

Hidden Utilities and T_EX

Headers

pdf

Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

pdf

Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

pdf

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

pdf

This version: April 2020. Original version: July 1994.

(Alt) In[]:=

```
MakePDF[]
```

pdf

```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

pdf

```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ , op_] := Collect[Normal@Series[_ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_] := SS[_ , Together];
```

pdf

```

In[ ]:= $N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
y[i_, j_] := y<>i<>j; b[i_] := b<>i; B[i_] := B<>i;
η[i_, j_] := η<>i<>j; β[i_] := β<>i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];

```

```

In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

pdf

The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

pdf

```

In[ ]:= Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
  ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
  η[i, j]* = y[i, j]
]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
  b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_{αβ})* := (u*)_{αβ};
F[u_i] := F[u_i] = ToExpression["F" <> ToString[u]]_i

```

pdf

```

In[ ]:= LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;

```

Upper and lower case variables:

pdf

```

In[*]:= Module[{i, s, p, c, d, χ, f, v},
  U21Rule = Union @@ Table[As[i, 1, $N - 1], {A[i]_s^p → e^{p α[i]_s}, B[i]_s^p → e^{-p ħ b[i]_s}}];
  12URule = Append[Union @@ Table[As[i, 1, $N - 1],
    {e^{d_- + c_- b[i]_s} → B[i]_s^{-c/ħ} e^d, e^{d_- + c_- α[i]_s} → A[i]_s^c e^d}], e^χ → e^{Expand[12U[χ]}];
  U21[ε_] := ε /. U21Rule;
  12U[ε_] := ε //. 12URule;
  Do[As[i, 1, $N - 1], With[{αi = α[i], Ai = A[i], bi = b[i], Bi = B[i]},
    Dαi_ [f_] := ∂αi_ f + Ai_s ∂Ai_s f;
    Dbi_ [f_] := ∂bi_ f - ħ Bi_s ∂Bi_s f;
  ]];
  Dv_ [f_] := ∂v f;
];

```

```

In[*]:= {U21Rule, 12URule}

```

```

Time: 0.00 seconds Out:=

```

$$\left\{ \left\{ B1_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b1_{s\$}}, B2_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b2_{s\$}}, B3_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b3_{s\$}}, B4_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b4_{s\$}}, B5_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b5_{s\$}}, \right. \right. \\
B6_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b6_{s\$}}, B7_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b7_{s\$}}, B8_{s\$}^{p\$} \rightarrow e^{-p\$ \hbar b8_{s\$}}, A1_{s\$}^{p\$} \rightarrow e^{p\$ \alpha1_{s\$}}, A2_{s\$}^{p\$} \rightarrow e^{p\$ \alpha2_{s\$}}, \\
A3_{s\$}^{p\$} \rightarrow e^{p\$ \alpha3_{s\$}}, A4_{s\$}^{p\$} \rightarrow e^{p\$ \alpha4_{s\$}}, A5_{s\$}^{p\$} \rightarrow e^{p\$ \alpha5_{s\$}}, A6_{s\$}^{p\$} \rightarrow e^{p\$ \alpha6_{s\$}}, A7_{s\$}^{p\$} \rightarrow e^{p\$ \alpha7_{s\$}}, A8_{s\$}^{p\$} \rightarrow e^{p\$ \alpha8_{s\$}} \left. \right\}, \\
\left\{ e^{d\$_- + c\$_- b1_{s\$}} \rightarrow e^{d\$} B1_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b2_{s\$}} \rightarrow e^{d\$} B2_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b3_{s\$}} \rightarrow e^{d\$} B3_{s\$}^{-\frac{c\$}{\hbar}}, \right. \\
e^{d\$_- + c\$_- b4_{s\$}} \rightarrow e^{d\$} B4_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b5_{s\$}} \rightarrow e^{d\$} B5_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b6_{s\$}} \rightarrow e^{d\$} B6_{s\$}^{-\frac{c\$}{\hbar}}, \\
e^{d\$_- + c\$_- b7_{s\$}} \rightarrow e^{d\$} B7_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- b8_{s\$}} \rightarrow e^{d\$} B8_{s\$}^{-\frac{c\$}{\hbar}}, e^{d\$_- + c\$_- \alpha1_{s\$}} \rightarrow e^{d\$} A1_{s\$}^{c\$}, \\
e^{d\$_- + c\$_- \alpha2_{s\$}} \rightarrow e^{d\$} A2_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha3_{s\$}} \rightarrow e^{d\$} A3_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha4_{s\$}} \rightarrow e^{d\$} A4_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha5_{s\$}} \rightarrow e^{d\$} A5_{s\$}^{c\$}, \\
e^{d\$_- + c\$_- \alpha6_{s\$}} \rightarrow e^{d\$} A6_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha7_{s\$}} \rightarrow e^{d\$} A7_{s\$}^{c\$}, e^{d\$_- + c\$_- \alpha8_{s\$}} \rightarrow e^{d\$} A8_{s\$}^{c\$}, e^{\chi_-} \rightarrow e^{\text{Expand}[12U[\chi]}] \left. \right\}$$

```

In[*]:= {Db3i_ /@ {b3i^5, B3i^7}, Dα3i_ /@ {α3i^5, A3i^7}}

```

```

Time: 0.01 seconds Out:=

```

$$\left\{ \left\{ 5 b3i^4, -7 \hbar B3i^7 \right\}, \left\{ 5 \alpha3i^4, 7 A3i^7 \right\} \right\}$$

Canonical Form:

pdf

```

In[*]:= CF[ε_] := PPCF@Module[{s1},
  s1 = If[$k == ∞,
    Expand[ε] /. e^{p_-} → e^{Factor@p},
    Expand[ε] /. e^{p_-} → e^{Factor@p} /. q^{n_-} → ∑_{k=0}^{sk} \frac{(\hbar \in \gamma n)^k}{k!} /. e^{k_-} /. k > $k → 0
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern,∞],CCF] /. CCF→Factor*)
];

```

pdf

Weights: (Need confirmation!!!)

pdf

```

In[ ]:= (*Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,i-j};
  Wt[ξ[i,j]]={0,j-i}; Wt[η[i,j]]={1,j-i};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[α[i]]={0,0}; Wt[β[i]]={1,0};
];
Wt[u_]:=Wt[u];*)
Do[As[i,1,$N-1],As[j,i+1,$N],
  Wt[x[i,j]]={1,i-j}; Wt[y[i,j]]={0,-(i-j)};
  Wt[ξ[i,j]]={0,-(i-j)}; Wt[η[i,j]]={1,(i-j)};
];
Do[As[i,1,$N-1],
  Wt[a[i]]={1,0}; Wt[b[i]]={0,0};
  Wt[α[i]]={0,0}; Wt[β[i]]={1,0};
];
Wt[u_]:=Wt[u];

```

pdf

```

In[ ]:= (* q~eh γ ∈; Ai[n]~eh ∈ ai; *)

```

pdf

```

In[ ]:= $n = 5; $k = ∞; $E := {$n, $k}

```

pdf

```

In[ ]:= $EPBWGens := Flatten@{
  Table[As[i,1,$n-1],As[j,i+1,$n],x[i,j]],
  Table[As[i,1,$n-1],{a[i],A[i][_]}]
}
$EPBWRule := Module[{k=0},Flatten@Table[{g→++k,gi_→{i,k}},{g,$EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x,y}/.$EPBWRule]; x_ < y_ := !OrderedQ[{y,x}/.$EPBWRule];
$PBWGens := DeleteCases[$EPBWGens,_[_]]

```

```

In[ ]:= $PBWGens*

```

Time: 0.01 seconds Out[]:=

```

{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}

```

pdf

```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a @ L[AB], k], r]
  }]],
  U[____], F
] /. F → CF

```

pdf

```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

pdf

```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

pdf

```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

pdf

```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

pdf

```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
S̄i[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i→i[S̄i[f] S̄i1[Ui1[fs]]]
}];
```

pdf

The $\mathbb{A}$ -algebra

pdf

```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij, Am[n]]
];
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[*]=

```
γ U0[x12] + U0[x12, a1]
```

pdf

We take the reduction rules RR for xij's from Yamane page 509, rewritten in Monoblog/260928:

$$\begin{aligned}
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \\
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= q^2 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} - q \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \\
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} + (q^{-2} - q^2) \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \\
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= q^{-2} \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \\
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} \\
 \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix} &= q^{-2} \begin{pmatrix} \text{---} & \text{---} \\ \text{---} & \text{---} \end{pmatrix}
 \end{aligned}$$

pdf

```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.08 seconds Out:=
(-1 - γ ∈ ħ) U2[x24] U3[x12] + (1 + 2 γ ∈ ħ) U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 18.11 seconds Out:=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```

Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}

```

Out[8]=

{ }

pdf

The Co-Product, Co-Associativity, and the Hopf Axiom

pdf

According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

pdf

In[8]:=

```

Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1 -> k[mj, j1 -> j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]]
]

```

```

In[*]:= Block[{$k = ∞, $n = 5}, Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]]

Time: 0.08 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[0], A1[-2], A4[0], A1[0], A2[0], A4[-2], A3[-2], A3[1], A3[2], A2[-1]}

Out[*]=
{ }

```

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[-1], A1[2], A3[-1], A4[2], A2[0], A4[2], A3[2], A4[1], A2[-2], A4[-1]}
Out[*]=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[*]:= $k = 1
Time: 0.02 seconds Out[*]=
1

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1/@Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[*]=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

pdf

The Antipode

pdf

In[]:=

```

Do[As[i, 1, $N - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]]; (*?*)
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q2 Uj[x[i, i + 1]]; (*?*)
];
Do[As[i, 1, $N - 2], As[n, i + 2, $N],
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]];
  Sj-[x[i, n]] = mj1, j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]]
]

```

```
In[*]:= Block[{ $k = ∞, $n = 5 }, Column@Module[{ Bas = $PBWGens, P },
  Table[As[P, Ui /@ Bas], P → {Si[P],  $\bar{S}_i$ [P]}]
]]
```

Time: 0.08 seconds Out[]:=

$$\begin{aligned}
U_i[x_{12}] &\rightarrow \left\{ -\frac{U_i[x_{12}]}{q^2}, -q^2 U_i[x_{12}] \right\} \\
U_i[x_{13}] &\rightarrow \left\{ -\frac{U_i[x_{13}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{23}], -q^6 U_i[x_{13}] + (-q^3 + q^7) U_i[x_{12}, x_{23}] \right\} \\
U_i[x_{14}] &\rightarrow \left\{ -\frac{U_i[x_{14}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{24}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{34}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{34}], \right. \\
&\quad \left. -q^{10} U_i[x_{14}] + (-q^7 + q^{11}) U_i[x_{12}, x_{24}] + \right. \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{13}, x_{34}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{12}, x_{23}, x_{34}] \right\} \\
U_i[x_{15}] &\rightarrow \left\{ -\frac{U_i[x_{15}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{25}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{35}] + \right. \\
&\quad \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{14}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{35}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{24}, x_{45}] + \right. \\
&\quad \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{13}, x_{34}, x_{45}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q\right) U_i[x_{12}, x_{23}, x_{34}, x_{45}], \\
&\quad -q^{14} U_i[x_{15}] + (-q^{11} + q^{15}) U_i[x_{12}, x_{25}] + (-q^{11} + q^{15}) U_i[x_{13}, x_{35}] + (-q^{11} + q^{15}) U_i[x_{14}, x_{45}] + \\
&\quad (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{23}, x_{35}] + (-q^8 + 2q^{12} - q^{16}) U_i[x_{12}, x_{24}, x_{45}] + \\
&\quad \left. (-q^8 + 2q^{12} - q^{16}) U_i[x_{13}, x_{34}, x_{45}] + (-q^5 + 3q^9 - 3q^{13} + q^{17}) U_i[x_{12}, x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{23}] &\rightarrow \left\{ -\frac{U_i[x_{23}]}{q^2}, -q^2 U_i[x_{23}] \right\} \\
U_i[x_{24}] &\rightarrow \left\{ -\frac{U_i[x_{24}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{34}], -q^6 U_i[x_{24}] + (-q^3 + q^7) U_i[x_{23}, x_{34}] \right\} \\
U_i[x_{25}] &\rightarrow \left\{ -\frac{U_i[x_{25}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{35}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{24}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{23}, x_{34}, x_{45}], \right. \\
&\quad \left. -q^{10} U_i[x_{25}] + (-q^7 + q^{11}) U_i[x_{23}, x_{35}] + \right. \\
&\quad \left. (-q^7 + q^{11}) U_i[x_{24}, x_{45}] + (-q^4 + 2q^8 - q^{12}) U_i[x_{23}, x_{34}, x_{45}] \right\} \\
U_i[x_{34}] &\rightarrow \left\{ -\frac{U_i[x_{34}]}{q^2}, -q^2 U_i[x_{34}] \right\} \\
U_i[x_{35}] &\rightarrow \left\{ -\frac{U_i[x_{35}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{34}, x_{45}], -q^6 U_i[x_{35}] + (-q^3 + q^7) U_i[x_{34}, x_{45}] \right\} \\
U_i[x_{45}] &\rightarrow \left\{ -\frac{U_i[x_{45}]}{q^2}, -q^2 U_i[x_{45}] \right\} \\
U_i[a_1] &\rightarrow \{-U_i[a_1], -U_i[a_1]\} \\
U_i[a_2] &\rightarrow \{-U_i[a_2], -U_i[a_2]\} \\
U_i[a_3] &\rightarrow \{-U_i[a_3], -U_i[a_3]\} \\
U_i[a_4] &\rightarrow \{-U_i[a_4], -U_i[a_4]\}
\end{aligned}$$

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→2,3 // S2 // m2,3→1)
  ]]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[-1], A2[-1], A4[1], A4[1], A3[2], A1[1], A2[1], A3[-1], A3[2], A4[-2]}

Out[*]=
{x12 -> 0, x13 -> 0, x14 -> 0, x15 -> 0, x23 -> 0, x24 -> 0, x25 -> 0, x34 -> 0, x35 -> 0, x45 -> 0,
  a1 -> 0, a2 -> 0, a3 -> 0, a4 -> 0, A4[-1] -> U1[], A2[-1] -> U1[], A4[1] -> U1[], A4[1] -> U1[],
  A3[2] -> U1[], A1[1] -> U1[], A2[1] -> U1[], A3[-1] -> U1[], A3[2] -> U1[], A4[-2] -> U1[]}

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> UCF[(P // S1 // S̄1) - P]
  ]]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[2], A2[-2], A1[-1], A1[1], A3[1], A3[-1], A4[1], A1[0], A3[1], A1[-1]}

Out[*]=
{}

```

pdf

The Generating Functions Engine

pdf

Canonical Forms:

pdf

```

In[*]:= LogReduce[ε_] :=
  ε /. c_ * Log[a_] -> Log@Factor[ac] /. Log[a_] + Log[b_] -> Log@Factor[a b];
CF[ε_E] := CF /@ MapAt[LogReduce, ε, 1];
CF[ε : E__[_]] := CF /@ ε;

```

pdf

E operations:

pdf

```

In[*]:=  $\mathcal{E}_E[\$] := \text{Length}[\mathcal{E}] - 1$ ;  $\mathbb{E}_E[\mathcal{ES}\_\_\_\_\_\_][\$] := \mathbb{E}[\mathcal{ES}][\$]$ ;
 $\mathcal{E}_E[k\_Integer] := \mathcal{E}[[k + 1]]$ ;  $\mathbb{E}_E[k\_Integer][k\_Integer] := \{\mathcal{ES}\}[[k + 1]]$ ;
 $\mathbb{E} /: \mathcal{E}_1\_E \equiv \mathcal{E}_2\_E := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \ \&, \mathcal{E}_1, \mathcal{E}_2, \text{And}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E}1s] \equiv \mathbb{E}[\mathcal{E}2s])$ ;
 $\mathbb{E} /: \mathcal{E}_1\_E * \mathcal{E}_2\_E := \mathbb{E}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}]$ ;

```

pdf

```

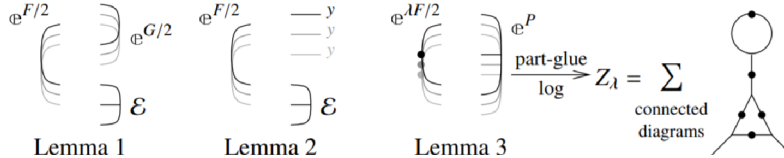
In[*]:=  $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] // \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] := \text{Module}[\{\text{is} = r1 \cap d2, \text{lvs}, \text{llvs}\},$ 
 $\text{lvs} = \text{Cases}[\{\mathcal{E}1s\}, (\text{Alternatives}@\text{LatinVars})_{i\_} /; \text{MemberQ}[\text{is}, i], \infty] \cup$ 
 $\text{Cases}[\{\mathcal{E}2s\}, (\text{Alternatives}@\text{GreekVars})_{i\_} /; \text{MemberQ}[\text{is}, i], \infty]^*$ ;
 $\text{llvs} = \text{lvs} /. u_{i\_} \rightarrow u_{\$i}$ ;
 $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])}@\text{Table}[\text{Zip}[\text{llvs} \cup \text{llvs}^*, \{\{F / @ \text{llvs}^*\} \cdot (F / @ \text{llvs}), \text{Times}[\mathbb{E}[\mathcal{E}1s] /. \text{Thread}[\text{lvs} \rightarrow \text{llvs}], \mathbb{E}[\mathcal{E}2s] /. \text{Thread}[\text{lvs}^* \rightarrow \text{llvs}^*]\}]]]$ 

```

pdf

Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

pdf



pdf

Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ii} ’s and the y_{ii} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

pdf

```

In[*]:=  $\text{Zip}_{vs\_}[\{\mathcal{F}\_, \mathcal{E}\_}\] := \text{Module}[\{\text{dvs} = \text{Select}[\text{vs}, (\text{Wt}[\#] == \{0, 0\} \vee \text{Wt}[\#] == \{1, 0\}) \ \&]\},$ 
 $\{\mathcal{F}, \mathcal{E}\} // \text{Zip1}_{vs}(* // \text{Zip2}_{\text{Complement}[\text{vs}, \text{dvs}]}*) // \text{Zip2}_{\text{dvs}} //$ 
 $\text{Zip3}_{\text{Complement}[\text{vs}, \text{dvs}]} // \text{Zip3}_{\text{dvs}}$ 
 $\] // \text{Last}$ 

```

pdf

Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

pdf

```

In[*]:= Zip1_{ } = Identity;
Zip1_{vs_@{\mathcal{F}_-, \mathbb{E}[Q_-, P_{---}]}} := Module[{I, F, G, u, v},
  (*Echo[{vs, \mathcal{F}, Q, P}];*)
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, \partial_{F[u], F[v]} \mathcal{F}, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, \partial_{u, v} Q, 0], {u, vs}, {v, vs}];
  (*Echo@MatrixForm[I - G.F];*)
  {CF[(F /@ vs) . (F.Inverse[I - G.F]) . (F /@ vs) / 2],
   \mathbb{E}[CF[Q - PowerExpand@Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]

```

pdf

Getting rid of linear terms.

Lemma 2. $\left\langle F : \mathcal{E}_{\oplus \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{E}^{\frac{1}{2} \sum_{i, j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

pdf

```

In[*]:= Zip2_{ } = Identity;
Zip2_{vs_@{\mathcal{F}_-, \mathbb{E}[Q_-, P_{---}]}} := Module[{F, Y, u, v},
  (*EchoLabel["Zip2"] [{vs, \mathcal{F}, Q, P}];*)
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[\partial_{F[u], F[v]} \mathcal{F}], 0], {u, vs}, {v, vs}];
  Y = Table[\partial_v Q, {v, vs}] /. (*AlsoUpper@*) Table[v \to 0, {v, vs}];
  CF /@ ({\mathcal{F}, \mathbb{E}[Q - Y.vs + Y.F.Y / 2, P]} /. (*AlsoUpper@*) Thread[vs \to vs + F.Y])
]

```

pdf

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \oplus^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i, j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda) (\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

pdf

```

In[*]:= Zip3vs@{ $\mathcal{F}$ _,  $\mathcal{E}$ _E} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  (*Echo[{"Zip3", vs,  $\mathcal{F}$ ,  $\mathcal{E}$ }]*)
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}$ @$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{F[u], F[v]} \mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = Du[Z[m, kk]];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = Dv[Z[m, kk, u]];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}$ @$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)} (Z[m, kk, u, v] +$ 
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}]]],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * F[u] * F[v] / 2, {u, vs}, {v, vs}],
    E @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}$ @$}]
  }) /. (*AlsoUpper@*) Table[v → 0, {v, vs}]
]

```

pdf

“Define” Code

pdf

```

In[*]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ =  $\mathcal{E}$ _] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[opnisp, $E, PPBoot@Block[{i, j, k}, opisp, $E =  $\mathcal{E}$ ; opnisp, $E]]];
    SD[opisp, op{is}, $E]; SD[opsis__, op{sis}]];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]

```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```
In[*]:= BeginProfile[]
```

Time: 0.06 seconds Out:=
ProfileRoot

```
In[*]:= $k = 1;
```

pdf

Ad, TT, Seepage

pdf

```
In[*]:= PotQ[0] = True;
PotQ[c_., U_ [aj_] (EUs : U_ [] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

pdf

```
In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := If[y == 0, 0, PPAdPoty[eUB[y,x]/y y]];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

pdf

```
In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

```
In[*]:= $k = 2
```

Time: 0.00 seconds Out:=
2

```
In[*]:= Ad[ε ħ Uj [x12] Uk []] [Uj [x23] Uk [a4]]
```

Time: 0.06 seconds Out:=

$$\left(-\epsilon \hbar - \gamma \epsilon^2 \hbar^2\right) U_j[x_{13}] U_k[a_4] + U_j[x_{23}] U_k[a_4] + 2 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, x_{23}] U_k[a_4]$$

```
In[*]:= PotQ[7 ħ Uj [x12] Uk [a3]]
```

Time: 0.02 seconds Out:=
 PotQ[7 ħ U_j [x₁₂] U_k [a₃]]

```
In[*]:= Ad[7 ħ Uj [x12] Uk [a3]] [Uj [x23] Uk [a4]]
```

Time: 0.09 seconds Out:=

$$U_j[x_{23}] U_k[a_4] + \left(-7 \hbar - 7 \gamma \epsilon \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{13}] U_k[a_3, a_4] +$$

$$\left(14 \gamma \epsilon \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{12}, x_{23}] U_k[a_3, a_4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{13}] U_k[a_3, a_3, a_4] +$$

$$98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{12}, x_{23}] U_k[a_3, a_3, a_4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x_{12}, x_{12}, x_{13}] U_k[a_3, a_3, a_3, a_4]$$

In[*]:= $\text{Ad}[\pi \mathbf{i} U_j[\mathbf{a1}]] [U_j[\mathbf{x23}] U_k[\mathbf{a4}]]$

Time: 0.05 seconds Out[]=

$$e^{\frac{i\pi\gamma}{2}} U_j[\mathbf{x23}] U_k[\mathbf{a4}]$$

pdf

In[*]:= $\text{TT}_{\lambda}[] = 0;$ (* Translated Tangent *)
 $\text{TT}_{\lambda}[\mathbf{xs}___, \mathbf{x}__] := \text{PP}_{\text{TT}} @ \text{UCF}[\text{Ad}[\mathbf{x}] [\text{TT}_{\lambda}[\mathbf{xs}]] + \partial_{\lambda} \mathbf{x}];$
 $\text{TT}_{\lambda}[\mathbf{xs_List}] := \text{TT}_{\lambda} @ \mathbf{xs}$

In[*]:= $\text{TT}_{\lambda}[\lambda \xi_{12} U_0[\mathbf{x12}] U_1[], \lambda \xi_{23} U_0[\mathbf{x23}] U_1[]]$

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[\mathbf{x12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[\mathbf{x13}] U_1[] + \xi_{23} U_0[\mathbf{x23}] U_1[]$$

pdf

In[*]:= $\text{O}_{i_}[\mathbf{poly_}] := \text{Total}[\text{CoefficientRules}[\mathbf{poly}, \#i \& / @ \$\text{PBWGens}] /. (\mathbf{p_} \rightarrow \mathbf{c_}) \Rightarrow \mathbf{c} U_i @ \text{Flatten} @ \text{MapThread}[\text{ConstantArray}, \{\$ \text{PBWGens}, \mathbf{p}\}]]];$
 $\text{O}_{i_is_}[\mathbf{poly_}] := \text{O}_{i_}[\text{O}_{is_}[\mathbf{poly}]];$
 $\mathbb{P}_{i_}[\mathbf{e_}] := \mathbf{e} / . U_i[\mathbf{xs}___] \Rightarrow \text{Times} @ @ (\#i \& / @ \{\mathbf{xs}\});$
 $\mathbb{P}_{i_is_}[\mathbf{poly_}] := \mathbb{P}_{i_}[\mathbb{P}_{is_}[\mathbf{poly}]];$

In[*]:= $\text{O}_i[\mathbf{x12}_i^2 \mathbf{x13}_i + 7 \mathbf{x23}_i^{17}]$

Time: 0.06 seconds Out[]=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}]$$

In[*]:= $\text{O}_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.03 seconds Out[]=

$$U_i[\mathbf{x12}, \mathbf{x12}, \mathbf{x13}] U_j[\mathbf{x24}] + 7 U_i[\mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}, \mathbf{x23}] U_j[\mathbf{x25}, \mathbf{x25}, \mathbf{x25}]$$

In[*]:= $\mathbb{P}_{i,j} @ \text{O}_{i,j}[\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3]$

Time: 0.06 seconds Out[]=

$$\mathbf{x12}_i^2 \mathbf{x13}_i \mathbf{x24}_j + 7 \mathbf{x23}_i^{17} \mathbf{x25}_j^3$$

pdf

In[*]:= $\text{Seepage}_{i_}[\{\}, \mathbf{c_}] = \mathbf{c};$
 $\text{Seepage}_{i_}[\xi\mathbf{s_List}, \mathbf{P_}] := \text{Module}[\{\mathbf{lx}\}, \mathbf{lx} = \$\text{PBWGens}[\text{Length} @ \xi\mathbf{s}]; \text{CF} @ \text{Collect}[\mathbf{P}, \mathbf{lx}_i, \mathbb{P}_i[\text{Ad}[\text{Last}[\xi\mathbf{s}] U_i[\mathbf{lx}]] [\text{O}_{i_}[\text{Seepage}_{i_}[\text{Most} @ \xi\mathbf{s}, \#]]]] \&]]];$
 $\text{Seepage}_{i_is_}[\xi\mathbf{s_List}, \xi\mathbf{ss_List}, \mathbf{P_}] := \text{Seepage}_{i_}[\xi\mathbf{s}, \text{Seepage}_{is_}[\xi\mathbf{ss}, \mathbf{P}]]$

In[*]:= **\$k = 1**

Time: 0.00 seconds Out[1]=

1

In[*]:= **\$PBWGens***

Time: 0.01 seconds Out[2]=

{ ξ_{12} , ξ_{13} , ξ_{14} , ξ_{15} , ξ_{23} , ξ_{24} , ξ_{25} , ξ_{34} , ξ_{35} , ξ_{45} , α_1 , α_2 , α_3 , α_4 }

In[*]:= **Seepage_i[\$PBWGens*, x35_i]**

Time: 0.16 seconds Out[3]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35_i} + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \in \xi_{45} \hbar x_{35_i} x_{45_i}$$

In[*]:= **Seepage_{i,j}[\$i & /@ \$PBWGens*, \$j & /@ \$PBWGens*, x35_i x45_j]**

Time: 0.25 seconds Out[4]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 + \alpha_3 j - \alpha_4 i - 2\alpha_4 j)} x_{35_i} x_{45_j} + 2 e^{-\frac{1}{2}(\alpha_2 - \alpha_3 + 3\alpha_4 i + 2\alpha_4 j)} \gamma \in \hbar x_{35_i} x_{45_i} x_{45_j} \xi_{45_i}$$

pdf

In[*]:=

```
Invertis__[$_] := Module[{A, B, U1, j}, (*  $\delta = A U[] + \epsilon B *$  *)
U1 = Times@@(U#[] & /@ {is});
A = Coefficient[$, U1] /.  $\epsilon \rightarrow 0$ ;
B = UCF[($ - A U1)];
UCF[
  A-1 (U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]
```

In[*]:= **\$k = 3;**

Invert_{i,j}[7 U_i[] U_j[] + ϵ U_i[x12] U_j[x23]]

Time: 0.09 seconds Out[5]=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \in U_i[x_{12}] U_j[x_{23}] + \frac{1}{343} \epsilon^2 U_i[x_{12}, x_{12}] U_j[x_{23}, x_{23}] - \frac{\epsilon^3 U_i[x_{12}, x_{12}, x_{12}] U_j[x_{23}, x_{23}, x_{23}]}{2401}$$

{\$k = 1, \$n = 3}

Time: 0.01 seconds Out[6]=

{2, 3}

pdf

In[]:=

```

TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
  PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      AppendTo[ss, s /. F_Function → ReplacePart[F, 2 → CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 → CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]

```

pdf

$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle &= \mathbb{C}\langle \lambda_{i,j}^{\pm 1} x \rangle \\
 TT(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f=f_i x_i, \quad g=\sum_{(i,j) \in I} g_{ij} w & e^f = F, \quad e^g = G, \quad R_\lambda = \mathbb{C}\langle FG \rangle, \quad \partial_\lambda F = F \partial_\lambda f \\
 TT(R_\lambda) &= R'_\lambda (\partial_\lambda R_\lambda) = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
 \end{aligned}$$

pdf

In[*]:=

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-k[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
  ];
  G = Series[εgg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ(ff.(#k & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], (α[m]i + α[m]j) a[m]k], 0],
    Total[Cases[fs, fp,-]] /. fp-,u- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]

```

pdf

In[]:=

```

Define[aΔti→j,k = PPaΔt[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
TTL = ES["TTL"] @
  TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
mons[-1] = {};
mons[p_] := mons[p - 1] ∪
  Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
ffj = Table[
  Sum[εp fp,uj[λ], {p, 0, $k}],
  {u, $PBWGens}
] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
ffk = ffj /. j → k;
fs = Flatten[Table[
  fp,ℙj,k[m],
  {p, 0, $k}, {m, mons[p]}
]];
gg = Sum[
  εp fp,ℙj,k[u][λ] ℙj,k[u],
  {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
];
G = Series[egg, {ε, 0, $k}];
S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
SG = ES["SG"] @ PPSG @ S[G];
TTR = ES["TTR"] @ PPTTR @ UCF[
  Inverttj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
diff = UCF[TTL - TTR];
eqns = Flatten[Table[
  Coefficient[Coefficient[diff, m], ε, p] == 0,
  {p, 0, $k}, {m, mons[p]}
]];
inits = (#[0] == 0) & /@ fs;
ss = TriangularDSolve[eqns, inits, fs, λ, CF];
E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
  If[p == 0, Sum[As[m, 1, $n - 1], α[m]i (a[m]j + a[m]k)], 0],
  Total[Cases[fs, fp,-]] /. fp,u => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
]
]]

```

```
In[*]:= { $k = 1, $n = 2 }; amti,j→k
(amt1,2→2 // amt2,3→1) ≡ (amt2,3→2 // amt1,2→1)
```

Time: 0.75 seconds Out[]=

$$E_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + x_{12k} \left(\xi_{12i} + e^{\alpha_{1i}} \xi_{12j} \right), 0 \right]$$

Out[*]=

True

```
In[*]:= { $k = 1, $n = 3 }; amti,j→k
(amt1,2→4 // amt4,3→5) ≡ (amt2,3→4 // amt1,4→5)
```

Time: 8.41 seconds Out[]=

$$E_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ \left. - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \right. \\ \left. 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right]$$

Out[*]=

True

In[*]:= { \$k = 2, \$n = 3 }; amt_{i,j→k}

(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})

Time: 104.00 seconds Out[=

$$\begin{aligned} & \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ & \quad x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i}, \\ & \quad - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + \\ & \quad x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \quad x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \quad x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \quad x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \quad x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \quad x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \quad \left. x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

» TTL $e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[x_{12}] U_k[] - 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j[x_{12}] U_k[a_1] +$
 $e^{\ll 1 \gg} \ll 2 \gg \ll 1 \gg + 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j[a_1] U_k[x_{12}]$
 » SG $U_j[] U_k[] + e^{-\gamma \alpha_{1i}} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k}} x_{12j}[\lambda] + e^{-\gamma \alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1,a_{1j}} x_{12k}[\lambda]$
 » TTR $\ll 1 \gg$

Out[*]=

$$\mathbb{E}_{\{i\} \rightarrow \{j,k\}} [(a_{1j} + a_{1k}) \alpha_{1i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i}, -2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i}]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

» **TTL** $e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \xi_{12_i} U_j[x_{12}] U_k[] + \ll 32 \gg + 2 e^{-\frac{3\gamma\alpha_{2i}}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_j[] U_k[x_{13}, x_{23}]$

» **SG** $U_j[] U_k[] + \ll 22 \gg + U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3\gamma\alpha_{2i}}{2}} \in f_{\emptyset, x_{23_k}}[\lambda] f_{1, x_{12_k} x_{23_k}}[\lambda] + e^{-\frac{\alpha_{1i}}{2}} \in f_{1, \ll 3 \gg_k \ll 1 \gg}[\lambda] \right)$

» **TTR** $\ll 1 \gg$

Out[*]=

$E_{\{i\} \rightarrow \{j, k\}} [$
 $(a_{1_j} + a_{1_k}) \alpha_{1_i} + (a_{2_j} + a_{2_k}) \alpha_{2_i} + x_{12_j} \xi_{12_i} + x_{12_k} \xi_{12_i} + x_{13_j} \xi_{13_i} + x_{13_k} \xi_{13_i} + x_{23_j} \xi_{23_i} + x_{23_k} \xi_{23_i},$
 $-2 a_{1_k} x_{12_j} \xi_{12_i} + 2 a_{1_j} x_{12_k} \xi_{12_i} - 2 a_{1_k} x_{13_j} \xi_{13_i} - 2 a_{2_k} x_{13_j} \xi_{13_i} + 2 a_{1_j} x_{13_k} \xi_{13_i} +$
 $2 a_{2_j} x_{13_k} \xi_{13_i} + x_{12_k} x_{13_j} \xi_{12_i} \xi_{13_i} - x_{12_j} x_{13_k} \xi_{12_i} \xi_{13_i} - 2 a_{2_k} x_{23_j} \xi_{23_i} + 2 a_{2_j} x_{23_k} \xi_{23_i} -$
 $x_{12_k} x_{23_j} \xi_{12_i} \xi_{23_i} + x_{13_k} x_{23_j} \xi_{13_i} \xi_{23_i} - x_{13_j} x_{23_k} \xi_{13_i} \xi_{23_i} + x_{12_j} x_{23_k} (4 \xi_{13_i} + \xi_{12_i} \xi_{23_i})]$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; (amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})

» **TTL** $e^{-\gamma\alpha_{1i}} \xi_{12_i} U_j[x_{12}] U_k[] - 2 e^{-\gamma\alpha_{1i}} \in \hbar \xi_{12_i} U_j[x_{12}] U_k[a_1] +$
 $e^{\ll 1 \gg} \ll 2 \gg \ll 1 \gg + 2 e^{-\gamma\alpha_{1i}} \in \hbar \xi_{12_i} U_j[a_1] U_k[x_{12}]$

» **SG** $U_j[] U_k[] + e^{-\gamma\alpha_{1i}} \in U_j[x_{12}] U_k[a_1] f_{1, a_{1k} x_{12_j}}[\lambda] + e^{-\gamma\alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1, a_{1j} x_{12_k}}[\lambda]$

» **TTR** $\ll 1 \gg$

Out[*]=

True

In[*]:= Block[{ \$k = 1, \$n = 3 }, Echo@{amt_{1,2→1}, aΔt_{1→1,2}};

(amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})]

» **TTL** $e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \xi_{12_i} U_j[x_{12}] U_k[] + \ll 32 \gg + 2 e^{-\frac{3\gamma\alpha_{2i}}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_j[] U_k[x_{13}, x_{23}]$

» **SG** $U_j[] U_k[] + e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \in U_j[x_{12}] U_k[a_1] f_{1, a_{1k} x_{12_j}}[\lambda] + \ll 21 \gg +$
 $U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3\gamma\alpha_{2i}}{2}} \in f_{\emptyset, x_{23_k}}[\lambda] f_{1, x_{12_k} x_{23_k}}[\lambda] + e^{-\frac{3\gamma\alpha_{2i}}{2}} \in f_{1, x_{13_k} x_{23_k}}[\lambda] \right)$

» **TTR** $U_j[x_{12}] U_k[x_{12}] \left(-e^{-\gamma(2\alpha_{1i}-\alpha_{2i})} \gamma \in f_{1, a_{1j} x_{12_k}}[\lambda] f_{\emptyset, x_{12_j}'}[\lambda] - e^{-\gamma(2\alpha_{1i}-\alpha_{2i})} \gamma \in f_{1, a_{1k} x_{12_j}}[\lambda] f_{\emptyset, x_{12_k}'}[\lambda] \right) +$
 $\ll 29 \gg + U_j[] U_k[x_{13}, x_{23}] (\ll 1 \gg)$

» $\left\{ E_{\{1,2\} \rightarrow \{1\}} \left[a_{1_1} (\alpha_{1_1} + \alpha_{1_2}) + a_{2_1} (\alpha_{2_1} + \alpha_{2_2}) + x_{12_1} \left(\xi_{12_1} + e^{\frac{1}{2}(2\alpha_{1_1}-\alpha_{2_1})} \xi_{12_2} \right) + \right. \right.$
 $x_{13_1} \left(\xi_{13_1} + e^{\frac{1}{2}(\alpha_{1_1}+\alpha_{2_1})} \xi_{13_2} - e^{\frac{1}{2}(2\alpha_{1_1}-\alpha_{2_1})} \xi_{12_2} \xi_{23_1} \right) + x_{23_1} \left(\xi_{23_1} + e^{\frac{1}{2}(-\alpha_{1_1}+2\alpha_{2_1})} \xi_{23_2} \right),$
 $-2 e^{\frac{1}{2}(2\alpha_{1_1}-\alpha_{2_1})} x_{12_1} x_{13_1} \xi_{12_2} \xi_{13_1} - e^{\frac{1}{2}(2\alpha_{1_1}-\alpha_{2_1})} x_{13_1} \xi_{12_2} \xi_{23_1} +$
 $2 e^{\frac{1}{2}(2\alpha_{1_1}-\alpha_{2_1})} x_{12_1} x_{23_1} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{1}{2}(\alpha_{1_1}+\alpha_{2_1})} x_{13_1} x_{23_1} \xi_{13_2} \xi_{23_1} \left. \right], E_{\{1\} \rightarrow \{1,2\}} [$
 $(a_{1_1} + a_{1_2}) \alpha_{1_1} + (a_{2_1} + a_{2_2}) \alpha_{2_1} + x_{12_1} \xi_{12_1} + x_{12_2} \xi_{12_1} + x_{13_1} \xi_{13_1} + x_{13_2} \xi_{13_1} + x_{23_1} \xi_{23_1} + x_{23_2} \xi_{23_1},$
 $-2 a_{1_2} x_{12_1} \xi_{12_1} + 2 a_{1_1} x_{12_2} \xi_{12_1} - 2 a_{1_2} x_{13_1} \xi_{13_1} - 2 a_{2_2} x_{13_1} \xi_{13_1} + 2 a_{1_1} x_{13_2} \xi_{13_1} +$
 $2 a_{2_1} x_{13_2} \xi_{13_1} + x_{12_2} x_{13_1} \xi_{12_1} \xi_{13_1} - x_{12_1} x_{13_2} \xi_{12_1} \xi_{13_1} - 2 a_{2_2} x_{23_1} \xi_{23_1} + 2 a_{2_1} x_{23_2} \xi_{23_1} -$
 $x_{12_2} x_{23_1} \xi_{12_1} \xi_{23_1} + x_{13_2} x_{23_1} \xi_{13_1} \xi_{23_1} - x_{13_1} x_{23_2} \xi_{13_1} \xi_{23_1} + x_{12_1} x_{23_2} (4 \xi_{13_1} + \xi_{12_1} \xi_{23_1})] \left. \right\}$

Out[*]=

True

pdf

In[]:=

```

Define[aSti = Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε;
  TTL = ES["TTL"]@TTλ@
    Echo@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Ui[x___] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp, ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp, ui[λ] /; τ[u] == "a" ⇒ If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp, Pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pi[u][λ] Pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = ES["SG"]@S[G];
  TTR = ES["TTR"]@UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = ES["eqns"]@Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], -α[m]i (a[m]i)], 0],
    Total[Cases[fs, fp, _]] /. fp, u ⇒ u (fp, u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

pdf

In[*]:=

```

Define[aSti = Module[{(*TTL, *)},
  TTL = TTλ@Reverse@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] :=
    mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], c_. Ui[x____] => Ui[x], ∞];
  ff = Table[
    Sum[εp fp,ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,ui[λ] /; τ[u] == "a" => If[p == 0, -(u*)i, 0];
  fs = Flatten[Table[
    fp, Pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pi[u][λ] Pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = S[G];
  TTR = UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], -α[m]i a[m]i], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```
In[*]:= $k = 1; $n = 3; aSti
aΔt1→2,3 // aSt2 // amt2,3→1
```

Time: 0.25 seconds Out[]=

$$\begin{aligned} & \mathbb{E}_{\{i\} \rightarrow \{i\}} \left[-a_{1i} \alpha_{1i} - a_{2i} \alpha_{2i} - e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} - e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + \right. \\ & \quad x_{13i} \left(-e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} - e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right), 2 e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} + \\ & \quad 2 e^{-\frac{3\alpha_{1i}}{2}} x_{12i} x_{13i} \xi_{12i} \xi_{13i} + 2 e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + 2 e^{-\frac{3\alpha_{2i}}{2}} x_{13i} x_{23i} \xi_{13i} \xi_{23i} + \\ & \quad 2 e^{-\alpha_{1i} - \alpha_{2i}} x_{13i}^2 \xi_{12i} \xi_{13i} \xi_{23i} + x_{12i} x_{23i} \left(4 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) + \\ & \quad \left. x_{13i} \left(2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 3 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) \right] \end{aligned}$$

Out[]=

$$\mathbb{E}_{\{1\} \rightarrow \{1\}} [0, 0]$$

pdf

```
In[*]:= (*Splitter B⊗A *)
sYi-→j-,k-,l-,m- := E{i}→{j,k,l,m} [
  Sum[η[r, s]i y[r, s]j + ξ[r, s]i x[r, s]m, {r, 1, $n - 1}, {s, r + 1, $n}] +
  Sum[α[r]i a[r]l + β[r]i b[r]k, {r, 1, $n - 1}], Sequence @@ ConstantArray[0, $k]
]
```

pdf

$$P: \mathbb{A} \otimes \mathbb{B} \rightarrow \mathbb{Q} \text{ and } R \in \mathbb{B} \otimes \mathbb{A}:$$

pdf

```
In[*]:= Pi-,j- := E{i,j}→{} @@ PadRight [
  {Sum[η[r, s]j ξ[r, s]i, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[β[r]j α[r]i, {r, 1, $n - 1}]},
  $k + 1]
Ri-,j- := E{i,j}→{i,j} @@ PadRight [
  {Sum[y[r, s]i x[r, s]j, {r, 1, $n - 1}, {s, r + 1, $n}] + Sum[b[r]i a[r]j, {r, 1, $n - 1}]},
  $k + 1]
```

```
In[*]:= {R1,2 // P2,3, R1,2 // P5,1}
```

Time: 0.12 seconds Out[]=

$$\begin{aligned} & \{ \mathbb{E}_{\{3\} \rightarrow \{1\}} [b_{11} \beta_{13} + b_{21} \beta_{23} + b_{31} \beta_{33} + b_{41} \beta_{43} + y_{121} \eta_{123} + y_{131} \eta_{133} + y_{141} \eta_{143} + \\ & \quad y_{151} \eta_{153} + y_{231} \eta_{233} + y_{241} \eta_{243} + y_{251} \eta_{253} + y_{341} \eta_{343} + y_{351} \eta_{353} + y_{451} \eta_{453}, 0], \\ & \mathbb{E}_{\{5\} \rightarrow \{2\}} [a_{12} \alpha_{15} + a_{22} \alpha_{25} + a_{32} \alpha_{35} + a_{42} \alpha_{45} + x_{122} \xi_{125} + x_{132} \xi_{135} + x_{142} \xi_{145} + \\ & \quad x_{152} \xi_{155} + x_{232} \xi_{235} + x_{242} \xi_{245} + x_{252} \xi_{255} + x_{342} \xi_{345} + x_{352} \xi_{355} + x_{452} \xi_{455}, 0] \} \end{aligned}$$

pdf

```
In[*]:= Define[bmtj,k→i = Module[{i0, i1, j1, k1}, (Ri0,i1 // aΔti1→j1,k1 // Pj1,j // Pk1,k) /. i0 → i],
  bΔtk→i,j = Module[{i1, j1, k1, k2}, ((Ri,i1 Rj,j1 // amti1,j1→k1) // Pk1,k2) /. k2 → k],
  bSti = Module[{i1, i2}, (Ri,i1 // aSti1 // Pi1,i2) /. i2 → i],
  bSti = Module[{i1, i2}, (Ri,i1 // aSti1 // Pi1,i2) /. i2 → i]
Define[Ri,j = Ri,j // bSti]
```

```
In[*]:= Block[{$k = 1, $n = 2}, bmt1,2→3]
```

$$\begin{aligned}
& \gg \text{TTL } e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[x_{12}] U_k[] - 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j[x_{12}] U_k[a_1] + \\
& \quad e^{-\gamma \alpha_{1i}} \xi_{12i} U_j[] U_k[x_{12}] + 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j[a_1] U_k[x_{12}] \\
& \gg \text{SG } U_j[] U_k[] + e^{-\gamma \alpha_{1i}} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k} x_{12j}}[\lambda] + e^{-\gamma \alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1,a_{1j} x_{12k}}[\lambda] \\
& \gg \text{TTR } U_j[x_{12}] U_k[x_{12}] \left(-e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1j} x_{12k}}[\lambda] f_{0,x_{12j}}'[\lambda] - e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1k} x_{12j}}[\lambda] f_{0,x_{12k}}'[\lambda] \right) + \\
& \quad \ll 3 \gg + e^{-\gamma \alpha_{1i}} \in U_j[a_1] U_k[x_{12}] f_{1,\ll 1 \gg} \ll 1 \gg'[\lambda]
\end{aligned}$$

Out[*]=

$$E_{\{1,2\} \rightarrow \{3\}} [b_{13} \beta_{11} + b_{13} \beta_{12} + y_{123} \eta_{121} + y_{123} \eta_{122}, -2 y_{123} \beta_{12} \eta_{121} + 2 y_{123} \beta_{11} \eta_{122}]$$

Testing A and B dually paired Hopf algebras

In[*]:= \$k = 1; \$n = 3;

co-associativity

In[*]:= {(aΔt_{1→1,2} // aΔt_{2→2,3}) ≡ (aΔt_{1→2,3} // aΔt_{2→1,2}), (bΔt_{1→1,2} // bΔt_{2→2,3}) ≡ (bΔt_{1→2,3} // bΔt_{2→1,2})}

$$\begin{aligned}
& \gg \text{TTL } e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \xi_{12i} U_j[x_{12}] U_k[] + \ll 32 \gg + 2 e^{-\frac{3\gamma\alpha_{2i}}{2}} \gamma \in \lambda \hbar \xi_{13i} \xi_{23i} U_j[] U_k[x_{13}, x_{23}] \\
& \gg \text{SG } U_j[] U_k[] + e^{-\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \in U_j[x_{12}] U_k[a_1] f_{1,a_{1k} x_{12j}}[\lambda] + \ll 21 \gg + \\
& \quad U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3\gamma\alpha_{2i}}{2}} \in f_{0,x_{23k}}[\lambda] f_{1,x_{12k} x_{23k}}[\lambda] + e^{-\frac{3\gamma\alpha_{2i}}{2}} \in f_{1,x_{13k} x_{23k}}[\lambda] \right) \\
& \gg \text{TTR } U_j[x_{12}] U_k[x_{12}] \left(-e^{-\gamma(2\alpha_{1i}-\alpha_{2i})} \gamma \in f_{1,a_{1j} x_{12k}}[\lambda] f_{0,x_{12j}}'[\lambda] - e^{-\gamma(2\alpha_{1i}-\alpha_{2i})} \gamma \in f_{1,a_{1k} x_{12j}}[\lambda] f_{0,x_{12k}}'[\lambda] \right) + \\
& \quad U_j[x_{23}] U_k[x_{23}] (\ll 1 \gg) + \ll 1 \gg + \ll 25 \gg + \ll 1 \gg + \ll 1 \gg + U_j[] U_k[x_{13}, x_{23}] (\ll 1 \gg)
\end{aligned}$$

Out[*]=

{True, True}

algebra morphism

In[*]:= {(aΔt_{i→1,2} aΔt_{j→3,4} // amt_{1,3→i} // amt_{2,4→j}) ≡ (amt_{i,j→k} // aΔt_{k→i,j}),
 (bΔt_{i→1,2} bΔt_{j→3,4} // bmt_{1,3→i} // bmt_{2,4→j}) ≡ (bmt_{i,j→k} // bΔt_{k→i,j})}

Time: 0.55 seconds Out[*]=

{True, True}

associativity

In[*]:= {(amt_{1,2→k} // amt_{k,3→k}) ≡ (amt_{2,3→k} // amt_{1,k→k}), (bmt_{1,2→k} // bmt_{k,3→k}) ≡ (bmt_{2,3→k} // bmt_{1,k→k})}

Time: 0.14 seconds Out[*]=

{True, True}

antipode

In[*]:= {aΔt_{i→1,2} // aSt₁ // amt_{1,2→1}, aΔt_{i→1,2} // aSt₂ // amt_{1,2→1},
 bΔt_{i→1,2} // bSt₁ // bmt_{1,2→1}, bΔt_{i→1,2} // bSt₂ // bmt_{1,2→1}}

$\gg \{-\alpha_{2i} U_i[a_2], -\alpha_{1i} U_i[a_1], (-\lambda \xi_{23i} + 2\gamma \in \lambda \hbar \xi_{23i}) U_i[x_{23}],$
 $(-\lambda \xi_{13i} + 2\gamma \in \lambda \hbar \xi_{13i}) U_i[x_{13}] + 4\gamma \in \lambda \hbar \xi_{13i} U_i[x_{12}, x_{23}], (-\lambda \xi_{12i} + 2\gamma \in \lambda \hbar \xi_{12i}) U_i[x_{12}]\}$
 $\gg \text{TTL } (-\xi_{12i} + 2\gamma \in \hbar \xi_{12i}) U_i[x_{12}] + (-\xi_{13i} + 2\gamma \in \hbar \xi_{13i} - \lambda \xi_{12i} \xi_{23i} + 3\gamma \in \lambda \hbar \xi_{12i} \xi_{23i}) U_i[x_{13}] +$
 $\ll 3 \gg + 2\gamma \in \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} U_i[x_{13}, x_{13}] + 2\gamma \in \lambda \hbar \xi_{13i} \xi_{23i} U_i[x_{13}, x_{23}]$
 $\gg \text{SG}$
 $U_i[] + e^{\frac{3\gamma\alpha_{1i}}{2}} \in U_i[x_{12}, x_{13}] f_{1,x_{12i}x_{13i}}[\lambda] + \ll 1 \gg \ll 1 \gg + e^{\frac{1}{2}\gamma(\alpha_{1i} + \ll 2 \gg_i)} \in \ll 1 \gg f_{1\ll 1 \gg \ll 1 \gg \ll 1 \gg \ll 1 \gg}[\lambda] +$
 $U_i[x_{13}, x_{23}] \left(e^{\frac{3\gamma\alpha_{2i}}{2}} \in f_{0,x_{23i}}[\lambda] f_{1,x_{12i}x_{23i}}[\lambda] + e^{\frac{3\gamma\alpha_{2i}}{2}} \in f_{1,x_{13i}x_{23i}}[\lambda] \right)$
 $\gg \text{TTR } U_i[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} f_{0,x_{12i}}'[\lambda] + e^{\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} \in f_{1,x_{12i}}'[\lambda] \right) + \ll 5 \gg +$
 $U_i[x_{13}, x_{23}] \left(2 e^{\frac{3\gamma\alpha_{2i}}{2}} \gamma \in \hbar f_{0,x_{23i}}[\lambda] f_{0,x_{13i}}'[\lambda] + e^{\frac{\ll 1 \gg}{2}} \in \ll 1 \gg \ll 1 \gg'[\lambda] + e^{\frac{\ll 1 \gg}{2}} \in f_{1\ll 1 \gg \ll 1 \gg'[\lambda] \right)$
 $\gg \text{eqns } \left\{ -\xi_{12i} - e^{\frac{1}{2}\gamma(2\alpha_{1i}-\alpha_{2i})} f_{0,x_{12i}}'[\lambda] = 0, \ll 1 \gg = 0, \ll 6 \gg, \ll 1 \gg = 0, 2\gamma \lambda \hbar \xi_{13i} \xi_{23i} - \right.$
 $\left. 2 e^{\frac{3\gamma\alpha_{2i}}{2}} \gamma \hbar f_{0,x_{23i}}[\lambda] f_{0,x_{13i}}'[\lambda] - e^{\frac{\ll 1 \gg}{2}} f_{0,\ll 1 \gg}[\lambda] f_{1\ll 1 \gg \ll 1 \gg'[\lambda] - e^{\frac{3\gamma\alpha_{2i}}{2}} f_{1,x_{13i}\ll 3 \gg_i'[\lambda] = 0 \right\}$

Out[*]=

$\{E_{\{i\} \rightarrow \{1\}}[0, 0], E_{\{i\} \rightarrow \{1\}}[0, 0], E_{\{i\} \rightarrow \{1\}}[0, 0], E_{\{i\} \rightarrow \{1\}}[0, 0]\}$

inverse antipode

$In[*] := \{aSt_i // \overline{aSt_i}, bSt_i // \overline{bSt_i}\}$

Time: 4.31 seconds Out[*]=

$\{E_{\{i\} \rightarrow \{i\}}[a_{1i}\alpha_{1i} + a_{2i}\alpha_{2i} + x_{12i}\xi_{12i} + x_{13i}\xi_{13i} + x_{23i}\xi_{23i}, 0],$
 $E_{\{i\} \rightarrow \{i\}}[b_{1i}\beta_{1i} + b_{2i}\beta_{2i} + y_{12i}\eta_{12i} + y_{13i}\eta_{13i} + y_{23i}\eta_{23i}, 0]\}$

Pairing

$In[*] := \{(\mathbf{amt}_{1,2 \rightarrow 3} // \mathbf{P}_{3,-3}) \equiv (\mathbf{bat}_{-3 \rightarrow -1,-2} // \mathbf{P}_{1,-1} // \mathbf{P}_{2,-2}),$
 $(\mathbf{bmt}_{1,2 \rightarrow 3} // \mathbf{P}_{-3,3}) \equiv (\mathbf{adt}_{-3 \rightarrow -1,-2} // \mathbf{P}_{-1,1} // \mathbf{P}_{-2,2}),$
 $(\mathbf{aSt}_1 // \mathbf{P}_{1,-1}) \equiv (\mathbf{bSt}_{-1} // \mathbf{P}_{1,-1}), (\overline{\mathbf{aSt}_1} // \mathbf{P}_{1,-1}) \equiv (\overline{\mathbf{bSt}_{-1}} // \mathbf{P}_{1,-1})\}$

Time: 0.23 seconds Out[*]=

{True, True, True, True}

Testing Double

pdf

$In[*] := \text{Define}[\mathbf{dm}_{i,j \rightarrow k} = \text{Module}[\{\mathbf{i1}, \mathbf{i2}, \mathbf{i3}, \mathbf{i4}\},$
 $\quad ((\mathbf{sy}_{i \rightarrow i4, i4, i1, i1} // \mathbf{adt}_{i1 \rightarrow i1, i2} // \mathbf{adt}_{i2 \rightarrow i2, i3} // \overline{\mathbf{ast}_{i3}}) (\mathbf{sy}_{j \rightarrow -i1, -i1, -i4, -i4} // \mathbf{bat}_{-i1 \rightarrow -i1, -i2} //$
 $\quad \mathbf{bat}_{-i2 \rightarrow -i2, -i3})) // (\mathbf{P}_{i1, -i3} \mathbf{P}_{i3, -i1} \mathbf{amt}_{i2, -i4 \rightarrow k} \mathbf{bmt}_{i4, -i2 \rightarrow k})]]$
 $\text{Define}[\mathbf{dS}_i = \mathbf{sy}_{i \rightarrow 1, 1, 2, 2} // (\mathbf{bSt}_1 \overline{\mathbf{aSt}_2}) // \mathbf{dm}_{2, 1 \rightarrow i},$
 $\quad \overline{\mathbf{dS}}_i = \mathbf{sy}_{i \rightarrow 1, 1, 2, 2} // (\overline{\mathbf{bSt}_1} \mathbf{aSt}_2) // \mathbf{dm}_{2, 1 \rightarrow i},$
 $\mathbf{d\Delta}_{i \rightarrow j, k} = (\mathbf{bat}_{i \rightarrow 1, 3} \mathbf{adt}_{i \rightarrow 4, 2}) // (\mathbf{dm}_{3, 4 \rightarrow k} \mathbf{dm}_{1, 2 \rightarrow j})]$
 $(*\text{Notice we take co-opposite on the A-side! to make the double}*)$

$In[*] := \text{Block}[\{\$k = 2, \$n = 3, \hbar = 1\}, \mathbf{dm}_{i,j \rightarrow k} // \mathbf{l2U} // \mathbf{CF}]$

$$\begin{aligned}
& \gg \text{TTL } e^{-\frac{1}{2} \gamma (2\alpha_{1i} - \alpha_{2i})} \xi_{12i} U_j[x_{12}] U_k[] + \ll 125 \gg + \\
& \left(2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 5\alpha_{2i})} \gamma^2 e^2 \lambda^2 \xi_{13i} \xi_{23i}^2 + \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_{1i} - 5\alpha_{2i})} \gamma^2 e^2 \lambda^3 \xi_{12i} \xi_{23i}^3 \right) U_j[] U_k[x_{13}, x_{23}, x_{23}] \\
& \gg \text{SG } U_j[] U_k[] + \ll 331 \gg + U_j[] U_k[x_{13}, x_{23}, x_{23}] \left(2 e^{\frac{1}{2} \gamma (\alpha_{1i} - 5\alpha_{2i})} \gamma e^2 f_{0, x_{23k}}[\lambda] f_{1, x_{13k} x_{23k}}[\lambda] + \right. \\
& \left. e^{\frac{1}{2} \gamma (\alpha_{1i} - 5\alpha_{2i})} e^2 f_{0, \ll 1 \gg}[\lambda] f_{2, x_{12k} \ll 1 \gg}[\lambda] + e^{\frac{1}{2} \gamma (\alpha_{1i} - 5\alpha_{2i})} e^2 f_{2, x_{13k} x_{23k}^2}[\lambda] \right)
\end{aligned}$$

Out[*]=

\$Aborted

In[*]:= **PrintProfile[]**

Out[*]=

```

ProfileRoot is root. Profiled time: 3717.61
(      1)      0/      0 above Boot
CF: called 628183 times, time in 2967.24/2967.24
(    166)  2496.600/ 2496.600 under TTR
(    108)  113.236/ 113.236 under SG
(    325)   1.077/   1.077 under TriangularDSolve
( 627584) 356.327/ 356.327 under UCF
UCF: called 352510 times, time in 349.914/706.241
(      8)   0.235/   0.593 under TTR
(      4)   0.125/   0.266 under SG
(   587)   5.397/  16.724 under AdNil
( 350246) 338.831/ 669.569 under NCM
(     15)   0.064/   5.970 under TT
(  1619)   5.231/  13.041 under UB
(      1)      0/   0.031 under amt[2, 3]
(     30)   0.031/   0.047 under aΔt[2, 3]
( 627584) 356.327/ 356.327 above CF
NCM: called 3241 times, time in 300.159/969.728
(      3)   45.017/  50.096 under TTR
(   3238) 255.142/ 919.632 under UB
( 350246) 338.831/ 669.569 above UCF
SG: called 2 times, time in 71.575/221.188
(      1)   3.593/   7.469 under amt[2, 3]
(      1)  67.982/ 213.719 under aΔt[2, 3]
(   199)   0.047/   9.763 above AdNil
(     12)   0.032/  26.332 above AdPotPlus
(      8)      0/   0.016 above AdPoty
(   108)  113.236/ 113.236 above CF
(      4)   0.125/   0.266 above UCF
TTR: called 2 times, time in 25.372/60.125
(      1)  25.372/  60.125 under amt[2, 3]
(      1)      0/      0 under aΔt[2, 3]
(   379)   0.548/ 377.601 above AdNil
(     19)   0.282/ 524.158 above AdPotPlus
(   166) 2496.600/ 2496.600 above CF
(      3)   45.017/  50.096 above NCM

```

```

(      8)      0.235/      0.593 above UCF
TriangularDSolve: called 1 times, time in 1.079/2.156
(      1)      1.079/      2.156 under amt[2, 3]
(    325)      1.077/      1.077 above CF
UB: called 1619 times, time in 0.768/933.441
(    723)      0.360/ 375.139 under AdNil
(    896)      0.408/ 558.302 under AdPoty
(   3238) 255.142/ 919.632 above NCM
(   1619)      5.231/   13.041 above UCF
AdNil: called 587 times, time in 0.595/392.458
(    379)      0.548/ 377.601 under TTR
(    199)      0.047/    9.763 under SG
(      9)         0/    5.094 under TT
(    723)      0.360/ 375.139 above UB
(    587)      5.397/   16.724 above UCF
amt[2, 3]: called 1 times, time in 0.344/82.453
(      1)      0.344/   82.453 under Boot
(      1)         0/   12.328 above TTL
(      1) 25.372/   60.125 above TTR
(      1)      3.593/    7.469 above SG
(      1)      1.079/    2.156 above TriangularDSolve
(      1)         0/    0.031 above UCF
AdPotPlus: called 37 times, time in 0.33/558.833
(     19)      0.282/ 524.158 under TTR
(     12)      0.032/  26.332 under SG
(      6)      0.016/    8.343 under TT
(    888)      0.217/ 558.503 above AdPoty
AdPoty: called 896 times, time in 0.217/558.519
(      8)         0/    0.016 under SG
(    888)      0.217/ 558.503 under AdPotPlus
(    896)      0.408/ 558.302 above UB
TT: called 15 times, time in 0.015/40.145
(      1)      0.015/   12.328 under TTL
(     13)         0/   20.723 under TT
(      1)         0/    7.094 under aΔt[2, 3]
(      9)         0/    5.094 above AdNil
(      6)      0.016/    8.343 above AdPotPlus
(     13)         0/   20.723 above TT
(     15)      0.064/    5.970 above UCF
aΔt[2, 3]: called 1 times, time in 0./0.
(      1)         0/         0 under Boot
(      1)         0/         0 above TTR
(      1) 67.982/ 213.719 above SG
(      1)         0/    7.094 above TT
(     30)      0.031/    0.047 above UCF
Boot: called 4 times, time in 0./82.453
(      3)         0/   82.453 under Boot

```

```

(      1)      0/      0 under ProfileRoot
(      3)      0/    82.453 above Boot
(      1)    0.344/    82.453 above amt[2, 3]
(      1)      0/      0 above aΔt[2, 3]
TTL: called 1 times, time in 0./12.328
(      1)      0/    12.328 under amt[2, 3]
(      1)    0.015/    12.328 above TT

```

associativity

```

In[*]:= (dm1,2→k // dmk,3→k) ≡ (dm2,3→k // dm1,k→k)
Time: 6.89 seconds Out[*]=
True

```

co-associativity

```

In[*]:= (dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→2,3 // dΔ2→1,2)
Time: 1.01 seconds Out[*]=
True

```

algebra morphism

```

In[*]:= (dΔ1→1,2 dΔ2→3,4 // (dm1,3→1 dm2,4→2)) ≡ (dm1,2→3 // dΔ3→1,2)
Time: 65.12 seconds Out[*]=
True

```

antipode

```

In[*]:= {dΔi→1,2 // dS1 // dm1,2→1, dΔi→1,2 // dS2 // dm1,2→1, dS1 // dS1}
Time: 339.55 seconds Out[*]=
{E{i}→{1}[0, 0], E{i}→{1}[0, 0], E{1}→{1}[a1 α1 + a2 α2 + b1 β1 +
b2 β2 + y12 η12 + y13 η13 + y23 η23 + x12 ξ12 + x13 ξ13 + x23 ξ23, 0]}

```

quasi-triangular axioms

```

In[*]:= (R1,3 // dΔ1→1,2) ≡ (R1,3 R2,4 // dm3,4→3)
(R1,3 // dΔ3→2,3) ≡ (R1,3 R0,2 // dm1,0→1)
(dΔi→k,j R1,2 // dmj,1→1 // dmk,2→2) ≡ (R1,2 dΔi→j,k // dm1,j→1 // dm2,k→2)
Time: 8.80 seconds Out[*]=
True

```

```

Out[*]=
True

```

```

Out[*]=
True

```

```

In[*]:= (R1,2 // aSt2) ≡ (R̄1,2)
Time: 0.06 seconds Out[*]=
True

```

Braid-like Reidemeister moves: R2,R3

$In[*]:= \left\{ R_{1,2} \bar{R}_{3,4} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2}, \bar{R}_{1,2} R_{3,4} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2}, \right.$
 $\left. (R_{1,2} R_{4,3} R_{5,6} // dm_{1,4 \rightarrow 1} // dm_{2,5 \rightarrow 5} // dm_{3,6 \rightarrow 3}) \equiv (R_{1,6} R_{2,3} R_{4,5} // dm_{1,4 \rightarrow 1} // dm_{2,5 \rightarrow 5} // dm_{3,6 \rightarrow 3}) \right\}$

Time: 12.20 seconds Out:=

$\{\mathbb{E}_{\{\} \rightarrow \{1,2\}} [\emptyset, \emptyset], \mathbb{E}_{\{\} \rightarrow \{1,2\}} [\emptyset, \emptyset], \text{True}\}$