

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

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This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

(Alt) In[]:=

```
MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ , op_] := PPSS@Collect[PPNormal@Normal@PPSeries@Series[_ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_] := SS[_ , Identity];
```

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```

In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
y[i_, j_] := y<>i<>j; b[i_] := b<>i; B[i_] := B<>i;
η[i_, j_] := η<>i<>j; β[i_] := β<>i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];

```

```

In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

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```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
  ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
  η[i, j]* = y[i, j]
]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
  b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_αβ)* := (u*)_αβ;
F[u_i_] := F[u_i] = ToExpression["F" <> ToString[u]]_i

```

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```

In[ ]:= LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;

```

Upper and lower case variables:

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```

In[ ]:= Module[{i, s, p, c, d, χ, f, v},
  U21Rule = Union @@ Table[As[i, 1, $N - 1], {A[i]_s^p → e^{p α[i]_s}, B[i]_s^p → e^{-p ħ b[i]_s}}];
  12URule = Append[Union @@ Table[As[i, 1, $N - 1],
    {e^{d_- + c_- b[i]_s} → B[i]_s^{-c/ħ} e^d, e^{d_- + c_- α[i]_s} → A[i]_s^c e^d}], e^χ → e^{Expand[12U[χ]}];
  U21[ε_] := ε /. U21Rule;
  12U[ε_] := ε //. 12URule;
  Do[As[i, 1, $N - 1], With[{αi = α[i], Ai = A[i], bi = b[i], Bi = B[i]},
    D_{αi_s}[f_] := ∂_{αi_s} f + Ai_s ∂_{Ai_s} f;
    D_{bi_s}[f_] := ∂_{bi_s} f - ħ Bi_s ∂_{Bi_s} f;
  ]];
  D_v[f_] := ∂_v f;
];

```

```

In[ ]:= {U21Rule, 12URule}

```

```

Time: 0.00 seconds Out:=

```

$$\left\{ \left\{ B1_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b1_{s\$}}, B2_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b2_{s\$}}, B3_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b3_{s\$}}, B4_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b4_{s\$}}, B5_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b5_{s\$}}, \right. \right. \\
B6_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b6_{s\$}}, B7_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b7_{s\$}}, B8_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b8_{s\$}}, A1_{s\$}^{p\$} \rightarrow e^{p\$ \alpha1_{s\$}}, A2_{s\$}^{p\$} \rightarrow e^{p\$ \alpha2_{s\$}}, \\
A3_{s\$}^{p\$} \rightarrow e^{p\$ \alpha3_{s\$}}, A4_{s\$}^{p\$} \rightarrow e^{p\$ \alpha4_{s\$}}, A5_{s\$}^{p\$} \rightarrow e^{p\$ \alpha5_{s\$}}, A6_{s\$}^{p\$} \rightarrow e^{p\$ \alpha6_{s\$}}, A7_{s\$}^{p\$} \rightarrow e^{p\$ \alpha7_{s\$}}, A8_{s\$}^{p\$} \rightarrow e^{p\$ \alpha8_{s\$}} \left. \right\}, \\
\left\{ e^{d\$_- + c\$_- b1_{s\$}} \rightarrow e^{d\$} B1_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b2_{s\$}} \rightarrow e^{d\$} B2_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b3_{s\$}} \rightarrow e^{d\$} B3_{s\$}^{-\frac{c\$}{\hbar}}, \right. \\
e^{d\$_- + c\$_- b4_{s\$}} \rightarrow e^{d\$} B4_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b5_{s\$}} \rightarrow e^{d\$} B5_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b6_{s\$}} \rightarrow e^{d\$} B6_{s\$}^{-\frac{c\$}{\hbar}}, \\
e^{d\$_- + c\$_- b7_{s\$}} \rightarrow e^{d\$} B7_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b8_{s\$}} \rightarrow e^{d\$} B8_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- \alpha1_{s\$}} \rightarrow e^{d\$} A1_{s\$}^{c\$}, \\
e^{d\$_- + c\$_- \alpha2_{s\$}} \rightarrow e^{d\$} A2_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha3_{s\$}} \rightarrow e^{d\$} A3_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha4_{s\$}} \rightarrow e^{d\$} A4_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha5_{s\$}} \rightarrow e^{d\$} A5_{s\$}^{c\$}, \\
e^{d\$_- + c\$_- \alpha6_{s\$}} \rightarrow e^{d\$} A6_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha7_{s\$}} \rightarrow e^{d\$} A7_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha8_{s\$}} \rightarrow e^{d\$} A8_{s\$}^{c\$}, e^{\chi_-} \rightarrow e^{\text{Expand}[12U[\chi]}] \left. \right\} \}$$

```

In[ ]:= {D_{b3_i} / @ {b3_i^5, B3_i^7}, D_{α3_i} / @ {α3_i^5, A3_i^7}}

```

```

Time: 0.01 seconds Out:=

```

$$\left\{ \left\{ 5 b3_i^4, -7 \hbar B3_i^7 \right\}, \left\{ 5 \alpha3_i^4, 7 A3_i^7 \right\} \right\}$$

Canonical Form:

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```

In[ ]:= CF[ε_] := PP_{CF} @ Module[{s1},
  s1 = If[$k == ∞,
    Expand[ε] /. e^{p_-} → e^{Factor@p},
    Expand[ε] /. e^{p_-} → e^{Factor@p} /. q^{n_-} → ∑_{k=0}^{$k} \frac{(\hbar \in \gamma n)^k}{k!} /. e^{k_-} /. k > $k → 0
  ]
  (*Expand@Collect[s1, Cases[s1, VarPattern, ∞], CCF] /. CCF → Factor*)
];

```

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Weights: (Need confirmation!!!)

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```

In[*]:= (*Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,i-j};
  Wt[ξ[i,j]]={0,j-i}; Wt[η[i,j]]={1,j-i};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[α[i]]={0,0}; Wt[β[i]]={1,0};
];
Wt[u_]:=Wt[u];*)
Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,-(i-j)};
  Wt[ξ[i,j]]={0,-(i-j)}; Wt[η[i,j]]={1,(i-j)};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[α[i]]={0,0}; Wt[β[i]]={1,0};
];
Wt[u_]:=Wt[u];

```

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```

In[*]:= (* q~eh γ ∈; Ai[n]~eh ∈ ai; *)

```

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```

In[*]:= $n = 5; $k = ∞; $E := {$n, $k}

```

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```

In[*]:= $EPBWGens := Flatten@{
  Table[As[i,1,$n-1],As[j,i+1,$n],x[i,j]],
  Table[As[i,1,$n-1],{a[i],A[i][_]}]
}
$EPBWRule := Module[{k=0},Flatten@Table[{g→++k,gi_→{i,k}},{g,$EPBWGens}]];
x_≤y_:=OrderedQ[{x,y}/.$EPBWRule]; x_<y_:=!OrderedQ[{y,x}/.$EPBWRule];
$PBWGens:=DeleteCases[$EPBWGens,_[_]]

```

```

In[*]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

```

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```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]
  }],
  U[____], F
] /. F → CF]

```

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```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

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```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
S̄i[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[S̄i[f] S̄i1[Ui1[fs]]]
}];
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij, Am[n]]
];
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[]:=

```
γ U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page 509, rewritten in Monoblog/260928:

$$\begin{aligned}
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= q^2 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) - q \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) + (q^{-2} - q^2) \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= q^{-2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) &= q^{-2} \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)
 \end{aligned}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.08 seconds Out:=
(-1 - γ ∈ ħ) U2[x24] U3[x12] + (1 + 2 γ ∈ ħ) U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 18.11 seconds Out:=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```

Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3->2 // m2,4->2 // m1,2->1) ==
      (P // m3,4->3 // m2,3->2 // m1,2->1) == (P // m1,2->1 // m3,4->3 // m1,3->1) ==
      (P // m1,2->1 // m1,3->1 // m1,4->1) == (P // m2,3->2 // m1,2->1 // m1,4->1)
  ]
]
» {x12, x13, x23, a1, a2}

```

Out[8]=

{ }

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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In[8]:=

```

Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1->k[mj, j1->j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]]
]

```

```

In[*]:= Block[{$k = ∞, $n = 5}, Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]]

Time: 0.08 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[0], A1[-2], A4[0], A1[0], A2[0], A4[-2], A3[-2], A3[1], A3[2], A2[-1]}

Out[*]=
{ }

```

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[-1], A1[2], A3[-1], A4[2], A2[0], A4[2], A3[2], A4[1], A2[-2], A4[-1]}
Out[*]=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[*]:= $k = 1
Time: 0.02 seconds Out[*]=
1

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[*]=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

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In[]:=

```

Do[As[i, 1, $N - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]]; (*?*)
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q2 Uj[x[i, i + 1]]; (*?*)
];
Do[As[i, 1, $N - 2], As[n, i + 2, $N],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]];
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]]
]

```

```
In[*]:= Block[{ $k = ∞, $n = 5 }, Column@Module[{ Bas = $PBWGens, P },
  Table[As[P, Ui /@ Bas], P → {Si[P],  $\bar{S}_i$ [P]}]
]]
```

Time: 0.08 seconds Out[]:=

$$\begin{aligned}
U_i[x_{12}] &\rightarrow \left\{ -\frac{U_i[x_{12}]}{q^2}, -q^2 U_i[x_{12}] \right\} \\
U_i[x_{13}] &\rightarrow \left\{ -\frac{U_i[x_{13}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{23}], -q^6 U_i[x_{13}] + (-q^3 + q^7) U_i[x_{12}, x_{23}] \right\} \\
U_i[x_{14}] &\rightarrow \left\{ -\frac{U_i[x_{14}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{24}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{34}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{34}], \right. \\
&\quad \left. -q^{10} U_i[x_{14}] + (-q^7 + q^{11}) U_i[x_{12}, x_{24}] + \right. \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{13}, x_{34}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{12}, x_{23}, x_{34}] \right\} \\
U_i[x_{15}] &\rightarrow \left\{ -\frac{U_i[x_{15}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{25}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{35}] + \right. \\
&\quad \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{14}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{35}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{24}, x_{45}] + \right. \\
&\quad \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{13}, x_{34}, x_{45}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q\right) U_i[x_{12}, x_{23}, x_{34}, x_{45}], \\
&\quad -q^{14} U_i[x_{15}] + (-q^{11} + q^{15}) U_i[x_{12}, x_{25}] + (-q^{11} + q^{15}) U_i[x_{13}, x_{35}] + (-q^{11} + q^{15}) U_i[x_{14}, x_{45}] + \\
&\quad (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{23}, x_{35}] + (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{24}, x_{45}] + \\
&\quad \left. (-q^8 + 2q^{12} - q^{16}) U_i[x_{13}, x_{34}, x_{45}] + (-q^5 + 3q^9 - 3q^{13} + q^{17}) U_i[x_{12}, x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{23}] &\rightarrow \left\{ -\frac{U_i[x_{23}]}{q^2}, -q^2 U_i[x_{23}] \right\} \\
U_i[x_{24}] &\rightarrow \left\{ -\frac{U_i[x_{24}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{34}], -q^6 U_i[x_{24}] + (-q^3 + q^7) U_i[x_{23}, x_{34}] \right\} \\
U_i[x_{25}] &\rightarrow \left\{ -\frac{U_i[x_{25}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{35}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{24}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{23}, x_{34}, x_{45}], \right. \\
&\quad \left. -q^{10} U_i[x_{25}] + (-q^7 + q^{11}) U_i[x_{23}, x_{35}] + \right. \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{24}, x_{45}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{34}] &\rightarrow \left\{ -\frac{U_i[x_{34}]}{q^2}, -q^2 U_i[x_{34}] \right\} \\
U_i[x_{35}] &\rightarrow \left\{ -\frac{U_i[x_{35}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{34}, x_{45}], -q^6 U_i[x_{35}] + (-q^3 + q^7) U_i[x_{34}, x_{45}] \right\} \\
U_i[x_{45}] &\rightarrow \left\{ -\frac{U_i[x_{45}]}{q^2}, -q^2 U_i[x_{45}] \right\} \\
U_i[a_1] &\rightarrow \{-U_i[a_1], -U_i[a_1]\} \\
U_i[a_2] &\rightarrow \{-U_i[a_2], -U_i[a_2]\} \\
U_i[a_3] &\rightarrow \{-U_i[a_3], -U_i[a_3]\} \\
U_i[a_4] &\rightarrow \{-U_i[a_4], -U_i[a_4]\}
\end{aligned}$$

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[-1], A2[-1], A4[1], A4[1], A3[2], A1[1], A2[1], A3[-1], A3[2], A4[-2]}

Out[*]=
{x12 -> 0, x13 -> 0, x14 -> 0, x15 -> 0, x23 -> 0, x24 -> 0, x25 -> 0, x34 -> 0, x35 -> 0, x45 -> 0,
  a1 -> 0, a2 -> 0, a3 -> 0, a4 -> 0, A4[-1] -> U1[], A2[-1] -> U1[], A4[1] -> U1[], A4[1] -> U1[],
  A3[2] -> U1[], A1[1] -> U1[], A2[1] -> U1[], A3[-1] -> U1[], A3[2] -> U1[], A4[-2] -> U1[]}

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> UCF[(P // S1 // S̄1) - P]
  ]
]
» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[2], A2[-2], A1[-1], A1[1], A3[1], A3[-1], A4[1], A1[0], A3[1], A1[-1]}

Out[*]=
{}

```

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The Generating Functions Engine

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Canonical Forms:

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```

In[*]:= LogReduce[ε_] :=
  ε /. c_ * Log[a_] -> Log@Factor[ac] /. Log[a_] + Log[b_] -> Log@Factor[a b];
  CF[ε_E] := CF /@ MapAt[LogReduce, ε, 1];
  CF[ε : E__[_]] := CF /@ ε;

```

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E operations:

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```

In[*]:= Zip1_{ } = Identity;
Zip1_{vs_@{\mathcal{F}_-, \mathbb{E}[Q_-, P_{---}]}} := Module[{I, F, G, u, v},
  (*Echo[{vs, \mathcal{F}, Q, P}];*)
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, \partial_{F[u], F[v]} \mathcal{F}, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, \partial_{u, v} Q, 0], {u, vs}, {v, vs}];
  (*Echo@MatrixForm[I - G.F];*)
  {CF[(F /@ vs) . (F.Inverse[I - G.F]) . (F /@ vs) / 2],
   \mathbb{E}[CF[Q - PowerExpand@Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]

```

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Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}_{\mathbb{E}^{\sum_{i \in B} Y_i Z_i}} \right\rangle_B = \mathbb{E}^{\frac{1}{2} \sum_{i, j \in B} F_{ij} Y_i Y_j} \left\langle F : \mathcal{E}_{|z_B \rightarrow z_B + F.Y_B} \right\rangle_B$.

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```

In[*]:= Zip2_{ } = Identity;
Zip2_{vs_@{\mathcal{F}_-, \mathbb{E}[Q_-, P_{---}]}} := Module[{F, Y, u, v},
  (*EchoLabel["Zip2"] [{vs, \mathcal{F}, Q, P}];*)
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[\partial_{F[u], F[v]} \mathcal{F}], 0], {u, vs}, {v, vs}];
  Y = Table[\partial_v Q, {v, vs}] /. (*AlsoUpper@*) Table[v \to 0, {v, vs}];
  CF /@ ({\mathcal{F}, \mathbb{E}[Q - Y.vs + Y.F.Y / 2, P]} /. (*AlsoUpper@*) Thread[vs \to vs + F.Y])
]

```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{E}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i, j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda) (\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

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```

In[*]:= Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3", vs,  $\mathcal{F}$ ,  $\mathcal{E}$ }]*)
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{F[u], F[v]} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] === 0, 0,  $\frac{F[u, v]}{2(m+1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
    E @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  ) /. (*AlsoUpper@*) Table[v → 0, {v, vs}]
]

```

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“Define” Code

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```

In[*]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ =  $\mathcal{E}$ _] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[opnisp, $E, PPBoot@Block[{i, j, k}, opisp, $E =  $\mathcal{E}$ ; opnisp, $E]]];
    SD[opisp, op{is}, $E]; SD[opsis__, op{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]

```

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The Generating Functions for the $\mathbb{A}$ operations

```
In[*]:= BeginProfile[]
```

```
Time: 0.06 seconds Out:=
ProfileRoot
```

```
In[*]:= $k = 1;
```

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Ad, TT, Seepage

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```
In[*]:= PotQ[0] = True;
PotQ[c_. U_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
In[*]:= Ad[x_][y_] /; PotQ[x] ∧ (Head[y] != Plus) := If[y == 0, 0, PPAdPoty[eUB[y,x]/y y]];
Ad[x_][y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

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```
In[*]:= Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
While[(t = UB[t, x]) != 0, s += t / k!; ++k];
UCF[s]
]
```

```
In[*]:= $k = 2
```

```
Time: 0.00 seconds Out:=
2
```

```
In[*]:= Ad[ε ħ Uj[x12] Uk[]][Uj[x23] Uk[a4]]
```

```
Time: 0.06 seconds Out:=
(-ε ħ - γ ε2 ħ2) Uj[x13] Uk[a4] + Uj[x23] Uk[a4] + 2 γ ε2 ħ2 Uj[x12, x23] Uk[a4]
```

```
In[*]:= PotQ[7 ħ Uj[x12] Uk[a3]]
```

```
Time: 0.02 seconds Out:=
PotQ[7 ħ Uj[x12] Uk[a3]]
```

```
In[*]:= Ad[7 ħ Uj[x12] Uk[a3]][Uj[x23] Uk[a4]]
```

```
Time: 0.09 seconds Out:=
Uj[x23] Uk[a4] + (-7 ħ - 7 γ ε ħ2 -  $\frac{7}{2}$  γ2 ε2 ħ3) Uj[x13] Uk[a3, a4] +
(14 γ ε ħ2 + 14 γ2 ε2 ħ3) Uj[x12, x23] Uk[a3, a4] - 98 γ2 ε2 ħ4 Uj[x12, x13] Uk[a3, a3, a4] +
98 γ2 ε2 ħ4 Uj[x12, x12, x23] Uk[a3, a3, a4] -  $\frac{686}{3}$  γ2 ε2 ħ5 Uj[x12, x12, x13] Uk[a3, a3, a3, a4]
```

In[*]:= $\text{Ad}[\pi \mathbf{i} U_j[\mathbf{a1}]] [U_j[\mathbf{x23}] U_k[\mathbf{a4}]]$

Time: 0.05 seconds Out[=

$$e^{\frac{i\pi\gamma}{2}} U_j[\mathbf{x23}] U_k[\mathbf{a4}]$$

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In[*]:= $\text{TT}_{\lambda}[] = 0;$ (* Translated Tangent *)
 $\text{TT}_{\lambda}[\mathbf{xs}___, \mathbf{x}__] := \text{PP}_{\text{TT}} @ \text{UCF}[\text{Ad}[\mathbf{x}] [\text{TT}_{\lambda}[\mathbf{xs}]] + \partial_{\lambda} \mathbf{x}];$
 $\text{TT}_{\lambda}[\mathbf{xs_List}] := \text{TT}_{\lambda} @ \mathbf{xs}$

In[*]:= $\text{TT}_{\lambda}[\lambda \xi_{12} U_0[\mathbf{x12}] U_1[], \lambda \xi_{23} U_0[\mathbf{x23}] U_1[]]$

Time: 0.08 seconds Out[=

$$\xi_{12} U_0[\mathbf{x12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[\mathbf{x13}] U_1[] + \xi_{23} U_0[\mathbf{x23}] U_1[]$$

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In[*]:= $\text{O}_i[\mathbf{poly}_] := \text{PP}_0 @ \text{Total}[\text{CoefficientRules}[\mathbf{poly}, \#_i \& / @ \$\text{PBWGens}] / .$
 $(p_ \rightarrow c_) \Rightarrow c U_i @ \text{Flatten} @ \text{MapThread}[\text{ConstantArray}, \{\$ \text{PBWGens}, p\}]$
 $];$
 $\text{O}_{i, \text{is}___}[\mathbf{poly}_] := \text{O}_i[\text{O}_{\text{is}}[\mathbf{poly}]];$
 $\mathbb{P}_i[\mathcal{E}_] := \mathcal{E} / . U_i[\mathbf{xs}___] \Rightarrow \text{Times} @ \#_i \& / @ \{\mathbf{xs}\};$
 $\mathbb{P}_{i, \text{is}___}[\mathbf{poly}_] := \mathbb{P}_i[\mathbb{P}_{\text{is}}[\mathbf{poly}]];$

In[*]:= $\text{O}_i[\mathbf{x12}_i^2 \mathbf{x13}_i + 7 \mathbf{x23}_i^{17}]$

Time: 0.06 seconds Out[=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}]$$

In[*]:= $\text{O}_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.03 seconds Out[=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] U_j[\mathbf{x24}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}] U_j[\mathbf{x25}, \mathbf{x25}, \mathbf{x25}]$$

In[*]:= $\mathbb{P}_{i,j} @ \text{O}_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.06 seconds Out[=

$$\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3$$

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In[*]:= $\text{Seepage}_i[\{\}, c_] = c;$
 $\text{Seepage}_i[\xi\mathbf{s}_\text{List}, P_] := \text{PP}_{\text{Seepage}} @ \text{Module}[\{\mathbf{lx}\},$
 $\mathbf{lx} = \$\text{PBWGens}[\text{Length} @ \xi\mathbf{s}];$
 $\text{CF} @ \text{Collect}[P, \mathbf{lx}_i, \mathbb{P}_i[\text{Ad}[\text{Last}[\xi\mathbf{s}] U_i[\mathbf{lx}]] [\text{O}_i[\text{Seepage}_i[\text{Most} @ \xi\mathbf{s}, \#]]]]$
 $\&]$
 $];$
 $\text{Seepage}_{i, \text{is}___}[\xi\mathbf{s}_\text{List}, \xi\mathbf{ss}_\text{List}, P_] := \text{Seepage}_i[\xi\mathbf{s}, \text{Seepage}_{\text{is}}[\xi\mathbf{ss}, P]]$

In[*]:= **\$k = 1**

Time: 0.00 seconds Out[1]=

1

In[*]:= **\$PBWgens***

Time: 0.01 seconds Out[2]=

{ $\xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{23}, \xi_{24}, \xi_{25}, \xi_{34}, \xi_{35}, \xi_{45}, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ }

In[*]:= **Seepage_i[\$PBWgens*, x35_i]**

Time: 0.16 seconds Out[3]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35_i} + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \in \xi_{45} \hbar x_{35_i} x_{45_i}$$

In[*]:= **Seepage_{i,j}[#_i & /@ \$PBWgens*, #_j & /@ \$PBWgens*, x35_i x45_j]**

Time: 0.25 seconds Out[4]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 + \alpha_3 - \alpha_4 - 2\alpha_4)} x_{35_i} x_{45_j} + 2 e^{-\frac{1}{2}(\alpha_2 - \alpha_3 + 3\alpha_4 + 2\alpha_4)} \gamma \in \hbar x_{35_i} x_{45_i} x_{45_j} \xi_{45_i}$$

pdf

In[*]:=

```
Invertis__[$_] := PPInvert@Module[{A, B, U1, j}, (*  $\mathcal{E} = A U[] + \epsilon B$  *)
U1 = Times@@(U#[] & /@ {is});
A = Coefficient[$, U1] /.  $\epsilon \rightarrow 0$ ;
B = UCF[($ - A U1)];
UCF[
  A-1 (U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]
```

In[*]:= **\$k = 3;**

Invert_{i,j}[7 U_i[] U_j[] + ϵ U_i[x12] U_j[x23]]

Time: 0.09 seconds Out[5]=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \epsilon U_i[x_{12}] U_j[x_{23}] +$$

$$\frac{1}{343} \epsilon^2 U_i[x_{12}, x_{12}] U_j[x_{23}, x_{23}] - \frac{\epsilon^3 U_i[x_{12}, x_{12}, x_{12}] U_j[x_{23}, x_{23}, x_{23}]}{2401}$$

{ $k = 1, n = 3$ }

Time: 0.01 seconds Out[6]=

{2, 3}

pdf

In[]:=

```

TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
  PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      AppendTo[ss, s /. F_Function → ReplacePart[F, 2 → CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 → CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]

```

pdf

$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle &= \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \\
 TT(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f=f_i x_i, \quad g=\sum_{(i,j) \in I} g_{ij} w_{ij} & e^f = F, \quad e^g = G, \quad R_\lambda = \mathbb{C}\langle FG \rangle, \quad \partial_\lambda F = F \partial_\lambda f \\
 TT(R_\lambda) &= R'_\lambda (\partial_\lambda R_\lambda) = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
 \end{aligned}$$

pdf

In[*]:=

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[0k[Seepagek[ff, UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ (ff. (#k & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]

```

pdf

In[*]:=

```

Define[aΔi→j,k = PP"aΔ"[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]);
  TTL = TTλ@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
  ffk = ffj /. j → k;
  fs = Flatten[Table[
    fp,ℙj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙj,k[u][λ] ℙj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[oj,k[Seepagej,k[ffj, ffk, UCF@B]]];
  SG = PP"SG"@S[G];
  BB = PP"BB"[∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G];
  SBB = PP"SBB"@S[BB];
  TTR = PP"TTR"@UCF[Invertj,k[SG] ⊗ SBB];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

`In[*]:= { $k = 1, $n = 2 }; amti,j→k`

$$(\text{amt}_{1,2 \rightarrow 2} // \text{amt}_{2,3 \rightarrow 1}) \equiv (\text{amt}_{2,3 \rightarrow 2} // \text{amt}_{1,2 \rightarrow 1})$$

`Time: 0.11 seconds Out[*]=`

$$\mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + x_{12k} \left(\xi_{12i} + e^{\alpha_{1i}} \xi_{12j} \right), 0 \right]$$

`Out[*]=`

True

`In[*]:= { $k = 1, $n = 3 }; amti,j→k`

$$(\text{amt}_{1,2 \rightarrow 4} // \text{amt}_{4,3 \rightarrow 5}) \equiv (\text{amt}_{2,3 \rightarrow 4} // \text{amt}_{1,4 \rightarrow 5})$$

`Time: 0.08 seconds Out[*]=`

$$\begin{aligned} \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ \left. x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \right. \\ \left. - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \right. \\ \left. 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right] \end{aligned}$$

`Out[*]=`

True

In[*]:= { \$k = 2, \$n = 3 }; amt_{i,j→k}

$$(amt_{1,2→4} // amt_{4,3→5}) \equiv (amt_{2,3→4} // amt_{1,4→5})$$

Time: 44.66 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ & \quad x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i}, \\ & \quad - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + \\ & \quad x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \quad x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \quad x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \quad x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \quad x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \quad x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \quad \left. x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; aΔt_{i→j,k}

$$(a\Delta t_{1→1,3} // a\Delta t_{1→1,2}) \equiv (a\Delta t_{1→1,2} // a\Delta t_{2→2,3})$$

Time: 0.23 seconds Out[]=

$$\mathbb{E}_{\{i\} \rightarrow \{j,k\}} [(a_{1j} + a_{1k}) \alpha_{1i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i}, -2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i}]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; aΔt_{i→j,k}

$$(a\Delta t_{1→1,3} // a\Delta t_{1→1,2}) \equiv (a\Delta t_{1→1,2} // a\Delta t_{2→2,3})$$

Time: 0.08 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{j,k\}} [\\ & \quad (a_{1j} + a_{1k}) \alpha_{1i} + (a_{2j} + a_{2k}) \alpha_{2i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i} + x_{13j} \xi_{13i} + x_{13k} \xi_{13i} + x_{23j} \xi_{23i} + x_{23k} \xi_{23i}, \\ & \quad - 2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i} - 2 a_{1k} x_{13j} \xi_{13i} - 2 a_{2k} x_{13j} \xi_{13i} + 2 a_{1j} x_{13k} \xi_{13i} + \\ & \quad 2 a_{2j} x_{13k} \xi_{13i} + x_{12k} x_{13j} \xi_{12i} \xi_{13i} - x_{12j} x_{13k} \xi_{12i} \xi_{13i} - 2 a_{2k} x_{23j} \xi_{23i} + 2 a_{2j} x_{23k} \xi_{23i} - \\ & \quad x_{12k} x_{23j} \xi_{12i} \xi_{23i} + x_{13k} x_{23j} \xi_{13i} \xi_{23i} - x_{13j} x_{23k} \xi_{13i} \xi_{23i} + x_{12j} x_{23k} (4 \xi_{13i} + \xi_{12i} \xi_{23i})] \end{aligned}$$

Out[*]=

True

```
In[*]:= { $k = 1, $n = 2 }; (amt1,2→1 // aΔt1→1,2) ≡ (aΔt2→3,4 // aΔt1→1,2 // amt1,3→1 // amt2,4→2)
```

Time: 0.11 seconds Out[=

True

```
In[*]:= Block[ { $k = 1, $n = 3 }, Echo@ { amt1,2→1, aΔt1→1,2 };
```

```
(amt1,2→1 // aΔt1→1,2) ≡ (aΔt2→3,4 // aΔt1→1,2 // amt1,3→1 // amt2,4→2) ]
```

$$\begin{aligned} & \gg \left\{ \mathbb{E}_{\{1,2\} \rightarrow \{1\}} \left[a_1 (\alpha_1 + \alpha_2) + a_2 (\alpha_2 + \alpha_2) + x_{12} \left(\xi_{12} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12} \right) + \right. \right. \\ & \quad x_{13} \left(\xi_{13} + e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12} \xi_{23} \right) + x_{23} \left(\xi_{23} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2)} \xi_{23} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{13} \xi_{12} \xi_{13} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{13} \xi_{12} \xi_{23} + \\ & \quad \left. 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} x_{12} x_{23} \xi_{12} \xi_{23} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} x_{13} x_{23} \xi_{13} \xi_{23} \right], \mathbb{E}_{\{1\} \rightarrow \{1,2\}} [\\ & \quad (a_1 + a_2) \alpha_1 + (a_2 + a_2) \alpha_2 + x_{12} \xi_{12} + x_{12} \xi_{12} + x_{13} \xi_{13} + x_{13} \xi_{13} + x_{23} \xi_{23} + x_{23} \xi_{23}, \\ & \quad - 2 a_1 x_{12} \xi_{12} + 2 a_1 x_{12} \xi_{12} - 2 a_2 x_{13} \xi_{13} - 2 a_2 x_{13} \xi_{13} + 2 a_1 x_{13} \xi_{13} + \\ & \quad 2 a_2 x_{13} \xi_{13} + x_{12} x_{13} \xi_{12} \xi_{13} - x_{12} x_{13} \xi_{12} \xi_{13} - 2 a_2 x_{23} \xi_{23} + 2 a_2 x_{23} \xi_{23} - \\ & \quad x_{12} x_{23} \xi_{12} \xi_{23} + x_{13} x_{23} \xi_{13} \xi_{23} - x_{13} x_{23} \xi_{13} \xi_{23} + x_{12} x_{23} (4 \xi_{13} + \xi_{12} \xi_{23})] \left. \right\} \end{aligned}$$

Out[*]=

True

pdf

In[*]:=

```

Define[aSti = Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Ui[x___] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp, ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp, ui[λ] /. τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp, pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, pi[u][λ] pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[Oi[Seepagei[ff, UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], -α[m]i (a[m]i)], 0],
    Total[Cases[fs, fp, _]] /. fp, u => u (fp, u /. ss)[λ] /. λ | ħ | γ → 1
  ]
]]

```

pdf

In[]:=

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x____] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp, Pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pi[u][λ] Pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[Oi[Seepagei[ff, UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```
In[*]:= $k = 1; $n = 3; aSti
aΔt1→2,3 // aSt2 // amt2,3→1
```

Time: 32.98 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{i\}} \left[-a_{1i} \alpha_{1i} - a_{2i} \alpha_{2i} - e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} - e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + \right. \\ & \quad x_{13i} \left(-e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} - e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right), 2 e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} + \\ & \quad 2 e^{-\frac{3\alpha_{1i}}{2}} x_{12i} x_{13i} \xi_{12i} \xi_{13i} + 2 e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + 2 e^{-\frac{3\alpha_{2i}}{2}} x_{13i} x_{23i} \xi_{13i} \xi_{23i} + \\ & \quad 2 e^{-\alpha_{1i} - \alpha_{2i}} x_{13i}^2 \xi_{12i} \xi_{13i} \xi_{23i} + x_{12i} x_{23i} \left(4 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) + \\ & \quad \left. x_{13i} \left(2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 3 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) \right] \end{aligned}$$

Out[]=

$$\mathbb{E}_{\{1\} \rightarrow \{1\}} [0, 0]$$

pdf

```
In[*]:= (*Splitter B⊗A *)
sYi-→j-,k-,l-,m- := E{i}→{j,k,l,m} [
  Sum[η[r, s]i y[r, s]j + ξ[r, s]i x[r, s]m, {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[α[r]i a[r]l + β[r]i b[r]k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

pdf

$P: \mathbb{A} \otimes \mathbb{B} \rightarrow \mathbb{Q}$ and $R \in \mathbb{B} \otimes \mathbb{A}$:

pdf

```
In[*]:= Pi-,j- := E{i,j}→{} @@ PadRight [
  {Sum[η[r, s]j ξ[r, s]i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]j α[r]i, {r, 1, $n - 1}]},
  $k + 1]
Ri-,j- := E{i}→{i,j} @@ PadRight [
  {Sum[y[r, s]i x[r, s]j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]i a[r]j, {r, 1, $n - 1}]},
  $k + 1]
```

```
In[*]:= {R1,2 // P2,3, R1,2 // P5,1}
```

Time: 0.12 seconds Out[]=

$$\begin{aligned} & \{ \mathbb{E}_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + b_{31} \beta_{33} + b_{41} \beta_{43} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{141} \eta_{143} + \\ & \quad y_{151} \eta_{153} + y_{231} \eta_{233} + y_{241} \eta_{243} + y_{251} \eta_{253} + y_{341} \eta_{343} + y_{351} \eta_{353} + y_{451} \eta_{453}, 0], \\ & \mathbb{E}_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + a_{32} \alpha_{35} + a_{42} \alpha_{45} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{142} \xi_{145} + \\ & \quad x_{152} \xi_{155} + x_{232} \xi_{235} + x_{242} \xi_{245} + x_{252} \xi_{255} + x_{342} \xi_{345} + x_{352} \xi_{355} + x_{452} \xi_{455}, 0] \} \end{aligned}$$

pdf

```
In[*]:= Define[bmtj,k→i = Module[{i0, i1, j1, k1}, (Ri0,i1 // aΔti1→j1,k1 // Pj1,j // Pk1,k) /. i0 → i],
  bΔtk→i,j = Module[{i1, j1, k1, k2}, ((Ri,i1 Rj,j1 // amti1,j1→k1) // Pk1,k2) /. k2 → k],
  bSti = Module[{i1, i2}, (Ri,i1 // aSti1 // Pi1,i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri,i1 // aSti1 // Pi1,i2) /. i2 → i]
Define[Ri,j = Ri,j // bSti]
```

In[*]:= **Block**[{**\$k** = 1, **\$n** = 2}, **bmt**_{1,2→3}]

Time: 0.22 seconds Out[]=

$$\mathbb{E}_{\{1,2\} \rightarrow \{3\}} [b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{121} + y_{123} \eta_{122}, -2 y_{123} \beta_{12} \eta_{121} + 2 y_{123} \beta_{11} \eta_{122}]$$

Testing **A** and **B** dually paired Hopf algebras

In[*]:= **\$k** = 1; **\$n** = 3;

co-associativity

In[*]:= {(**a** Δ **t**_{1→1,2} // **a** Δ **t**_{2→2,3}) $\equiv$ (**a** Δ **t**_{1→2,3} // **a** Δ **t**_{2→1,2}), (**b** Δ **t**_{1→1,2} // **b** Δ **t**_{2→2,3}) $\equiv$ (**b** Δ **t**_{1→2,3} // **b** Δ **t**_{2→1,2}) }

Time: 0.22 seconds Out[]=

{True, True}

algebra morphism

In[*]:= {(**a** Δ **t**_{i→1,2} **a** Δ **t**_{j→3,4} // **a****m****t**_{1,3→i} // **a****m****t**_{2,4→j}) $\equiv$ (**a****m****t**_{i,j→k} // **a** Δ **t**_{k→i,j}),
(**b** Δ **t**_{i→1,2} **b** Δ **t**_{j→3,4} // **b****m****t**_{1,3→i} // **b****m****t**_{2,4→j}) $\equiv$ (**b****m****t**_{i,j→k} // **b** Δ **t**_{k→i,j}) }

Time: 0.44 seconds Out[]=

{True, True}

associativity

In[*]:= {(**a****m****t**_{1,2→k} // **a****m****t**_{k,3→k}) $\equiv$ (**a****m****t**_{2,3→k} // **a****m****t**_{1,k→k}), (**b****m****t**_{1,2→k} // **b****m****t**_{k,3→k}) $\equiv$ (**b****m****t**_{2,3→k} // **b****m****t**_{1,k→k}) }

Time: 0.12 seconds Out[]=

{True, True}

antipode

In[*]:= {**a** Δ **t**_{i→1,2} // **a****S****t**₁ // **a****m****t**_{1,2→1}, **a** Δ **t**_{i→1,2} // **a****S****t**₂ // **a****m****t**_{1,2→1},
b Δ **t**_{i→1,2} // **b****S****t**₁ // **b****m****t**_{1,2→1}, **b** Δ **t**_{i→1,2} // **b****S****t**₂ // **b****m****t**_{1,2→1} }

Time: 0.89 seconds Out[]=

{ $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$, $\mathbb{E}_{\{i\} \rightarrow \{1\}} [0, 0]$ }

inverse antipode

In[*]:= {**a****S****t**_i // $\overline{\mathbf{aSt}_i}$, **b****S****t**_i // $\overline{\mathbf{bSt}_i}$ }

Time: 3.53 seconds Out[]=

{ $\mathbb{E}_{\{i\} \rightarrow \{i\}} [a_{1i} \alpha_{1i} + a_{2i} \alpha_{2i} + x_{12i} \xi_{12i} + x_{13i} \xi_{13i} + x_{23i} \xi_{23i}, 0]$,
 $\mathbb{E}_{\{i\} \rightarrow \{i\}} [b_{1i} \beta_{1i} + b_{2i} \beta_{2i} + y_{12i} \eta_{12i} + y_{13i} \eta_{13i} + y_{23i} \eta_{23i}, 0]$ }

Pairing

In[*]:= {(**a****m****t**_{1,2→3} // **P**_{3,-3}) $\equiv$ (**b** Δ **t**_{-3→-1,-2} // **P**_{1,-1} // **P**_{2,-2}),
(**b****m****t**_{1,2→3} // **P**_{-3,3}) $\equiv$ (**a** Δ **t**_{-3→-1,-2} // **P**_{-1,1} // **P**_{-2,2}),
(**a****S****t**₁ // **P**_{1,-1}) $\equiv$ ($\overline{\mathbf{bSt}_{-1}}$ // **P**_{1,-1}), ($\overline{\mathbf{aSt}_1}$ // **P**_{1,-1}) $\equiv$ ($\overline{\mathbf{bSt}_{-1}}$ // **P**_{1,-1}) }

Time: 0.23 seconds Out[]=

{True, True, True, True}

Testing Double

pdf

```
In[ ]:= Define[dmi,j→k = Module[{i1, i2, i3, i4},
  ((sYi→i4,i4,i1,i1 // aΔti1→i1,i2 // aΔti2→i2,i3 // aSti3) (sYj→-i1,-i1,-i4,-i4 // bΔt-i1→-i1,-i2 //
    bΔt-i2→-i2,-i3) // (Pi1,-i3 Pi3,-i1 amti2,-i4→k bmti4,-i2→k))]
Define[dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dSi = sYi→1,1,2,2 // (bSt1 aSt2) // dm2,1→i,
  dΔi→j,k = (bΔti→1,3 aΔti→4,2) // (dm3,4→k dm1,2→j)]
(*Notice we take co-opposite on the A-side! to make the double*)
```

```
In[ ]:= {$k = 1, $n = 3, h = 1};
dmi,j→k // 12U // CF // Short
```

Time: 47.70 seconds Out[]=

$$\mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} \alpha_{1i} + a_{1k} \alpha_{1j} + a_{2k} \alpha_{2i} + \langle\langle 33 \rangle\rangle + \frac{x_{23k} \mathcal{A}_{2i} \xi_{23j}}{\sqrt{\mathcal{A}_{1i}}}, \right. \\ \left. - 2 y_{12k} \beta_{1j} \eta_{12i} + \langle\langle 344 \rangle\rangle + \frac{3 \langle\langle 6 \rangle\rangle \xi_{23j}}{B_{1k} \mathcal{A}_{1i}} \right]$$

associativity

```
In[ ]:= (dm1,2→k // dmk,3→k) ≡ (dm2,3→k // dm1,k→k)
```

Time: 7.22 seconds Out[]=

True

co-associativity

```
In[ ]:= (dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→2,3 // dΔ2→1,2)
```

Time: 1.86 seconds Out[]=

True

algebra morphism

```
In[ ]:= (dΔ1→1,2 dΔ2→3,4 // (dm1,3→1 dm2,4→2)) ≡ (dm1,2→3 // dΔ3→1,2)
```

Time: 82.44 seconds Out[]=

True

antipode

```
In[ ]:= {dΔi→1,2 // dS1 // dm1,2→1, dΔi→1,2 // dS2 // dm1,2→1, dS1 // dS1}
```

Time: 400.97 seconds Out[]=

$$\{ \mathbb{E}_{\{i\} \rightarrow \{1\}} [\mathbf{0}, \mathbf{0}], \mathbb{E}_{\{i\} \rightarrow \{1\}} [\mathbf{0}, \mathbf{0}], \mathbb{E}_{\{1\} \rightarrow \{1\}} [a_{11} \alpha_{11} + a_{21} \alpha_{21} + b_{11} \beta_{11} + \\ b_{21} \beta_{21} + y_{121} \eta_{121} + y_{131} \eta_{131} + y_{231} \eta_{231} + x_{121} \xi_{121} + x_{131} \xi_{131} + x_{231} \xi_{231}, \mathbf{0}] \}$$

quasi-triangular axioms

```
In[*]:= (R1,3 // dΔ1→1,2) ≡ (R1,3 R2,4 // dm3,4→3)
(R1,3 // dΔ3→2,3) ≡ (R1,3 R0,2 // dm1,0→1)
(dΔi→k,j R1,2 // dmj,1→1 // dmk,2→2) ≡ (R1,2 dΔi→j,k // dm1,j→1 // dm2,k→2)
```

Time: 9.41 seconds Out[]=

True

Out[*]=

True

Out[*]=

True

```
In[*]:= (R1,2 // aSt2) ≡ (R̄1,2)
```

Time: 0.08 seconds Out[]=

True

Braid-like Reidemeister moves: R2,R3

```
In[*]:= {R1,2 R̄3,4 // dm1,3→1 // dm2,4→2, R̄1,2 R3,4 // dm1,3→1 // dm2,4→2,
(R1,2 R4,3 R5,6 // dm1,4→1 // dm2,5→5 // dm3,6→3) ≡ (R1,6 R2,3 R4,5 // dm1,4→1 // dm2,5→5 // dm3,6→3) }
```

Time: 12.17 seconds Out[]=

{E_{{ }→{1,2}} [0, 0], E_{{ }→{1,2}} [0, 0], True}

```
In[*]:= Block[{ $k = 2, $n = 3, h̄ = 1 }, dmi,j→k // 12U // CF]
```

Time: 18271.69 seconds Out[]=

```
In[*]:= PrintProfile[]
```

Time: 0.06 seconds Out[]=

ProfileRoot is root. Profiled time: 16908.9

(6) 53.466/ 16629.843 above Boot

(24675) 279.082/ 279.082 above CF

CF: called 974127 times, time in 15807.8/15807.8

(16977) 152.488/ 152.488 under Boot

(24675) 279.082/ 279.082 under ProfileRoot

(483) 15170.329/ 15170.329 under Seepage

(11573) 8.895/ 8.895 under TriangularDSolve

(920419) 196.959/ 196.959 under UCF

NCM: called 5266 times, time in 817.677/1204.65

(4) 659.641/ 713.328 under TTR

(3) 1.636/ 3.265 under Boot

(3) 37.519/ 40.437 under Invert

(5256) 118.881/ 447.616 under UB

(506363) 201.465/ 386.969 above UCF

UCF: called 510163 times, time in 206.803/403.762

(8) 0.063/ 0.391 under TTR

(4) 1.110/ 2.078 under SBB

(8) 0.095/ 0.220 under SG

(960) 1.328/ 3.669 under AdNil

(69) 0.079/ 0.157 under Boot

(14) 0.267/ 0.516 under Invert

```

( 506363) 201.465/ 386.969 under NCM
( 45) 0.172/ 4.406 under TT
( 2628) 2.083/ 4.919 under UB
( 1) 0/ 0.015 under amt[1, 3]
( 1) 0.015/ 0.047 under amt[2, 3]
( 31) 0.016/ 0.031 under aΔt[1, 3]
( 31) 0.110/ 0.344 under aΔt[2, 3]
( 920419) 196.959/ 196.959 above CF
Boot: called 19 times, time in 54.106/49403.5
( 13) 0.640/ 32773.671 under Boot
( 6) 53.466/ 16629.843 under ProfileRoot
( 13) 0.640/ 32773.671 above Boot
( 16977) 152.488/ 152.488 above CF
( 3) 0/ 0.313 above Invert
( 3) 1.636/ 3.265 above NCM
( 6) 0.016/ 19.202 above Seepage
( 3) 0.346/ 0.501 above TriangularDSolve
( 3) 0/ 0.296 above TT
( 69) 0.079/ 0.157 above UCF
( 6) 0.109/ 0.109 above ①
( 1) 0.063/ 4.734 above amt[1, 3]
( 1) 0.250/ 47.031 above amt[2, 3]
( 1) 0.267/ 31.500 above aΔt[1, 3]
( 1) 2.125/ 16316.141 above aΔt[2, 3]
①: called 1027 times, time in 11.739/11.739
( 2) 0.109/ 0.109 under TTR
( 4) 3.281/ 3.281 under SBB
( 6) 0.329/ 0.329 under SG
( 6) 0.109/ 0.109 under Boot
( 1009) 7.911/ 7.911 under Seepage
TriangularDSolve: called 7 times, time in 3.652/12.547
( 3) 0.346/ 0.501 under Boot
( 1) 0.282/ 0.328 under amt[1, 3]
( 1) 0.591/ 1.203 under amt[2, 3]
( 1) 0.235/ 0.515 under aΔt[1, 3]
( 1) 2.198/ 10.000 under aΔt[2, 3]
( 11573) 8.895/ 8.895 above CF
Seepage: called 483 times, time in 2.917/22420.
( 2) 0.015/ 31.172 under TTR
( 4) 0.841/ 15424.063 under SBB
( 6) 0.062/ 151.811 under SG
( 6) 0.016/ 19.202 under Boot
( 465) 1.983/ 6793.770 under Seepage
( 933) 0.344/ 110.047 above AdNil
( 62) 0.267/ 335.013 above AdPotPlus
( 14) 0/ 0.031 above AdPoty
( 483) 15170.329/ 15170.329 above CF

```

```

( 465)      1.983/ 6793.770 above Seepage
( 1009)     7.911/   7.911 above 0
aΔt[2, 3]: called 1 times, time in 2.125/16316.1
( 1)      2.125/ 16316.141 under Boot
( 1)      0.062/   0.062 above BB
( 1)      0.015/  748.453 above TTR
( 1)      0.062/ 15402.703 above SBB
( 1)      0/    147.563 above SG
( 1)      2.198/   10.000 above TriangularDSolve
( 1)      0/    4.891 above TT
( 31)     0.110/   0.344 above UCF
UB: called 2628 times, time in 0.464/452.999
( 1182)    0.278/   110.346 under AdNil
( 1446)    0.186/   342.653 under AdPoty
( 5256)   118.881/  447.616 above NCM
( 2628)    2.083/    4.919 above UCF
AdNil: called 960 times, time in 0.344/114.359
( 933)     0.344/   110.047 under Seepage
( 27)      0/    4.312 under TT
( 1182)    0.278/   110.346 above UB
( 960)     1.328/    3.669 above UCF
AdPotPlus: called 74 times, time in 0.298/343.122
( 62)      0.267/   335.013 under Seepage
( 12)      0.031/    8.109 under TT
( 1432)    0.202/   342.824 above AdPoty
aΔt[1, 3]: called 1 times, time in 0.267/31.5
( 1)      0.267/   31.500 under Boot
( 1)      0.078/   0.078 above BB
( 1)      0/    1.703 above TTR
( 1)      0.016/   26.797 above SBB
( 1)      0/    1.109 above SG
( 1)      0.235/   0.515 above TriangularDSolve
( 1)      0/    1.000 above TT
( 31)     0.016/   0.031 above UCF
amt[2, 3]: called 1 times, time in 0.25/47.031
( 1)      0.250/   47.031 under Boot
( 1)      0/    9.125 above TTL
( 1)      0.016/   32.953 above TTR
( 1)      0/    3.453 above SG
( 1)      0.591/    1.203 above TriangularDSolve
( 1)      0.015/   0.047 above UCF
AdPoty: called 1446 times, time in 0.202/342.855
( 1432)    0.202/   342.824 under AdPotPlus
( 14)      0/    0.031 under Seepage
( 1446)    0.186/   342.653 above UB
BB: called 2 times, time in 0.14/0.14
( 1)      0.078/   0.078 under aΔt[1, 3]

```

```

(      1)      0.062/      0.062 under aΔt[2, 3]
SBB: called 2 times, time in 0.078/15429.5
(      1)      0.016/      26.797 under aΔt[1, 3]
(      1)      0.062/ 15402.703 under aΔt[2, 3]
(      4)      0.841/ 15424.063 above Seepage
(      4)      1.110/      2.078 above UCF
(      4)      3.281/      3.281 above ①
amt[1, 3]: called 1 times, time in 0.063/4.734
(      1)      0.063/      4.734 under Boot
(      1)      0/      1.515 above TTL
(      1)      0/      2.578 above TTR
(      1)      0/      0.235 above SG
(      1)      0.282/      0.328 above TriangularDSolve
(      1)      0/      0.015 above UCF
TTR: called 4 times, time in 0.031/785.687
(      1)      0/      2.578 under amt[1, 3]
(      1)      0.016/      32.953 under amt[2, 3]
(      1)      0/      1.703 under aΔt[1, 3]
(      1)      0.015/      748.453 under aΔt[2, 3]
(      4)      0.016/      40.656 above Invert
(      4)      659.641/      713.328 above NCM
(      2)      0.015/      31.172 above Seepage
(      8)      0.063/      0.391 above UCF
(      2)      0.109/      0.109 above ①
Invert: called 7 times, time in 0.016/40.969
(      4)      0.016/      40.656 under TTR
(      3)      0/      0.313 under Boot
(      3)      37.519/      40.437 above NCM
(      14)      0.267/      0.516 above UCF
TT: called 45 times, time in 0./35.167
(      2)      0/      10.640 under TTL
(      3)      0/      0.296 under Boot
(      38)      0/      18.340 under TT
(      1)      0/      1.000 under aΔt[1, 3]
(      1)      0/      4.891 under aΔt[2, 3]
(      27)      0/      4.312 above AdNil
(      12)      0.031/      8.109 above AdPotPlus
(      38)      0/      18.340 above TT
(      45)      0.172/      4.406 above UCF
SG: called 4 times, time in 0./152.36
(      1)      0/      0.235 under amt[1, 3]
(      1)      0/      3.453 under amt[2, 3]
(      1)      0/      1.109 under aΔt[1, 3]
(      1)      0/      147.563 under aΔt[2, 3]
(      6)      0.062/      151.811 above Seepage
(      8)      0.095/      0.220 above UCF
(      6)      0.329/      0.329 above ①

```

TTL: called 2 times, time in 0./10.64

(	1)	0/	1.515	under amt[1, 3]
(	1)	0/	9.125	under amt[2, 3]
(	2)	0/	10.640	above TT