}, f_{1,x13\,x23}\}$

In[∘]:=
```
gg = Sum[
  ϵ^p f_{p,Times@@u}[λ] Times @@ u,
  {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
 ]
```
Time: 0.05 seconds Out[∘]=

$x12\,x13\,\epsilon f_{1,x12\,x13}[\lambda] + x13^2 \epsilon f_{1,x13^2}[\lambda] + x12\,x23\,\epsilon f_{1,x12\,x23}[\lambda] + x13\,x23\,\epsilon f_{1,x13\,x23}[\lambda]$

In[∘]:=
```
G = Series[e^gg, {ϵ, 0, $k}]
```
Time: 0.08 seconds Out[∘]=

$1 + \left(x12\,x13\,f_{1,x12\,x13}[\lambda] + x13^2 f_{1,x13^2}[\lambda] + x12\,x23\,f_{1,x12\,x23}[\lambda] + x13\,x23\,f_{1,x13\,x23}[\lambda]\right)\epsilon + O[\epsilon]^2$

In[∘]:=
```
𝕆_i_[poly_] := Total[
  CoefficientRules[poly, $PBWGens] /.
   (p_ → c_) :> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
 ]
```

In[∘]:=
```
𝕆_i[x12^2 x13 + 7 x23^17]
```
Time: 0.02 seconds Out[∘]=

$U_i[x12, x12, x13] + 7\,U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$

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$$L_\lambda = \mathbb{OE}(\lambda \xi_{i,1} x)\, \mathbb{OE}(\lambda \xi_{i,2} x) \qquad \text{Seepage is defined by: } \underbrace{\mathbb{OE}(f_i x_i)}_{\text{off}}\, \mathbb{O}(\text{Seepage}(\{f_i\}, A)) = \mathbb{O}(e^{f_i x_i} A)$$

$$\Pi(L_\lambda) = L^{-1}\partial_\lambda L_\lambda \qquad S(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \mathbb{OE}(f+g), \quad f = f_i x_i \quad g = \sum_{\substack{w\\ |w|\neq 1}} g_w w \qquad e^f = F \quad e^g = G \qquad R_\lambda = \mathbb{O}(FG) \qquad \partial_\lambda F = F\partial_\lambda f$$

$$\Pi(R_\lambda) = R_\lambda^{-1}(\partial_\lambda R_\lambda) = \mathbb{O}(FG)^{-1}\mathbb{O}(\partial_\lambda FG) = S(G)^{-1}\mathbb{O}(F)^{-1}\mathbb{O}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)^{-1} S(\partial_\lambda f G + \partial_\lambda G).$$

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```
In[∘]:= Seepageᵢ_ [{}, c_] = c;
        Seepageᵢ_ [ξs_List, P_] := Module[{lx},
          lx = $PBWGens[[Length@ξs]];
          CF@Collect[P, lx, (Ad[Last[ξs] Uᵢ[lx]][𝕆ᵢ[Seepageᵢ[Most@ξs, #]]] /. Uᵢ → Times) &]
         ]
```

```
In[∘]:= $PBWGens
```

Time: 0.01 seconds Out[∘]=

{x12, x13, x23, a1, a2}

```
In[∘]:= Seepageᵢ[#* & /@ $PBWGens, x12]
```

Time: 0.14 seconds Out[∘]=

$$e^{\alpha 1 \gamma - \frac{\alpha 2 \gamma}{2}} x12 + e^{\frac{1}{2}(\alpha 1+\alpha 2)\gamma} x13 + e^{\frac{1}{2}(\alpha 1+\alpha 2)\gamma} x13\, \gamma \in \hbar + 2 e^{\frac{3\alpha 1 \gamma}{2}} x12\, x13\, \gamma \in \hbar + 2 e^{(\alpha 1+\alpha 2)\gamma} x13^2 \gamma \in \hbar - 2 e^{\frac{1}{2}(\alpha 1+\alpha 2)\gamma} x12\, x23\, \gamma \in \hbar$$

```
In[∘]:= ff
```

Time: 0.02 seconds Out[∘]=

$$\{f_{0,x12}[\lambda] + \in f_{1,x12}[\lambda],\ f_{0,x13}[\lambda] + \in f_{1,x13}[\lambda],\ f_{0,x23}[\lambda] + \in f_{1,x23}[\lambda],\ \alpha 1_i + \alpha 1_j + \in f_{1,a1}[\lambda],\ \alpha 2_i + \alpha 2_j + \in f_{1,a2}[\lambda]\}$$

```
In[∘]:= SS@UCF@G
```

Time: 0.05 seconds Out[∘]=

$$1 + \in \left(x12\, x13\, f_{1,x12\,x13}[\lambda] + x13^2 f_{1,x13^2}[\lambda] + x12\, x23\, f_{1,x12\,x23}[\lambda] + x13\, x23\, f_{1,x13\,x23}[\lambda]\right)$$

```
In[∘]:= Seepageᵢ[ff, SS@UCF@G]
```

Time: 0.31 seconds Out[∘]=

$$1 + e^{\frac{3}{2}\gamma(\alpha 1_i+\alpha 1_j)} x12\, x13 \in f_{1,x12\,x13}[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} x13^2 \in f_{1,x12\,x13}[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} x13^2 \in f_{1,x13^2}[\lambda] + e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} x12\, x23 \in f_{1,x12\,x23}[\lambda] + e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} x13\, x23 \in f_{1,x12\,x23}[\lambda] + e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} x13\, x23 \in f_{1,x13\,x23}[\lambda]$$

```
In[∘]:= 𝕆ᵢ@Seepageᵢ[ff, SS@UCF@G]
```

Time: 0.31 seconds Out[∘]=

$$U_i[\,] + e^{\frac{3}{2}\gamma(\alpha 1_i+\alpha 1_j)} \in U_i[x12, x13]\, f_{1,x12\,x13}[\lambda] + U_i[x13, x13]\left(e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in f_{1,x12\,x13}[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in f_{1,x13^2}[\lambda]\right) + e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in U_i[x12, x23]\, f_{1,x12\,x23}[\lambda] + U_i[x13, x23]\left(e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} \in f_{1,x12\,x23}[\lambda] + e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} \in f_{1,x13\,x23}[\lambda]\right)$$

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In[ ]:=
```
𝒮i_[B_] := UCF[SS@𝕆i[Seepagei[ff, SS@UCF@B]]];
```

In[ ]:=
```
Short[𝒮G = 𝒮k[G], 10]
```

Time: 0.45 seconds Out[ ]=

$$U_k[\,] + e^{\frac{3}{2}\gamma(\alpha 1_i+\alpha 1_j)} \in U_k[x12, x13]\, f_{1,x12\,x13}[\lambda] + U_k[x13, x13]\left(e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in f_{1,x12\,x13}[\lambda] + e^{\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in f_{1,x13^2}[\lambda]\right) + e^{\frac{1}{2}\gamma(\alpha 1_i+\alpha 1_j+\alpha 2_i+\alpha 2_j)} \in U_k[x12, x23]\, f_{1,x12\,x23}[\lambda] + U_k[x13, x23]\left(e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} \in f_{1,x12\,x23}[\lambda] + e^{\frac{3}{2}\gamma(\alpha 2_i+\alpha 2_j)} \in f_{1,x13\,x23}[\lambda]\right)$$