

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

(Alt) In[]:=

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```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
BeginProfile[]
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

(Alt) Out[]=

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(Alt) In[]:=

```
MakePDF[]
```

(Alt) In[]:=

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```
 $\delta_{ij} := \text{KroneckerDelta}[i, j];$ 
```

(Alt) In[]:=

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```
SetAttributes[{SS, HoldAll};
SS[ $\mathcal{E}$ , op_] := PPSS@Collect[PPNormal@Normal@PPSeries@Series[ $\mathcal{E}$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\mathcal{E}$ ] := SS[ $\mathcal{E}$ , Identity];
```

(Alt) In[]:=
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```
$N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; ℳ[i_] := ℳ <> i;
y[i_, j_] := y <> i <> j; b[i_] := b <> i; B[i_] := B <> i;
η[i_, j_] := η <> i <> j; β[i_] := β <> i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];
```

In[]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

(Alt) In[]:=
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```
Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
  ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
  η[i, j]* = y[i, j]
]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
  b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_{αβ})* := (u*)_{αβ};
F[u_i] := F[u_i] = ToExpression["F" <> ToString[u]]_i
```

(Alt) In[]:=
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```
LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;
```

Upper and lower case variables:

(Alt) In[] :=
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```
Module[{i, s, p, c, d, χ, f, v},
  U21Rule = Union @@ Table[As[i, 1, $N - 1], {A[i]_s^p → e^{p α[i]_s}, B[i]_s^p → e^{-p ħ b[i]_s}}];
  l2URule = Append[Union @@ Table[As[i, 1, $N - 1],
    {e^{d_- + c_- . b[i]_s} → B[i]_s^{-c/ħ} e^d, e^{d_- + c_- . α[i]_s} → A[i]_s^c e^d}], e^χ → e^{Expand[l2U[χ]}]];
  U21[ε_] := ε /. U21Rule;
  l2U[ε_] := ε //. l2URule;
  Do[As[i, 1, $N - 1], With[{αi = α[i], Ai = A[i], bi = b[i], Bi = B[i]},
    Dαi_s[f_] := ∂αi_s f + Ai_s ∂Ai_s f;
    Dbi_s[f_] := ∂bi_s f - ħ Bi_s ∂Bi_s f;
  ]];
  Dv[f_] := ∂v f;
];
```

In[] := {U21Rule, l2URule}

Time: 0.00 seconds Out[] =

$$\left\{ \left\{ B1_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b1_{s\$}}, B2_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b2_{s\$}}, B3_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b3_{s\$}}, B4_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b4_{s\$}}, B5_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b5_{s\$}}, \right. \right.$$

$$B6_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b6_{s\$}}, B7_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b7_{s\$}}, B8_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b8_{s\$}}, A1_{s\$}^{p\$} \rightarrow e^{p\$ \alpha1_{s\$}}, A2_{s\$}^{p\$} \rightarrow e^{p\$ \alpha2_{s\$}},$$

$$A3_{s\$}^{p\$} \rightarrow e^{p\$ \alpha3_{s\$}}, A4_{s\$}^{p\$} \rightarrow e^{p\$ \alpha4_{s\$}}, A5_{s\$}^{p\$} \rightarrow e^{p\$ \alpha5_{s\$}}, A6_{s\$}^{p\$} \rightarrow e^{p\$ \alpha6_{s\$}}, A7_{s\$}^{p\$} \rightarrow e^{p\$ \alpha7_{s\$}}, A8_{s\$}^{p\$} \rightarrow e^{p\$ \alpha8_{s\$}} \left. \right\},$$

$$\left\{ e^{d\$_- + c\$_- . b1_{s\$}} \rightarrow e^{d\$} B1_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . b2_{s\$}} \rightarrow e^{d\$} B2_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . b3_{s\$}} \rightarrow e^{d\$} B3_{s\$}^{-\frac{c\$}{\hbar}}, \right.$$

$$e^{d\$_- + c\$_- . b4_{s\$}} \rightarrow e^{d\$} B4_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . b5_{s\$}} \rightarrow e^{d\$} B5_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . b6_{s\$}} \rightarrow e^{d\$} B6_{s\$}^{-\frac{c\$}{\hbar}},$$

$$e^{d\$_- + c\$_- . b7_{s\$}} \rightarrow e^{d\$} B7_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . b8_{s\$}} \rightarrow e^{d\$} B8_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- . \alpha1_{s\$}} \rightarrow e^{d\$} A1_{s\$}^{c\$},$$

$$e^{d\$_- + c\$_- . \alpha2_{s\$}} \rightarrow e^{d\$} A2_{s\$}^{c\$}, e^{d\$_- + c\$_- . \alpha3_{s\$}} \rightarrow e^{d\$} A3_{s\$}^{c\$}, e^{d\$_- + c\$_- . \alpha4_{s\$}} \rightarrow e^{d\$} A4_{s\$}^{c\$}, e^{d\$_- + c\$_- . \alpha5_{s\$}} \rightarrow e^{d\$} A5_{s\$}^{c\$},$$

$$e^{d\$_- + c\$_- . \alpha6_{s\$}} \rightarrow e^{d\$} A6_{s\$}^{c\$}, e^{d\$_- + c\$_- . \alpha7_{s\$}} \rightarrow e^{d\$} A7_{s\$}^{c\$}, e^{d\$_- + c\$_- . \alpha8_{s\$}} \rightarrow e^{d\$} A8_{s\$}^{c\$}, e^{\chi} \rightarrow e^{\text{Expand}[l2U[\chi]]} \left. \right\}$$

In[] := {D_{b3_i} / @ {b3_i⁵, B3_i⁷}, D_{α3_i} / @ {α3_i⁵, A3_i⁷}}

Time: 0.01 seconds Out[] =

$$\left\{ \left\{ 5 b3_i^4, -7 \hbar B3_i^7 \right\}, \left\{ 5 \alpha3_i^4, 7 A3_i^7 \right\} \right\}$$

Canonical Form:

(Alt) In[]:=
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```
CF[ $\mathcal{E}$ _] := PPCF@Module[{s1},
  s1 = If[$k ==  $\infty$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor} @ p}$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor} @ p}$  /.  $q^{n-} \rightarrow \sum_{k=0}^{\$k} \frac{(\hbar \in \gamma n)^k}{k!}$  /.  $e^{k-} \rightarrow 0$  ;  $k > \$k \rightarrow 0$ 
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern, $\infty$ ],CCF] /. CCF→Factor*)
]
```

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Weights: (Need confirmation!!!)

(Alt) In[]:=
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```
(*Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,i-j};
  Wt[ $\xi$ [i,j]]={0,j-i}; Wt[ $\eta$ [i,j]]={1,j-i};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[ $\alpha$ [i]]={0,0}; Wt[ $\beta$ [i]]={1,0};
];
Wt[u_]:=Wt[u];*)
Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,-(i-j)};
  Wt[ $\xi$ [i,j]]={0,-(i-j)}; Wt[ $\eta$ [i,j]]={1,(i-j)};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[ $\alpha$ [i]]={0,0}; Wt[ $\beta$ [i]]={1,0};
];
Wt[u_]:=Wt[u];
```

(Alt) In[]:=
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```
(* q~ $e^{\hbar \gamma \in}$ ; Ai[n]~ $e^{\hbar \in ai}$ ; *)
```

(Alt) In[]:=
pdf

```
$n = 5; $k =  $\infty$ ; $E := {$n, $k}
```

(Alt) In[]:=
pdf

```
$EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi → {i, k}}, {g, $EPBWGens}]];
x- ≤ y- := OrderedQ[{x, y} /. $EPBWRule]; x- < y- := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

In[]:= \$PBWGens*

Time: 0.01 seconds Out[]:=

{ξ₁₂, ξ₁₃, ξ₁₄, ξ₁₅, ξ₂₃, ξ₂₄, ξ₂₅, ξ₃₄, ξ₃₅, ξ₄₅, α₁, α₂, α₃, α₄}

(Alt) In[]:=
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```
UCF[ε-] := PPUCF@Module[{F, k}, Expand@Collect[
  ε /. {Ui[L---, _[0], r---] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L---, AB[n-], r---] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]]
  }],
  U-[---], F
] /. F → CF]
```

(Alt) In[]:=
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```
ms-[0] = 0;
ms-[xPlus] := UCF[ms /@ x];
mi→j-[ε-] := ε /. Ui → Uj;
mi→j→k-[c-. Ui[x---] Uj[-]] := c Uk[x];
mi→j→k-[c-. Ui[-] Uj[y---]] := c Uk[y];
mi→j→k-[c-. Ui[xx---, x[n1-]] Uj[x[n2-], yy---]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi→j→k;
mi→j→k-[c-. Ui[xx---, x-] Uj[y-, yy---]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi→j→i) Uj[yy] /. UCF /. mi→j→k) /. UCF
];
ms-[ε-] := ms [UCF[ε]];
```

(Alt) In[]:=
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```
Supp[ε-] := Union@Cases[{ε}, Ui[---] ⇒ i, ∞];
```

(Alt) In[]:=
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```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ[i][z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /. ek· /; k > $k &⇒ 0];
  Do[z = mσ[i], i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];
```

(Alt) In[]:=
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```
Δi→j, k[ε_] := UCF@Module[{j1, k1}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[f_, fs___] ⇒ mk, k1→k@mj, j1→j[Δj, k[f] Δi→j1, k1[Ui[fs]]]
}];
Δi→j, k, l[ε_] := ε // Δi→j, k // Δk→l
```

(Alt) In[]:=
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```
Si[ε_] := UCF@Module[{i1}, ε /. {
  Ui[f_, fs___] ⇒ mi1, i→i[Si[f] Si1[Ui1[fs]]]
}];
S̄i[ε_] := UCF@Module[{i1}, ε /. {
  Ui[f_, fs___] ⇒ mi1, i→i[S̄i[f] S̄i1[Ui1[fs]]]
}];
```

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The $\mathbb{A}$ -algebra

(Alt) In[]:=
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```
RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai[m_], Aj[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = L@xij; {m} = L@am;
  Uk[xij, am] + γ (δm, i - δm+1, i - δm, j + δm+1, j) Uk[xij] / 2
];
RRk[Am[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = L@xij; {m} = L@Am; q-n (δm, i - δm+1, i - δm, j + δm+1, j) Uk[xij, Am[n]]
]
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

```
Time: 0.05 seconds Out[ ]=
```

```
γ U0[x12] + U0[x12, a1]
```

```
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```

We take the reduction rules RR for xij's from Yamane page 509, rewritten in Monoblog/260928:

$$\begin{aligned}
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= (\overrightarrow{\cdot} \overrightarrow{\cdot}) \\
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= q^2 (\overrightarrow{\cdot} \overrightarrow{\cdot}) - q (\overrightarrow{\cdot} \overrightarrow{\cdot}) \\
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= (\overrightarrow{\cdot} \overrightarrow{\cdot}) + (q^{-2} - q^2) (\overrightarrow{\cdot} \overrightarrow{\cdot}) \\
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= q^{-2} (\overrightarrow{\cdot} \overrightarrow{\cdot}) \\
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= (\overrightarrow{\cdot} \overrightarrow{\cdot}) \\
 (\overrightarrow{\cdot} \overrightarrow{\cdot}) &= q^{-2} (\overrightarrow{\cdot} \overrightarrow{\cdot})
 \end{aligned}$$

```
(Alt) In[*]:=
```

```
pdf
```

```
RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]
```

```
In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2
```

```
Time: 0.08 seconds Out[ ]=
```

```
(-1 - γ ∈ ħ) U2[x24] U3[x12] + (1 + 2 γ ∈ ħ) U2[x23, x34] U3[x12]
```

```
In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
```

```
Time: 0.05 seconds Out[ ]=
```

```
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]
```

```
In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
```

```
Time: 18.11 seconds Out[ ]=
```

```
{ }
```

```

In[ ]:= Block[{ $k = ∞, $n = 3}, Module[{ Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{ }

Module[{ Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
  ]
]
» {x12, x13, x23, a1, a2}

Out[ ]:=
{ }

```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

(Alt) In[]:=
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```

Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1→k[mj, j1→j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]
]

```

```

In[*]:= Block[{ $k = ∞, $n = 5}, Column@Module[{ Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]]

Time: 0.08 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (− $\frac{1}{q^2}$  + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (− $\frac{1}{q^2}$  + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (− $\frac{1}{q^2}$  + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (− $\frac{1}{q^2}$  + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (− $\frac{1}{q^2}$  + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (− $\frac{1}{q^2}$  + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (− $\frac{1}{q^2}$  + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (− $\frac{1}{q^2}$  + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (− $\frac{1}{q^2}$  + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (− $\frac{1}{q^2}$  + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{ Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[0], A1[-2], A4[0], A1[0], A2[0], A4[-2], A3[-2], A3[1], A3[2], A2[-1]}

Out[*]=
{ }

```

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[-1], A1[2], A3[-1], A4[2], A2[0], A4[2], A3[2], A4[1], A2[-2], A4[-1]}
Out[*]=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[*]:= $k = 1
Time: 0.02 seconds Out[*]=
1

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[*]=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

pdf

The Antipode

(Alt) In[]:=
pdf

```

Do[As[i, 1, $N - 1],
  Sj_ [a[i]] = -Uj [a[i]];
  Sj_ [A[i][n_]] = Uj [A[i][-n]];
  Sj_ [x[i, i + 1]] = -q-2 Uj [x[i, i + 1]]; (*?*)
  Sj_ [a[i]] = -Uj [a[i]];
  Sj_ [A[i][n_]] = Uj [A[i][-n]];
  Sj_ [x[i, i + 1]] = -q2 Uj [x[i, i + 1]]; (*?*)
];
Do[As[i, 1, $N - 2], As[n, i + 2, $N],
  Sj_ [x[i, n]] = mj1,j→j [
    q Sj [x[i, n - 1]] Sj1 [x[n - 1, n]] - q-1 Sj [x[n - 1, n]] Sj1 [x[i, n - 1]];
  Sj_ [x[i, n]] = mj1,j→j [
    q Sj [x[i, n - 1]] Sj1 [x[n - 1, n]] - q-1 Sj [x[n - 1, n]] Sj1 [x[i, n - 1]]
  ]
];

```

```
In[*]:= Block[{ $k = ∞, $n = 5 }, Column@Module[{ Bas = $PBWGens, P },
  Table[As[P, Ui /@ Bas], P → {Si[P],  $\bar{S}_i$ [P]}]
]]
```

Time: 0.08 seconds Out[]:=

$$\begin{aligned}
U_i[x_{12}] &\rightarrow \left\{ -\frac{U_i[x_{12}]}{q^2}, -q^2 U_i[x_{12}] \right\} \\
U_i[x_{13}] &\rightarrow \left\{ -\frac{U_i[x_{13}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{12}, x_{23}], -q^6 U_i[x_{13}] + (-q^3 + q^7) U_i[x_{12}, x_{23}] \right\} \\
U_i[x_{14}] &\rightarrow \\
&\left\{ -\frac{U_i[x_{14}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{12}, x_{24}] + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{13}, x_{34}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x_{12}, x_{23}, x_{34}], \right. \\
&\quad -q^{10} U_i[x_{14}] + (-q^7 + q^{11}) U_i[x_{12}, x_{24}] + \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{13}, x_{34}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{12}, x_{23}, x_{34}] \right\} \\
U_i[x_{15}] &\rightarrow \left\{ -\frac{U_i[x_{15}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{12}, x_{25}] + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{13}, x_{35}] + \right. \\
&\quad \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{14}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x_{12}, x_{23}, x_{35}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x_{12}, x_{24}, x_{45}] + \\
&\quad \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x_{13}, x_{34}, x_{45}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x_{12}, x_{23}, x_{34}, x_{45}], \\
&\quad -q^{14} U_i[x_{15}] + (-q^{11} + q^{15}) U_i[x_{12}, x_{25}] + (-q^{11} + q^{15}) U_i[x_{13}, x_{35}] + (-q^{11} + q^{15}) U_i[x_{14}, x_{45}] + \\
&\quad (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{23}, x_{35}] + (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{24}, x_{45}] + \\
&\quad \left. (-q^8 + 2q^{12} - q^{16}) U_i[x_{13}, x_{34}, x_{45}] + (-q^5 + 3q^9 - 3q^{13} + q^{17}) U_i[x_{12}, x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{23}] &\rightarrow \left\{ -\frac{U_i[x_{23}]}{q^2}, -q^2 U_i[x_{23}] \right\} \\
U_i[x_{24}] &\rightarrow \left\{ -\frac{U_i[x_{24}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{23}, x_{34}], -q^6 U_i[x_{24}] + (-q^3 + q^7) U_i[x_{23}, x_{34}] \right\} \\
U_i[x_{25}] &\rightarrow \\
&\left\{ -\frac{U_i[x_{25}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{23}, x_{35}] + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{24}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x_{23}, x_{34}, x_{45}], \right. \\
&\quad -q^{10} U_i[x_{25}] + (-q^7 + q^{11}) U_i[x_{23}, x_{35}] + \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{24}, x_{45}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{34}] &\rightarrow \left\{ -\frac{U_i[x_{34}]}{q^2}, -q^2 U_i[x_{34}] \right\} \\
U_i[x_{35}] &\rightarrow \left\{ -\frac{U_i[x_{35}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x_{34}, x_{45}], -q^6 U_i[x_{35}] + (-q^3 + q^7) U_i[x_{34}, x_{45}] \right\} \\
U_i[x_{45}] &\rightarrow \left\{ -\frac{U_i[x_{45}]}{q^2}, -q^2 U_i[x_{45}] \right\} \\
U_i[a_1] &\rightarrow \{-U_i[a_1], -U_i[a_1]\} \\
U_i[a_2] &\rightarrow \{-U_i[a_2], -U_i[a_2]\} \\
U_i[a_3] &\rightarrow \{-U_i[a_3], -U_i[a_3]\} \\
U_i[a_4] &\rightarrow \{-U_i[a_4], -U_i[a_4]\}
\end{aligned}$$

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[-1], A2[-1], A4[1], A4[1], A3[2], A1[1], A2[1], A3[-1], A3[2], A4[-2]}

Out[*]=
{x12 -> 0, x13 -> 0, x14 -> 0, x15 -> 0, x23 -> 0, x24 -> 0, x25 -> 0, x34 -> 0, x35 -> 0, x45 -> 0,
  a1 -> 0, a2 -> 0, a3 -> 0, a4 -> 0, A4[-1] -> U1[], A2[-1] -> U1[], A4[1] -> U1[], A4[1] -> U1[],
  A3[2] -> U1[], A1[1] -> U1[], A2[1] -> U1[], A3[-1] -> U1[], A3[2] -> U1[], A4[-2] -> U1[]}

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> UCF[(P // S1 // S1-) - P]
  ]
]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[2], A2[-2], A1[-1], A1[1], A3[1], A3[-1], A4[1], A1[0], A3[1], A1[-1]}

Out[*]=
{}

```

pdf

The Generating Functions Engine

pdf

Canonical Forms:

(Alt) In[*]:=

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```

LogReduce[ $\mathcal{E}$ ] :=
   $\mathcal{E} /. c\_ * \text{Log}[a\_]$  -> Log@Factor[ac] /. Log[a_] + Log[b_] -> Log@Factor[a b];
CF[ $\mathcal{E} : \mathbb{E}$ ] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E} : \mathbb{E} \_ \_ \_$ ] := CF /@  $\mathcal{E}$ ;

```

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E operations:

(Alt) In[]:=
pdf

```

 $\mathcal{E}_E[\$] := \text{Length}[\mathcal{E}] - 1; \mathbb{E}_E[\mathcal{E}\_\_\_\_\_\_] [\$] := \mathbb{E}[\mathcal{E}\_\_\_\_\_\_] [\$];$ 
 $\mathcal{E}_E[k\_Integer] := \mathcal{E}[[k+1]]; \mathbb{E}_E[\mathcal{E}\_\_\_\_\_\_] [k\_Integer] := \{\mathcal{E}\_\_\_\_\_\_][[k+1]];$ 
 $\mathbb{E} /: \mathcal{E}1\_E \equiv \mathcal{E}2\_E := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \ \&, \mathcal{E}1, \mathcal{E}2, \text{And}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E}1s] \equiv \mathbb{E}[\mathcal{E}2s]);$ 
 $\mathbb{E} /: \mathcal{E}1\_E * \mathcal{E}2\_E := \mathbb{E} @ \text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)} @ @ (\mathbb{E}[\mathcal{E}1s] \mathbb{E}[\mathcal{E}2s]);$ 

```

(Alt) In[]:=
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```

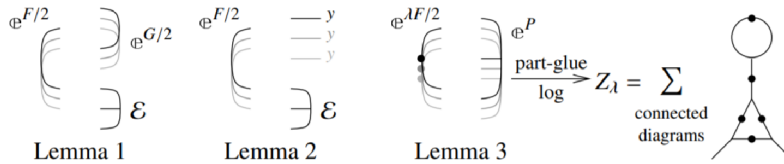
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] // \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] := \text{Module}[\{\text{is} = r1 \cap d2, \text{lvs}, \text{llvs}\},$ 
 $\text{lvs} = \text{Cases}[\{\mathcal{E}1s\}, (\text{Alternatives} @ \text{LatinVars})_{i\_} /; \text{MemberQ}[\text{is}, i], \infty] \cup$ 
 $\text{Cases}[\{\mathcal{E}2s\}, (\text{Alternatives} @ \text{GreekVars})_{i\_} /; \text{MemberQ}[\text{is}, i], \infty]^*;$ 
 $\text{llvs} = \text{lvs} /. u_{i\_} \Rightarrow u_{\$i};$ 
 $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])} @ @ (\text{Zip}_{\text{llvs} \cup \text{llvs}^*}[\{(F / @ \text{llvs}^*) \cdot (F / @ \text{llvs})\}, \text{Times}[$ 
 $\mathbb{E}[\mathcal{E}1s] /. \text{Thread}[\text{lvs} \rightarrow \text{llvs}],$ 
 $\mathbb{E}[\mathcal{E}2s] /. \text{Thread}[\text{lvs}^* \rightarrow \text{llvs}^*]$ 
 $\text{}]\text{}])$ 
 $\text{}]$ 

```

pdf

Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

pdf



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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

(Alt) In[]:=
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```

 $\text{Zip}_{vs\_}[\{\mathcal{F}\_\_, \mathcal{E}\_\_\_\_\_\_\}] := \text{Module}[\{\text{dvs} = \text{Select}[\text{vs}, (\text{Wt}[\#] == \{0, 0\} \vee \text{Wt}[\#] == \{1, 0\}) \ \&]\},$ 
 $\{\mathcal{F}\_\_, \mathcal{E}\_\_\_\_\_\_\} // \text{Zip1}_{vs}(* // \text{Zip2}_{\text{Complement}[vs, \text{dvs}]} *) // \text{Zip2}_{\text{dvs}} //$ 
 $\text{Zip3}_{\text{Complement}[vs, \text{dvs}]} // \text{Zip3}_{\text{dvs}}$ 
 $\text{}] // \text{Last}$ 

```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \otimes^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

(Alt) In[]:=
pdf

```

Zip1{} = Identity;
Zip1_{vs_} @ {F_, E[Q_, P___]} := Module[{I, F, G, u, v},
  (*Echo[{vs,F,Q,P}];*)
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, D_F[u],F[v] F, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, D_{u,v} Q, 0], {u, vs}, {v, vs}];
  (*Echo@MatrixForm[I-G.F];*)
  {CF[(F/@vs).(F.Inverse[I-G.F]).(F/@vs)/2],
   E[CF[Q - PowerExpand@Log[Det[I - G.F]]/2 - vs.G.vs/2], P]}
]

```

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Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E} \otimes^{\sum_{i \in B} y_i z_i} \right\rangle_B = \otimes^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.(Alt) In[]:=
pdf

```

Zip2{} = Identity;
Zip2_{vs_} @ {F_, E[Q_, P___]} := Module[{F, Y, u, v},
  (*EchoLabel["Zip2"][{vs,F,Q,P}];*)
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[D_F[u],F[v] F], 0], {u, vs}, {v, vs}];
  Y = Table[D_v Q, {v, vs}] /. (*AlsoUpper@*) Table[v -> 0, {v, vs}];
  CF /@ ({F, E[Q - Y.vs + Y.F.Y/2, P]} /. (*AlsoUpper@*) Thread[vs -> vs + F.Y])
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \otimes^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

(Alt) In[]:=
pdf

```
Zip3vs_@{ $\mathcal{F}_$ ,  $\mathcal{E}_\mathbb{E}$ } := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3", vs,  $\mathcal{F}$ ,  $\mathcal{E}$ }] ; *)
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}@\$$ }] ;
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{F[u], F[v]} \mathcal{F}$ , 0] ;
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]] ;
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]] ;
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}@\$$ , ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] === 0, 0,  $\frac{F[u, v]}{2 (m + 1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}]]],
      {u, vs}, {v, vs}] ;
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ] ;
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
     $\mathbb{E}@@$ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}@\$$ }
  ] /. (*AlsoUpper@*) Table[v → 0, {v, vs}]
]
```

pdf

“Define” Code

(Alt) In[]:=
pdf

```
SetAttributes[Define, HoldAll] ;
Define[def_, defs__] := (Define[def] ; Define[defs] ;) ;
Define[op_is__ =  $\mathcal{E}_$ ] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E_, PPboot@Block[{i, j, k}, op_isp, $E =  $\mathcal{E}$  ; op_nisp, $E]]] ;
    SD[op_isp, op_{is, $E}] ; SD[op_sis__, op_{sis}] ;
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]]
```

pdf

The Generating Functions for the $\mathbb{A}$ operations

(Alt) In[]:=

```
$k = 1;
```

pdf

Ad, TT, Seepage

(Alt) In[]:=

pdf

```
PotQ[0] = True;
PotQ[c_. U_i_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

(Alt) In[]:=

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```
Ad[x_][y_] /; PotQ[x] ∧ (Head[y] != Plus) := If[y === 0, 0, PPAdPoty[eUB[y,x]/y y]];
Ad[x_][y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

(Alt) In[]:=

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```
Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

In[]:= \$k = 2

Time: 0.00 seconds Out[]:=

2

In[]:= Ad[ε ħ U_j[x₁₂] U_k[]][U_j[x₂₃] U_k[a₄]]

Time: 0.06 seconds Out[]:=

$$\left(-\epsilon \hbar - \gamma \epsilon^2 \hbar^2\right) U_j[x_{13}] U_k[a_4] + U_j[x_{23}] U_k[a_4] + 2 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, x_{23}] U_k[a_4]$$
In[]:= PotQ[7 ħ U_j[x₁₂] U_k[a₃]]

Time: 0.02 seconds Out[]:=

PotQ[7 ħ U_j[x₁₂] U_k[a₃]]In[]:= Ad[7 ħ U_j[x₁₂] U_k[a₃]][U_j[x₂₃] U_k[a₄]]

Time: 0.09 seconds Out[]:=

$$U_j[x_{23}] U_k[a_4] + \left(-7 \hbar - 7 \gamma \epsilon \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{13}] U_k[a_3, a_4] +$$

$$\left(14 \gamma \epsilon \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{12}, x_{23}] U_k[a_3, a_4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{13}] U_k[a_3, a_3, a_4] +$$

$$98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{12}, x_{23}] U_k[a_3, a_3, a_4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x_{12}, x_{12}, x_{13}] U_k[a_3, a_3, a_3, a_4]$$

$In[*]:= \text{Ad}[\pi \mathbf{i} U_j[\mathbf{a1}]] [U_j[\mathbf{x23}] U_k[\mathbf{a4}]]$

Time: 0.05 seconds Out:=

$$e^{\frac{i\pi\gamma}{2}} U_j[\mathbf{x23}] U_k[\mathbf{a4}]$$

(Alt) In[*]:=

pdf

```
TT_λ[] = 0; (* Translated Tangent *)
TT_λ[XS___, X_] := PP_TT@UCF[Ad[X][TT_λ[XS]] + ∂_λ X];
TT_λ[XS_List] := TT_λ@@XS
```

$In[*]:= \text{TT}_\lambda[\lambda \xi_{12} U_0[\mathbf{x12}] U_1[], \lambda \xi_{23} U_0[\mathbf{x23}] U_1[]]$

Time: 0.08 seconds Out:=

$$\xi_{12} U_0[\mathbf{x12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[\mathbf{x13}] U_1[] + \xi_{23} U_0[\mathbf{x23}] U_1[]$$

(Alt) In[*]:=

pdf

```
O_i[poly_] := PP_0@Total[
  CoefficientRules[poly, #i & /@ $PBWGens] /.
    (p_ -> c_) -> c U_i@@Flatten@MapThread[ConstantArray, {$PBWGens, p}]
];
O_i,is_[poly_] := O_i[O_is[poly]];
P_i[ε_] := ε /. U_i[XS___] -> Times@@(#i & /@ {XS});
P_i,is_[poly_] := P_i[P_is[poly]];
```

$In[*]:= O_i[\mathbf{x12}_i^2 \mathbf{x13}_i + 7 \mathbf{x23}_i^{17}]$

Time: 0.06 seconds Out:=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}]$$

$In[*]:= O_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.03 seconds Out:=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] U_j[\mathbf{x24}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}] U_j[\mathbf{x25}, \mathbf{x25}, \mathbf{x25}]$$

$In[*]:= P_{i,j}@O_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.06 seconds Out:=

$$\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3$$

(Alt) In[*]:=

pdf

```
Seepage_i[{ }, c_] = c;
Seepage_i[ξS_List, P_] := PP_Seepage@Module[{lx},
  lx = $PBWGens[[Length@ξS]];
  CF@Collect[P, lx_i, P_i[Ad[Last[ξS] U_i[lx]] [O_i[Seepage_i[Most@ξS, #]]]]
    &]
];
Seepage_i,is_[ξS_List, ξSS_List, P_] := Seepage_i[ξS, Seepage_is[ξSS, P]]
```

In[*]:= **\$k = 1**

Time: 0.00 seconds Out[1]=

1

In[*]:= **\$PBWGens***

Time: 0.01 seconds Out[2]=

{ ξ_{12} , ξ_{13} , ξ_{14} , ξ_{15} , ξ_{23} , ξ_{24} , ξ_{25} , ξ_{34} , ξ_{35} , ξ_{45} , α_1 , α_2 , α_3 , α_4 }

In[*]:= **Seepage_i[\$PBWGens*, x35_i]**

Time: 0.16 seconds Out[3]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35_i} + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \gamma \in \xi_{45} \hbar x_{35_i} x_{45_i}$$

In[*]:= **Seepage_{i,j}[#_i & /@ \$PBWGens*, #_j & /@ \$PBWGens*, x35_i x45_j]**

Time: 0.25 seconds Out[4]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 + \alpha_3_j - \alpha_4_i - 2\alpha_4_j)} x_{35_i} x_{45_j} + 2 e^{-\frac{1}{2}(\alpha_2 - \alpha_3 + 3\alpha_4_i + 2\alpha_4_j)} \gamma \gamma \in \hbar x_{35_i} x_{45_i} x_{45_j} \xi_{45_i}$$

(Alt) In[*]:=
pdf

```
Invertis__[ $\mathcal{E}$ _] := PPInvert@Module[{A, B, U1, j}, (*  $\mathcal{E} = A U[] + \epsilon B *$  *)
  U1 = Times@@(U#[] & /@ {is});
  A = Coefficient[ $\mathcal{E}$ , U1] /.  $\epsilon \rightarrow 0$ ;
  B = UCF[( $\mathcal{E} - A U1$ )];
  UCF[
    A-1(U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]
```

In[*]:= **\$k = 3;**

Invert_{i,j}[7 U_i[] U_j[] + ϵ U_i[x12] U_j[x23]]

Time: 0.09 seconds Out[5]=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \epsilon U_i[x_{12}] U_j[x_{23}] + \frac{1}{343} \epsilon^2 U_i[x_{12}, x_{12}] U_j[x_{23}, x_{23}] - \frac{\epsilon^3 U_i[x_{12}, x_{12}, x_{12}] U_j[x_{23}, x_{23}, x_{23}]}{2401}$$

{ $k = 1$, $n = 3$ }

Time: 0.01 seconds Out[6]=

{2, 3}

(Alt) In[]:=
pdf

```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
  PPTriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      AppendTo[ss, s /. F_Function → ReplacePart[F, 2 → CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 → CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]
```

pdf

$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i_1} x \rangle \mathbb{C}\langle \lambda_{i_2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i x_i \rangle \mathcal{O}(\text{Seepage}(\{f_i\}, A)) &= \mathcal{O}(e^{f_i x_i} A) \\
 \text{TT}(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f = f_i x_i, \quad g = \sum_{|w| \geq 1} g_w w & e^f = F, \quad e^g = G, \quad R_\lambda = \mathcal{O}(FG), \quad \partial_\lambda F = F \partial_\lambda f \\
 \text{TT}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathcal{O}(FG)' \mathcal{O}(\partial_\lambda FG) = S(G)' \mathcal{O}(F)' \mathcal{O}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
 \end{aligned}$$

(Alt) In[]:=
pdf

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uk[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk/@$PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[0k[Seepagek[ff, UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⌗ S[(∂λ(ff.(#k & /@$PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], (α[m]i + α[m]j) a[m]k, 0],
    Total[Cases[fs, fp,u]] /. fp,u => u (fp,u /. ss)[λ] /. λ | h | γ → 1]
  ]
]

```

(Alt) In[]:=
pdf

```

Define[aΔti→j,k = PP"aΔt"[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]);
  TTL = TTλ@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
  ffk = ffj /. j -> k;
  fs = Flatten[Table[
    fp,ℙj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙj,k[u][λ] ℙj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[0j,k[Seepagej,k[ffj, ffk, UCF@B]]];
  SG = PP"SG"@S[G];
  BB = PP"BB"[∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λG];
  SBB = PP"SBB"@S[BB];
  TTR = PP"TTR"@UCF[Invertj,k[SG] ⊗ SBB];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ -> 1]
  ]
]]

```

`In[*]:= { $k = 1, $n = 2 }; amti,j→k`

$$(\text{amt}_{1,2 \rightarrow 2} // \text{amt}_{2,3 \rightarrow 1}) \equiv (\text{amt}_{2,3 \rightarrow 2} // \text{amt}_{1,2 \rightarrow 1})$$

`Time: 0.11 seconds Out[*]=`

$$\mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + x_{12k} \left(\xi_{12i} + e^{\alpha_{1i}} \xi_{12j} \right), 0 \right]$$

`Out[*]=`

True

`In[*]:= { $k = 1, $n = 3 }; amti,j→k`

$$(\text{amt}_{1,2 \rightarrow 4} // \text{amt}_{4,3 \rightarrow 5}) \equiv (\text{amt}_{2,3 \rightarrow 4} // \text{amt}_{1,4 \rightarrow 5})$$

`Time: 0.08 seconds Out[*]=`

$$\begin{aligned} \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ \left. - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \right. \\ \left. 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right] \end{aligned}$$

`Out[*]=`

True

In[*]:= { \$k = 2, \$n = 3 }; amt_{i,j→k}

$$(\text{amt}_{1,2 \rightarrow 4} // \text{amt}_{4,3 \rightarrow 5}) \equiv (\text{amt}_{2,3 \rightarrow 4} // \text{amt}_{1,4 \rightarrow 5})$$

Time: 44.66 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ & \quad x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i}, \\ & \quad - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + \\ & \quad x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \quad x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \quad x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \quad x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \quad x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \quad x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \quad \left. x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; aΔt_{i→j,k}

$$(a\Delta t_{1 \rightarrow 1,3} // a\Delta t_{1 \rightarrow 1,2}) \equiv (a\Delta t_{1 \rightarrow 1,2} // a\Delta t_{2 \rightarrow 2,3})$$

Time: 0.23 seconds Out[]=

$$\mathbb{E}_{\{i\} \rightarrow \{j,k\}} [(a_{1j} + a_{1k}) \alpha_{1i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i}, -2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i}]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; aΔt_{i→j,k}

$$(a\Delta t_{1 \rightarrow 1,3} // a\Delta t_{1 \rightarrow 1,2}) \equiv (a\Delta t_{1 \rightarrow 1,2} // a\Delta t_{2 \rightarrow 2,3})$$

Time: 0.08 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{j,k\}} [\\ & \quad (a_{1j} + a_{1k}) \alpha_{1i} + (a_{2j} + a_{2k}) \alpha_{2i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i} + x_{13j} \xi_{13i} + x_{13k} \xi_{13i} + x_{23j} \xi_{23i} + x_{23k} \xi_{23i}, \\ & \quad - 2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i} - 2 a_{1k} x_{13j} \xi_{13i} - 2 a_{2k} x_{13j} \xi_{13i} + 2 a_{1j} x_{13k} \xi_{13i} + \\ & \quad 2 a_{2j} x_{13k} \xi_{13i} + x_{12k} x_{13j} \xi_{12i} \xi_{13i} - x_{12j} x_{13k} \xi_{12i} \xi_{13i} - 2 a_{2k} x_{23j} \xi_{23i} + 2 a_{2j} x_{23k} \xi_{23i} - \\ & \quad x_{12k} x_{23j} \xi_{12i} \xi_{23i} + x_{13k} x_{23j} \xi_{13i} \xi_{23i} - x_{13j} x_{23k} \xi_{13i} \xi_{23i} + x_{12j} x_{23k} (4 \xi_{13i} + \xi_{12i} \xi_{23i})] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; (amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})

Time: 0.11 seconds Out[=

True

In[*]:= Block[{ \$k = 1, \$n = 3 }, Echo@ { amt_{1,2→1}, aΔt_{1→1,2} };

(amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})]

$$\begin{aligned} & \gg \left\{ \mathbb{E}_{\{1,2\} \rightarrow \{1\}} \left[a_1 (\alpha_1 + \alpha_2) + a_2 (\alpha_2 + \alpha_2) + x_{12} \left(\xi_{12} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12} \right) + \right. \right. \\ & \quad x_{13} \left(\xi_{13} + e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12} \xi_{23} \right) + x_{23} \left(\xi_{23} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2)} \xi_{23} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{13} \xi_{12} \xi_{13} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \xi_{12} \xi_{23} + \\ & \quad \left. 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} x_{13} x_{23} \xi_{13} \xi_{23} \right], \mathbb{E}_{\{1\} \rightarrow \{1,2\}} [\\ & \quad (a_1 + a_2) \alpha_1 + (a_2 + a_2) \alpha_2 + x_{12} \xi_{12} + x_{12} \xi_{12} + x_{13} \xi_{13} + x_{13} \xi_{13} + x_{23} \xi_{23} + x_{23} \xi_{23}, \\ & \quad - 2 a_1 x_{12} \xi_{12} + 2 a_1 x_{12} \xi_{12} - 2 a_2 x_{13} \xi_{13} - 2 a_2 x_{13} \xi_{13} + 2 a_1 x_{13} \xi_{13} + \\ & \quad 2 a_2 x_{13} \xi_{13} + x_{12} x_{13} \xi_{12} \xi_{13} - x_{12} x_{13} \xi_{12} \xi_{13} - 2 a_2 x_{23} \xi_{23} + 2 a_2 x_{23} \xi_{23} - \\ & \quad \left. x_{12} x_{23} \xi_{12} \xi_{23} + x_{13} x_{23} \xi_{13} \xi_{23} - x_{13} x_{23} \xi_{13} \xi_{23} + x_{12} x_{23} (4 \xi_{13} + \xi_{12} \xi_{23}) \right] \} \end{aligned}$$

Out[*]=

True

(Alt) In[]:=
pdf

```

Define[aSti = Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]]; ε);
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Ui[x___] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp, ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp, ui[λ] /; τ[u] == "a" ⇒ If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp, Pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pi[u][λ] Pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], -α[m]i (a[m]i)], 0],
    Total[Cases[fs, fp, _]] /. fp, u ⇒ u (fp, u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

(Alt) In[]:=
pdf

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c-. Ui[x---] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp,ℙi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙi[u][λ] ℙi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp-,u- => u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1]
  ]
]]

```

In[*]:= **\$k = 1; \$n = 3; aSt_i**
aΔt_{1→2,3} // aSt₂ // amt_{2,3→1}

Time: 32.98 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{i\}} \left[-a_{1i} \alpha_{1i} - a_{2i} \alpha_{2i} - e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} - e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + \right. \\ & \quad x_{13i} \left(-e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} - e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right), 2 e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} + \\ & \quad 2 e^{-\frac{3\alpha_{1i}}{2}} x_{12i} x_{13i} \xi_{12i} \xi_{13i} + 2 e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + 2 e^{-\frac{3\alpha_{2i}}{2}} x_{13i} x_{23i} \xi_{13i} \xi_{23i} + \\ & \quad 2 e^{-\alpha_{1i} - \alpha_{2i}} x_{13i}^2 \xi_{12i} \xi_{13i} \xi_{23i} + x_{12i} x_{23i} \left(4 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) + \\ & \quad \left. x_{13i} \left(2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 3 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) \right] \end{aligned}$$

Out[]=

$$\mathbb{E}_{\{1\} \rightarrow \{1\}} [0, 0]$$

(Alt) In[*]:= pdf

```
(*Splitter B⊗A *)
SYi→j, k, l, m :=  $\mathbb{E}_{\{i\} \rightarrow \{j, k, l, m\}}$  [
  Sum[ $\eta[r, s]_i y[r, s]_j + \xi[r, s]_i x[r, s]_m$ , {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[ $\alpha[r]_i a[r]_l + \beta[r]_i b[r]_k$ , {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

pdf

$$P: \mathbb{A} \otimes \mathbb{B} \rightarrow \mathbb{Q} \text{ and } R \in \mathbb{B} \otimes \mathbb{A}:$$

(Alt) In[*]:= pdf

```
Pi, j :=  $\mathbb{E}_{\{i, j\} \rightarrow \{\}}$  @@ PadRight [
  {Sum[ $\eta[r, s]_j \xi[r, s]_i$ , {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[ $\beta[r]_j \alpha[r]_i$ , {r, 1, $n - 1}]},
  $k + 1]
Ri, j :=  $\mathbb{E}_{\{\} \rightarrow \{i, j\}}$  @@ PadRight [
  {Sum[ $y[r, s]_i x[r, s]_j$ , {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[ $b[r]_i a[r]_j$ , {r, 1, $n - 1}]},
  $k + 1]
```

In[*]:= {R_{1,2} // P_{2,3}, R_{1,2} // P_{5,1}}

Time: 0.12 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + b_{31} \beta_{33} + b_{41} \beta_{43} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{141} \eta_{143} + \\ & \quad y_{151} \eta_{153} + y_{231} \eta_{233} + y_{241} \eta_{243} + y_{251} \eta_{253} + y_{341} \eta_{343} + y_{351} \eta_{353} + y_{451} \eta_{453}, 0], \\ & \mathbb{E}_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + a_{32} \alpha_{35} + a_{42} \alpha_{45} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{142} \xi_{145} + \\ & \quad x_{152} \xi_{155} + x_{232} \xi_{235} + x_{242} \xi_{245} + x_{252} \xi_{255} + x_{342} \xi_{345} + x_{352} \xi_{355} + x_{452} \xi_{455}, 0] \end{aligned}$$

(Alt) In[*]:= pdf

```
Define[bmtj, k→i = Module[{i0, i1, j1, k1}, (Ri0, i1 // aΔti1→j1, k1 // Pj1, j // Pk1, k) /. i0 → i],
  bΔtk→i, j = Module[{i1, j1, k1, k2}, ((Ri, i1 Rj, j1 // amti1, j1→k1 // Pk1, k2) /. k2 → k),
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri, i1 // aSti1 // Pi1, i2) /. i2 → i]]
Define[R̄i, j = Ri, j // bSti]
```

In[*]:= **Block**[{**\$k** = 1, **\$n** = 2}, **bmt**_{1,2→3}]

Time: 0.22 seconds Out[]=

$$\mathbb{E}_{\{1,2\} \rightarrow \{3\}} [b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{121} + y_{123} \eta_{122}, -2 y_{123} \beta_{12} \eta_{121} + 2 y_{123} \beta_{11} \eta_{122}]$$

Testing **A** and **B** dually paired Hopf algebras

In[*]:= **\$k** = 1; **\$n** = 3;

co-associativity

In[*]:= {(**aΔt**_{1→1,2} // **aΔt**_{2→2,3}) ≡ (**aΔt**_{1→2,3} // **aΔt**_{2→1,2}), (**bΔt**_{1→1,2} // **bΔt**_{2→2,3}) ≡ (**bΔt**_{1→2,3} // **bΔt**_{2→1,2}) }

Time: 0.22 seconds Out[]=

{True, True}

algebra morphism

In[*]:= {(**aΔt**_{i→1,2} **aΔt**_{j→3,4} // **amt**_{1,3→i} // **amt**_{2,4→j}) ≡ (**amt**_{i,j→k} // **aΔt**_{k→i,j}),
(**bΔt**_{i→1,2} **bΔt**_{j→3,4} // **bmt**_{1,3→i} // **bmt**_{2,4→j}) ≡ (**bmt**_{i,j→k} // **bΔt**_{k→i,j}) }

Time: 0.44 seconds Out[]=

{True, True}

associativity

In[*]:= {(**amt**_{1,2→k} // **amt**_{k,3→k}) ≡ (**amt**_{2,3→k} // **amt**_{1,k→k}), (**bmt**_{1,2→k} // **bmt**_{k,3→k}) ≡ (**bmt**_{2,3→k} // **bmt**_{1,k→k}) }

Time: 0.12 seconds Out[]=

{True, True}

antipode

In[*]:= {**aΔt**_{i→1,2} // **aSt**₁ // **amt**_{1,2→1}, **aΔt**_{i→1,2} // **aSt**₂ // **amt**_{1,2→1},
bΔt_{i→1,2} // **bSt**₁ // **bmt**_{1,2→1}, **bΔt**_{i→1,2} // **bSt**₂ // **bmt**_{1,2→1}}

Time: 0.89 seconds Out[]=

{ $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$ }

inverse antipode

In[*]:= {**aSt**_i // **aSt**_i, **bSt**_i // **bSt**_i}

Time: 3.53 seconds Out[]=

{ $\mathbb{E}_{\{i\} \rightarrow \{i\}} [a_{1i} \alpha_{1i} + a_{2i} \alpha_{2i} + x_{12i} \xi_{12i} + x_{13i} \xi_{13i} + x_{23i} \xi_{23i}, 0]$,
 $\mathbb{E}_{\{i\} \rightarrow \{i\}} [b_{1i} \beta_{1i} + b_{2i} \beta_{2i} + y_{12i} \eta_{12i} + y_{13i} \eta_{13i} + y_{23i} \eta_{23i}, 0]$ }

Pairing

In[*]:= {(**amt**_{1,2→3} // **P**_{3,-3}) ≡ (**bΔt**_{-3→-1,-2} // **P**_{1,-1} // **P**_{2,-2}),
(**bmt**_{1,2→3} // **P**_{-3,3}) ≡ (**aΔt**_{-3→-1,-2} // **P**_{-1,1} // **P**_{-2,2}),
(**aSt**₁ // **P**_{1,-1}) ≡ (**bSt**₋₁ // **P**_{1,-1}), (**aSt**₁ // **P**_{1,-1}) ≡ (**bSt**₋₁ // **P**_{1,-1}) }

Time: 0.23 seconds Out[]=

{True, True, True, True}

Testing Double

(Alt) In[]:=
pdf

```
Define[dmi,j→k = Module[{i1, i2, i3, i4},
  ((sYi→i4,i4,i1,i1 // aΔti1→i1,i2 // aΔti2→i2,i3 // aSti3) (sYj→-i1,-i1,-i4,-i4 // bΔt-i1→-i1,-i2 //
    bΔt-i2→-i2,-i3) // (Pi1,-i3 Pi3,-i1 amti2,-i4→k bmti4,-i2→k))]
Define[dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dΔi→j,k = (bΔti→1,3 aΔti→4,2) // (dm3,4→k dm1,2→j)]
(*Notice we take co-opposite on the A-side! to make the double*)
```

(Alt) In[]:=

```
{$k = 1, $n = 3, $h = 1};
dmi,j→k // l2U // CF
```

Time: 70.25 seconds Out[]:=

$$\begin{aligned}
 & E_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} \alpha_{1i} + a_{1k} \alpha_{1j} + a_{2k} \alpha_{2i} + a_{2k} \alpha_{2j} + b_{1k} \beta_{1i} + b_{1k} \beta_{1j} + b_{2k} \beta_{2i} + b_{2k} \beta_{2j} + y_{12k} \eta_{12i} + \right. \\
 & \frac{y_{12k} \sqrt{\mathcal{A}_{2i}} \eta_{12j}}{\mathcal{A}_{1i}} + y_{13k} \eta_{13i} + \frac{y_{13k} \eta_{13j}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + y_{23k} \eta_{23i} + \frac{y_{23k} \sqrt{\mathcal{A}_{1i}} \eta_{23j}}{\mathcal{A}_{2i}} + x_{12k} \xi_{12i} - \\
 & \frac{\sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{A}_{1i}} + \frac{\sqrt{\mathcal{B}_{2k}} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{B}_{1k} \mathcal{A}_{1i}} - \frac{\sqrt{\mathcal{B}_{2k}} y_{23k} \eta_{13j} \xi_{12i}}{\mathcal{B}_{1k} \sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + \frac{x_{12k} \mathcal{A}_{1i} \xi_{12j}}{\sqrt{\mathcal{A}_{2i}}} + \\
 & x_{13k} \xi_{13i} - \frac{\eta_{13j} \xi_{13i}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + \frac{\eta_{13j} \xi_{13i}}{\sqrt{\mathcal{B}_{1k}} \sqrt{\mathcal{B}_{2k}} \sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + x_{13k} \sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}} \xi_{13j} + \\
 & x_{23k} \xi_{23i} + \frac{y_{12k} \eta_{13j} \xi_{23i}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} - \frac{\sqrt{\mathcal{A}_{1i}} \eta_{23j} \xi_{23i}}{\mathcal{A}_{2i}} + \frac{\sqrt{\mathcal{B}_{1k}} \sqrt{\mathcal{A}_{1i}} \eta_{23j} \xi_{23i}}{\mathcal{B}_{2k} \mathcal{A}_{2i}} - \\
 & \frac{\eta_{13j} \xi_{12i} \xi_{23i}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + \frac{\sqrt{\mathcal{B}_{2k}} \eta_{13j} \xi_{12i} \xi_{23i}}{\mathcal{B}_{1k} \sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} - \frac{x_{13k} \mathcal{A}_{1i} \xi_{12j} \xi_{23i}}{\sqrt{\mathcal{A}_{2i}}} + \frac{x_{23k} \mathcal{A}_{2i} \xi_{23j}}{\sqrt{\mathcal{A}_{1i}}}, \\
 & - 2 y_{12k} \beta_{1j} \eta_{12i} + \frac{2 y_{12k} \sqrt{\mathcal{A}_{2i}} \beta_{1i} \eta_{12j}}{\mathcal{A}_{1i}} - 2 y_{13k} \beta_{1j} \eta_{13i} - 2 y_{13k} \beta_{2j} \eta_{13i} + \\
 & \frac{y_{12k} y_{13k} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \eta_{13i}}{\mathcal{A}_{1i}} + \frac{2 y_{13k} \beta_{1i} \eta_{13j}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + \frac{2 y_{13k} \beta_{2i} \eta_{13j}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} - \frac{y_{12k} y_{13k} \eta_{12i} \eta_{13j}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} - \\
 & 2 y_{23k} \beta_{2j} \eta_{23i} - \frac{y_{12k} y_{23k} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \eta_{23i}}{\mathcal{A}_{1i}} + \frac{y_{13k} y_{23k} \eta_{13j} \eta_{23i}}{\sqrt{\mathcal{A}_{1i}} \sqrt{\mathcal{A}_{2i}}} + \frac{2 y_{23k} \sqrt{\mathcal{A}_{1i}} \beta_{2i} \eta_{23j}}{\mathcal{A}_{2i}} + \\
 & \frac{4 y_{13k} \sqrt{\mathcal{A}_{1i}} \eta_{12i} \eta_{23j}}{\mathcal{A}_{2i}} + \frac{y_{12k} y_{23k} \sqrt{\mathcal{A}_{1i}} \eta_{12i} \eta_{23j}}{\mathcal{A}_{2i}} - \frac{y_{13k} y_{23k} \sqrt{\mathcal{A}_{1i}} \eta_{13i} \eta_{23j}}{\mathcal{A}_{2i}} + 4 x_{12k} \beta_{1j} \xi_{12i} - \\
 & \frac{2 \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{A}_{1i}} - \frac{2 a_{1k} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{A}_{1i}} + \frac{2 \sqrt{\mathcal{B}_{2k}} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{B}_{1k} \mathcal{A}_{1i}} - \frac{2 a_{1k} \sqrt{\mathcal{B}_{2k}} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{B}_{1k} \mathcal{A}_{1i}} + \\
 & \frac{2 x_{12k} y_{12k} \sqrt{\mathcal{A}_{2i}} \eta_{12j} \xi_{12i}}{\mathcal{A}_{1i}} - \frac{2 \sqrt{\mathcal{A}_{2i}} \beta_{1j} \eta_{12j} \xi_{12i}}{\mathcal{A}_{1i}} + \frac{2 \sqrt{\mathcal{B}_{2k}} \sqrt{\mathcal{A}_{2i}} \beta_{1j} \eta_{12j} \xi_{12i}}{\mathcal{B}_{1k} \mathcal{A}_{1i}} -
 \end{aligned}$$

$$\begin{aligned}
& \frac{2 \sqrt{B2_k} y12_k \sqrt{A2_i} \eta12_i \eta12_j \xi12_i}{B1_k A1_i} + \frac{2 \sqrt{B2_k} y12_k A2_i \eta12_j^2 \xi12_i}{B1_k A1_i^2} - \frac{\sqrt{B2_k} y13_k \sqrt{A2_i} \eta12_j \eta13_i \xi12_i}{B1_k A1_i} + \\
& \frac{x12_k y13_k \eta13_j \xi12_i}{\sqrt{A1_i} \sqrt{A2_i}} - \frac{2 \sqrt{B2_k} y23_k \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} + \frac{2 a1_k \sqrt{B2_k} y23_k \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \\
& \frac{2 \sqrt{B2_k} y23_k \beta1_j \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{2 \sqrt{B2_k} y23_k \beta2_i \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{4 \sqrt{B2_k} y13_k \eta12_i \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} + \\
& \frac{\sqrt{B2_k} y12_k y23_k \eta12_i \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{\sqrt{B2_k} y13_k \eta12_j \eta13_j \xi12_i}{B1_k A1_i^{3/2}} - \frac{2 \sqrt{B2_k} y12_k y23_k \eta12_j \eta13_j \xi12_i}{B1_k A1_i^{3/2}} + \\
& \frac{2 \sqrt{B2_k} y13_k y23_k \eta13_i \eta13_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{\sqrt{B2_k} y13_k y23_k \eta13_j^2 \xi12_i}{B1_k A1_i A2_i} + \frac{\sqrt{B2_k} y23_k \sqrt{A2_i} \eta12_j \eta23_i \xi12_i}{B1_k A1_i} - \\
& \frac{\sqrt{B2_k} y23_k^2 \eta13_j \eta23_i \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{x12_k y23_k \sqrt{A1_i} \eta23_j \xi12_i}{A2_i} + \frac{\sqrt{B2_k} y23_k \eta12_j \eta23_j \xi12_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} + \\
& \frac{\sqrt{B2_k} y23_k^2 \eta13_j \eta23_j \xi12_i}{B1_k A2_i^{3/2}} - \frac{2 x12_k \sqrt{A2_i} \eta12_j \xi12_i^2}{A1_i} + \frac{2 \sqrt{B2_k} x12_k \sqrt{A2_i} \eta12_j \xi12_i^2}{B1_k A1_i} - \\
& \frac{2 \sqrt{B2_k} A2_i \eta12_j^2 \xi12_i^2}{B1_k A1_i^2} + \frac{2 B2_k A2_i \eta12_j^2 \xi12_i^2}{B1_k^2 A1_i^2} - \frac{\sqrt{B2_k} x12_k y23_k \eta13_j \xi12_i^2}{B1_k \sqrt{A1_i} \sqrt{A2_i}} + \\
& \frac{2 \sqrt{B2_k} y23_k \eta12_j \eta13_j \xi12_i^2}{B1_k A1_i^{3/2}} - \frac{3 B2_k y23_k \eta12_j \eta13_j \xi12_i^2}{B1_k^2 A1_i^{3/2}} + \frac{B2_k y23_k^2 \eta13_j^2 \xi12_i^2}{B1_k^2 A1_i A2_i} - 2 x12_k \eta12_j \xi12_i \xi12_j - \\
& \frac{2 \sqrt{B2_k} x12_k \eta12_j \xi12_i \xi12_j}{B1_k} + \frac{2 \sqrt{B2_k} x12_k y23_k \sqrt{A1_i} \eta13_j \xi12_i \xi12_j}{B1_k A2_i} + 4 x13_k \beta1_j \xi13_i + \\
& 4 x13_k \beta2_j \xi13_i + \frac{4 \sqrt{B2_k} x23_k \sqrt{A2_i} \eta12_j \xi13_i}{B1_k A1_i} + \frac{x13_k y12_k \sqrt{A2_i} \eta12_j \xi13_i}{A1_i} - \frac{6 \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} - \\
& \frac{2 a1_k \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} - \frac{2 a2_k \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} + \frac{2 \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} - \frac{2 a1_k \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} - \\
& \frac{2 a2_k \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} + \frac{4 \sqrt{B2_k} \eta13_j \xi13_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} + \frac{4 x12_k y12_k \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} + \frac{2 x13_k y13_k \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} - \\
& \frac{4 \sqrt{B2_k} x23_k y23_k \eta13_j \xi13_i}{B1_k \sqrt{A1_i} \sqrt{A2_i}} - \frac{2 \beta1_j \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} + \frac{2 \beta1_j \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} - \frac{2 \beta2_j \eta13_j \xi13_i}{\sqrt{A1_i} \sqrt{A2_i}} + \\
& \frac{2 \beta2_j \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} - \frac{y12_k \eta12_i \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} - \frac{2 y12_k \eta12_j \eta13_j \xi13_i}{A1_i^{3/2}} + \\
& \frac{y12_k \eta12_j \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} A1_i^{3/2}} + \frac{4 \sqrt{B2_k} y12_k \eta12_j \eta13_j \xi13_i}{B1_k A1_i^{3/2}} - \frac{2 y13_k \eta13_i \eta13_j \xi13_i}{\sqrt{B1_k} \sqrt{B2_k} \sqrt{A1_i} \sqrt{A2_i}} +
\end{aligned}$$

$$\begin{aligned}
& \frac{2 y_{13_k} \eta_{13_j^2} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{4 \sqrt{B_{2_k}} y_{12_k} y_{23_k} \eta_{13_j^2} \xi_{13_i}}{B_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{y_{23_k} \eta_{13_j} \eta_{23_i} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{4 x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{13_i}}{\mathcal{A}_{2_i}} + \frac{x_{13_k} y_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{13_i}}{\mathcal{A}_{2_i}} + \frac{4 \eta_{12_j} \eta_{23_j} \xi_{13_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{4 \sqrt{B_{2_k}} \eta_{12_j} \eta_{23_j} \xi_{13_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{y_{23_k} \eta_{13_j} \eta_{23_j} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \mathcal{A}_{2_i}^{3/2}} + \frac{4 \sqrt{B_{2_k}} y_{23_k} \eta_{13_j} \eta_{23_j} \xi_{13_i}}{B_{1_k} \mathcal{A}_{2_i}^{3/2}} - \frac{2 x_{13_k} \sqrt{\mathcal{A}_{2_i}} \eta_{12_j} \xi_{12_i} \xi_{13_i}}{\mathcal{A}_{1_i}} - \\
& \frac{4 x_{12_k} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 x_{12_k} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{4 \sqrt{B_{2_k}} x_{12_k} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{\sqrt{B_{2_k}} x_{13_k} y_{23_k} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{\mathcal{A}_{1_i}^{3/2}} + \frac{2 \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{B_{1_k}^{3/2} \mathcal{A}_{1_i}^{3/2}} - \\
& \frac{2 \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \mathcal{A}_{1_i}^{3/2}} - \frac{6 \sqrt{B_{2_k}} \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{B_{1_k} \mathcal{A}_{1_i}^{3/2}} + \frac{4 B_{2_k} \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{13_i}}{B_{1_k}^2 \mathcal{A}_{1_i}^{3/2}} - \\
& \frac{3 y_{23_k} \eta_{13_j^2} \xi_{12_i} \xi_{13_i}}{B_{1_k}^{3/2} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{4 \sqrt{B_{2_k}} y_{23_k} \eta_{13_j^2} \xi_{12_i} \xi_{13_i}}{B_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{4 B_{2_k} y_{23_k} \eta_{13_j^2} \xi_{12_i} \xi_{13_i}}{B_{1_k}^2 \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \\
& \frac{2 x_{12_k} x_{13_k} \mathcal{A}_{1_i} \xi_{12_j} \xi_{13_i}}{\sqrt{\mathcal{A}_{2_i}}} - \frac{4 \sqrt{B_{2_k}} x_{13_k} \eta_{12_j} \xi_{12_j} \xi_{13_i}}{B_{1_k}} - \frac{x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i}}{\mathcal{A}_{2_i}} - \\
& \frac{x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \mathcal{A}_{2_i}} + \frac{4 \sqrt{B_{2_k}} x_{13_k} y_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i}}{B_{1_k} \mathcal{A}_{2_i}} - \frac{2 x_{13_k} \eta_{13_j} \xi_{13_i}^2}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 x_{13_k} \eta_{13_j} \xi_{13_i}^2}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 \eta_{13_j^2} \xi_{13_i}^2}{B_{1_k} B_{2_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{2 \eta_{13_j^2} \xi_{13_i}^2}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{x_{13_k} \mathcal{A}_{2_i} \eta_{12_j} \xi_{12_i} \xi_{13_j}}{\sqrt{\mathcal{A}_{1_i}}} - \\
& \frac{\sqrt{B_{2_k}} x_{13_k} \mathcal{A}_{2_i} \eta_{12_j} \xi_{12_i} \xi_{13_j}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}}} + \frac{\sqrt{B_{2_k}} x_{13_k} y_{23_k} \eta_{13_j} \xi_{12_i} \xi_{13_j}}{B_{1_k}} - 2 x_{13_k} \eta_{13_j} \xi_{13_i} \xi_{13_j} - \\
& \frac{2 x_{13_k} \eta_{13_j} \xi_{13_i} \xi_{13_j}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}}} + 4 x_{23_k} \beta_{2_j} \xi_{23_i} - \frac{x_{23_k} y_{12_k} \sqrt{\mathcal{A}_{2_i}} \eta_{12_j} \xi_{23_i}}{\mathcal{A}_{1_i}} + \frac{3 y_{12_k} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 a_{2_k} y_{12_k} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{x_{23_k} y_{13_k} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 y_{12_k} \beta_{1_i} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 y_{12_k} \beta_{2_j} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{y_{12_k} y_{13_k} \eta_{13_i} \eta_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{y_{12_k} y_{23_k} \eta_{13_j} \eta_{23_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}}{\mathcal{A}_{2_i}} - \frac{2 a_{2_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}}{\mathcal{A}_{2_i}} + \\
& \frac{2 \sqrt{B_{1_k}} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} - \frac{2 a_{2_k} \sqrt{B_{1_k}} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} + \frac{2 x_{23_k} y_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}}{\mathcal{A}_{2_i}} - \\
& \frac{2 \sqrt{\mathcal{A}_{1_i}} \beta_{2_j} \eta_{23_j} \xi_{23_i}}{\mathcal{A}_{2_i}} + \frac{2 \sqrt{B_{1_k}} \sqrt{\mathcal{A}_{1_i}} \beta_{2_j} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} + \frac{\sqrt{B_{1_k}} y_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{12_i} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} -
\end{aligned}$$

$$\begin{aligned}
& \frac{2 y_{12_k} \eta_{12_j} \eta_{23_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{\sqrt{B_{1_k}} y_{12_k} \eta_{12_j} \eta_{23_j} \xi_{23_i}}{B_{2_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{\sqrt{B_{1_k}} y_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_i} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} + \\
& \frac{2 y_{13_k} \eta_{13_j} \eta_{23_j} \xi_{23_i}}{\mathcal{A}_{2_i}^{3/2}} + \frac{\sqrt{B_{1_k}} y_{13_k} \eta_{13_j} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}^{3/2}} - \frac{2 \sqrt{B_{1_k}} y_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_i} \eta_{23_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} + \\
& \frac{2 \sqrt{B_{1_k}} y_{23_k} \mathcal{A}_{1_i} \eta_{23_j}^2 \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}^2} + \frac{2 x_{23_k} \sqrt{\mathcal{A}_{2_i}} \eta_{12_j} \xi_{12_i} \xi_{23_i}}{\mathcal{A}_{1_i}} - \frac{5 \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{2 a_{1_k} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 a_{2_k} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{5 \sqrt{B_{2_k}} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 a_{1_k} \sqrt{B_{2_k}} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 a_{2_k} \sqrt{B_{2_k}} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 x_{12_k} y_{12_k} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \sqrt{B_{2_k}} x_{23_k} y_{23_k} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{2 \beta_{1_j} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 \sqrt{B_{2_k}} \beta_{1_j} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \beta_{2_j} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 \sqrt{B_{2_k}} \beta_{2_j} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \sqrt{B_{2_k}} y_{12_k} \eta_{12_i} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{4 \sqrt{B_{2_k}} y_{12_k} \eta_{12_j} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \mathcal{A}_{1_i}^{3/2}} - \\
& \frac{\sqrt{B_{2_k}} y_{13_k} \eta_{13_i} \eta_{13_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{\sqrt{B_{2_k}} y_{13_k} \eta_{13_j}^2 \xi_{12_i} \xi_{23_i}}{B_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{2 \sqrt{B_{2_k}} y_{12_k} y_{23_k} \eta_{13_j}^2 \xi_{12_i} \xi_{23_i}}{B_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \\
& \frac{\sqrt{B_{2_k}} y_{23_k} \eta_{13_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \sqrt{B_{1_k}} x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}} + \frac{2 \eta_{12_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 \sqrt{B_{1_k}} \eta_{12_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{B_{2_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \eta_{12_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{\sqrt{B_{1_k}} \sqrt{B_{2_k}} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 \sqrt{B_{2_k}} \eta_{12_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{\sqrt{B_{2_k}} y_{23_k} \eta_{13_j} \eta_{23_j} \xi_{12_i} \xi_{23_i}}{B_{1_k} \mathcal{A}_{2_i}^{3/2}} - \frac{2 x_{12_k} \eta_{13_j} \xi_{12_i}^2 \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{2 \sqrt{B_{2_k}} x_{12_k} \eta_{13_j} \xi_{12_i}^2 \xi_{23_i}}{B_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{4 \sqrt{B_{2_k}} \eta_{12_j} \eta_{13_j} \xi_{12_i}^2 \xi_{23_i}}{B_{1_k} \mathcal{A}_{1_i}^{3/2}} + \frac{4 B_{2_k} \eta_{12_j} \eta_{13_j} \xi_{12_i}^2 \xi_{23_i}}{B_{1_k}^2 \mathcal{A}_{1_i}^{3/2}} + \frac{2 \sqrt{B_{2_k}} y_{23_k} \eta_{13_j}^2 \xi_{12_i}^2 \xi_{23_i}}{B_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \\
& \frac{3 B_{2_k} y_{23_k} \eta_{13_j}^2 \xi_{12_i}^2 \xi_{23_i}}{B_{1_k}^2 \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{x_{13_k} \mathcal{A}_{1_i} \xi_{12_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{2_i}}} + \frac{2 x_{12_k} x_{23_k} \mathcal{A}_{1_i} \xi_{12_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{4 x_{13_k} \mathcal{A}_{1_i} \beta_{2_j} \xi_{12_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{2_i}}} + x_{13_k} y_{12_k} \eta_{12_j} \xi_{12_j} \xi_{23_i} - \frac{x_{12_k} y_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{23_i}}{\mathcal{A}_{2_i}} - \\
& \frac{x_{13_k} y_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{23_i}}{\mathcal{A}_{2_i}} + \frac{x_{12_k} \mathcal{A}_{1_i}^{3/2} \eta_{23_j} \xi_{12_j} \xi_{23_i}}{\mathcal{A}_{2_i}^{3/2}} + \frac{\sqrt{B_{1_k}} x_{12_k} \mathcal{A}_{1_i}^{3/2} \eta_{23_j} \xi_{12_j} \xi_{23_i}}{B_{2_k} \mathcal{A}_{2_i}^{3/2}} - \\
& \frac{2 x_{13_k} y_{23_k} \mathcal{A}_{1_i}^{3/2} \eta_{23_j} \xi_{12_j} \xi_{23_i}}{\mathcal{A}_{2_i}^{3/2}} + \frac{2 \sqrt{B_{2_k}} x_{13_k} \eta_{12_j} \xi_{12_i} \xi_{12_j} \xi_{23_i}}{B_{1_k}} -
\end{aligned}$$

$$\begin{aligned}
& \frac{x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_i} \xi_{12_j} \xi_{23_i}}{\mathcal{A}_{2_i}} - \frac{3 \sqrt{\mathcal{B}_{2_k}} x_{12_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_i} \xi_{12_j} \xi_{23_i}}{\mathcal{B}_{1_k} \mathcal{A}_{2_i}} - \frac{2 x_{23_k} \eta_{13_j} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{4 \sqrt{\mathcal{B}_{2_k}} x_{23_k} \eta_{13_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \frac{x_{13_k} y_{12_k} \eta_{13_j} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \frac{2 y_{12_k} \eta_{13_j}^2 \xi_{13_i} \xi_{23_i}}{\mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \\
& \frac{y_{12_k} \eta_{13_j}^2 \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{B}_{1_k}} \sqrt{\mathcal{B}_{2_k}} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{4 \sqrt{\mathcal{B}_{2_k}} y_{12_k} \eta_{13_j}^2 \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{2 \sqrt{\mathcal{B}_{1_k}} x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}} + \\
& \frac{2 \eta_{13_j} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\mathcal{A}_{2_i}^{3/2}} + \frac{2 \eta_{13_j} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{2_k}^{3/2} \mathcal{A}_{2_i}^{3/2}} - \frac{2 \sqrt{\mathcal{B}_{1_k}} \eta_{13_j} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}^{3/2}} + \\
& \frac{2 \eta_{13_j} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{B}_{1_k}} \sqrt{\mathcal{B}_{2_k}} \mathcal{A}_{2_i}^{3/2}} - \frac{4 \sqrt{\mathcal{B}_{2_k}} \eta_{13_j} \eta_{23_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k} \mathcal{A}_{2_i}^{3/2}} - \frac{2 x_{13_k} \eta_{13_j} \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 \eta_{13_j}^2 \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{2 \eta_{13_j}^2 \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k}^{3/2} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{2 \eta_{13_j}^2 \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{B}_{1_k}} \sqrt{\mathcal{B}_{2_k}} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \\
& \frac{6 \sqrt{\mathcal{B}_{2_k}} \eta_{13_j}^2 \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{4 \mathcal{B}_{2_k} \eta_{13_j}^2 \xi_{12_i} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k}^2 \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{3 x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i} \xi_{23_i}}{\mathcal{A}_{2_i}} + \\
& \frac{x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i} \xi_{23_i}}{\sqrt{\mathcal{B}_{1_k}} \sqrt{\mathcal{B}_{2_k}} \mathcal{A}_{2_i}} - \frac{4 \sqrt{\mathcal{B}_{2_k}} x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_j} \xi_{13_i} \xi_{23_i}}{\mathcal{B}_{1_k} \mathcal{A}_{2_i}} - \\
& 2 x_{13_k} x_{23_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}} \xi_{13_j} \xi_{23_i} + x_{13_k} y_{12_k} \eta_{13_j} \xi_{13_j} \xi_{23_i} - \frac{x_{13_k} \mathcal{A}_{1_i} \eta_{23_j} \xi_{13_j} \xi_{23_i}}{\sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{\sqrt{\mathcal{B}_{1_k}} x_{13_k} \mathcal{A}_{1_i} \eta_{23_j} \xi_{13_j} \xi_{23_i}}{\mathcal{B}_{2_k} \sqrt{\mathcal{A}_{2_i}}} - 2 x_{13_k} \eta_{13_j} \xi_{12_i} \xi_{13_j} \xi_{23_i} + \frac{x_{23_k} y_{12_k} \eta_{13_j} \xi_{23_i}^2}{\sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} - \\
& \frac{2 x_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}^2}{\mathcal{A}_{2_i}} + \frac{2 \sqrt{\mathcal{B}_{1_k}} x_{23_k} \sqrt{\mathcal{A}_{1_i}} \eta_{23_j} \xi_{23_i}^2}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}} + \frac{\sqrt{\mathcal{B}_{1_k}} y_{12_k} \eta_{13_j} \eta_{23_j} \xi_{23_i}^2}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}^{3/2}} + \\
& \frac{2 \mathcal{B}_{1_k} \mathcal{A}_{1_i} \eta_{23_j}^2 \xi_{23_i}^2}{\mathcal{B}_{2_k}^2 \mathcal{A}_{2_i}^2} - \frac{2 \sqrt{\mathcal{B}_{1_k}} \mathcal{A}_{1_i} \eta_{23_j}^2 \xi_{23_i}^2}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}^2} + \frac{2 \sqrt{\mathcal{B}_{2_k}} x_{23_k} \eta_{13_j} \xi_{12_i} \xi_{23_i}^2}{\mathcal{B}_{1_k} \sqrt{\mathcal{A}_{1_i}} \sqrt{\mathcal{A}_{2_i}}} + \\
& \frac{2 \sqrt{\mathcal{B}_{2_k}} y_{12_k} \eta_{13_j}^2 \xi_{12_i} \xi_{23_i}^2}{\mathcal{B}_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} - \frac{2 \sqrt{\mathcal{B}_{2_k}} \eta_{13_j}^2 \xi_{12_i}^2 \xi_{23_i}^2}{\mathcal{B}_{1_k} \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \frac{2 \mathcal{B}_{2_k} \eta_{13_j}^2 \xi_{12_i}^2 \xi_{23_i}^2}{\mathcal{B}_{1_k}^2 \mathcal{A}_{1_i} \mathcal{A}_{2_i}} + \\
& \frac{x_{13_k} \mathcal{A}_{1_i}^{3/2} \eta_{23_j} \xi_{12_j} \xi_{23_i}^2}{\mathcal{A}_{2_i}^{3/2}} - \frac{3 \sqrt{\mathcal{B}_{1_k}} x_{13_k} \mathcal{A}_{1_i}^{3/2} \eta_{23_j} \xi_{12_j} \xi_{23_i}^2}{\mathcal{B}_{2_k} \mathcal{A}_{2_i}^{3/2}} + \frac{x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_i} \xi_{12_j} \xi_{23_i}^2}{\mathcal{A}_{2_i}} + \\
& \frac{\sqrt{\mathcal{B}_{2_k}} x_{13_k} \sqrt{\mathcal{A}_{1_i}} \eta_{13_j} \xi_{12_i} \xi_{12_j} \xi_{23_i}^2}{\mathcal{B}_{1_k} \mathcal{A}_{2_i}} + \frac{x_{23_k} \mathcal{A}_{2_i}^{3/2} \eta_{12_j} \xi_{12_i} \xi_{23_j}}{\mathcal{A}_{1_i}^{3/2}} + \frac{\sqrt{\mathcal{B}_{2_k}} x_{23_k} \mathcal{A}_{2_i}^{3/2} \eta_{12_j} \xi_{12_i} \xi_{23_j}}{\mathcal{B}_{1_k} \mathcal{A}_{1_i}^{3/2}} -
\end{aligned}$$

$$\begin{aligned}
& \frac{\sqrt{B_{2k}} x_{23k} y_{23k} \sqrt{A_{2i}} \eta_{13j} \xi_{12i} \xi_{23j}}{B_{1k} A_{1i}} - \frac{x_{23k} \sqrt{A_{2i}} \eta_{13j} \xi_{13i} \xi_{23j}}{A_{1i}} - \frac{x_{23k} \sqrt{A_{2i}} \eta_{13j} \xi_{13i} \xi_{23j}}{\sqrt{B_{1k}} \sqrt{B_{2k}} A_{1i}} + \\
& \frac{2 x_{23k} y_{12k} \sqrt{A_{2i}} \eta_{13j} \xi_{23i} \xi_{23j}}{A_{1i}} - 2 x_{23k} \eta_{23j} \xi_{23i} \xi_{23j} - \frac{2 \sqrt{B_{1k}} x_{23k} \eta_{23j} \xi_{23i} \xi_{23j}}{B_{2k}} - \\
& \left. \frac{x_{23k} \sqrt{A_{2i}} \eta_{13j} \xi_{12i} \xi_{23i} \xi_{23j}}{A_{1i}} + \frac{3 \sqrt{B_{2k}} x_{23k} \sqrt{A_{2i}} \eta_{13j} \xi_{12i} \xi_{23i} \xi_{23j}}{B_{1k} A_{1i}} \right]
\end{aligned}$$

associativity

$$In[*]:= (dm_{1,2 \rightarrow k} // dm_{k,3 \rightarrow k}) \equiv (dm_{2,3 \rightarrow k} // dm_{1,k \rightarrow k})$$

Time: 4.80 seconds Out[]=

True

co-associativity

$$In[*]:= (d\Delta_{1 \rightarrow 1,2} // d\Delta_{2 \rightarrow 2,3}) \equiv (d\Delta_{1 \rightarrow 2,3} // d\Delta_{2 \rightarrow 1,2})$$

Time: 1.11 seconds Out[]=

True

algebra morphism

$$In[*]:= (d\Delta_{1 \rightarrow 1,2} d\Delta_{2 \rightarrow 3,4} // (dm_{1,3 \rightarrow 1} dm_{2,4 \rightarrow 2})) \equiv (dm_{1,2 \rightarrow 3} // d\Delta_{3 \rightarrow 1,2})$$

Time: 57.36 seconds Out[]=

True

antipode

$$In[*]:= \{d\Delta_{i \rightarrow 1,2} // dS_1 // dm_{1,2 \rightarrow 1}, d\Delta_{i \rightarrow 1,2} // dS_2 // dm_{1,2 \rightarrow 1}, dS_1 // \overline{dS_1}\}$$

Time: 308.56 seconds Out[]=

$$\{E_{\{i\} \rightarrow \{1\}} [\emptyset, \emptyset], E_{\{i\} \rightarrow \{1\}} [\emptyset, \emptyset], E_{\{1\} \rightarrow \{1\}} [a_{11} \alpha_{11} + a_{21} \alpha_{21} + b_{11} \beta_{11} + b_{21} \beta_{21} + y_{12_1} \eta_{12_1} + y_{13_1} \eta_{13_1} + y_{23_1} \eta_{23_1} + x_{12_1} \xi_{12_1} + x_{13_1} \xi_{13_1} + x_{23_1} \xi_{23_1}, \emptyset]\}$$

quasi-triangular axioms

$$\begin{aligned}
In[*]:= & (R_{1,3} // d\Delta_{1 \rightarrow 1,2}) \equiv (R_{1,3} R_{2,4} // dm_{3,4 \rightarrow 3}) \\
& (R_{1,3} // d\Delta_{3 \rightarrow 2,3}) \equiv (R_{1,3} R_{\emptyset,2} // dm_{1,\emptyset \rightarrow 1}) \\
& (d\Delta_{i \rightarrow k,j} R_{1,2} // dm_{j,1 \rightarrow 1} // dm_{k,2 \rightarrow 2}) \equiv (R_{1,2} d\Delta_{i \rightarrow j,k} // dm_{1,j \rightarrow 1} // dm_{2,k \rightarrow 2})
\end{aligned}$$

Time: 8.66 seconds Out[]=

True

Out[]=

True

Out[]=

True

$$In[*]:= (R_{1,2} // aSt_2) \equiv (\overline{R}_{1,2})$$

Time: 0.06 seconds Out[]=

True

Braid-like Reidemeister moves: R2,R3

$$\text{In}[*]:= \left\{ \mathbf{R}_{1,2} \bar{\mathbf{R}}_{3,4} // \mathbf{dm}_{1,3 \rightarrow 1} // \mathbf{dm}_{2,4 \rightarrow 2}, \bar{\mathbf{R}}_{1,2} \mathbf{R}_{3,4} // \mathbf{dm}_{1,3 \rightarrow 1} // \mathbf{dm}_{2,4 \rightarrow 2}, \right. \\ \left. (\mathbf{R}_{1,2} \mathbf{R}_{4,3} \mathbf{R}_{5,6} // \mathbf{dm}_{1,4 \rightarrow 1} // \mathbf{dm}_{2,5 \rightarrow 5} // \mathbf{dm}_{3,6 \rightarrow 3}) \equiv (\mathbf{R}_{1,6} \mathbf{R}_{2,3} \mathbf{R}_{4,5} // \mathbf{dm}_{1,4 \rightarrow 1} // \mathbf{dm}_{2,5 \rightarrow 5} // \mathbf{dm}_{3,6 \rightarrow 3}) \right\}$$

Time: 12.34 seconds Out[=

$$\{\mathbb{E}_{\{\} \rightarrow \{1,2\}} [\emptyset, \emptyset], \mathbb{E}_{\{\} \rightarrow \{1,2\}} [\emptyset, \emptyset], \text{True}\}$$

(Alt) In[*]:=

$$\text{Block}[\{\$k = 2, \$n = 3, \hbar = 1\}, \mathbf{dm}_{i,j \rightarrow k} // \mathbf{12U} // \mathbf{CF}]$$

Time: 52.62 seconds Out[=

\$Aborted