Hidden Utilities and $\TeX$

Headers

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Pensieve header: Quantum \$sl_n\$ following Rosso and Yamane.

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Notes.
1. Rosso is “An Analogue of P.B.W. Theorem and the Universal \$R\$-matrix for \$U_\hbar(sl(N+1))\$’’, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N”.
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso’s X_i and Y_j. Rosso’s $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i,\ j] \leftrightarrow$ xij. Likewise Rosso’s $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i,\ j] \leftrightarrow$ yij.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i.

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In[◦]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

pdf

```
This version: April 2020. Original version: July 1994.
```

In[◦]:=
```
MakePDF[]
```

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In[◦]:=
```
δ_{ij__} := KroneckerDelta[ij];
```

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In[◦]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
```

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In[◦]:=
```
CF[ℰ_] := PP_CF@If[$k == ∞,
    Expand[Expand[ℰ] /. 𝕖^p_ :> 𝕖^(Factor@p)],
    Expand[Expand[ℰ] /. 𝕖^p_ :> 𝕖^(Factor@p) /. q^n_. :> Sum[(ℏ ϵ γ n)^k / k!, {k, 0, $k}]] /. ϵ^k_. /; k > $k :> 0
  ];
```

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```
In[•]:= x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
        ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝒜[i_] := 𝒜 <> i;
```

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```
In[•]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
        ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
        τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

```
In[•]:= {τ[x[3, 4]], ι[x[3, 4]]} // FullForm
Time: 0.03 seconds Out[•]=
        List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for *R* at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[•]:= (* q~ℯ^(ℏ γ ϵ); Ai[n]~ℯ^(ℏ ϵ ai); *)
```

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```
In[•]:= $n = 5; $k = ∞; $E := {$n, $k}
```

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```
In[•]:= $EPBWGens := Flatten@{
          Table[As[i, 1, $n - 1], As[j, i + 1, $n],
           x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
          Table[As[i, 1, $n - 1],
           A[i]* = 𝒜[i]; 𝒜[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i]; {a[i], A[i][_]}]
         }
        l_List* := #* & /@ l;
        $EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
        x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
        $PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[•]:= $PBWGens*
Time: 0.01 seconds Out[•]=
        {ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}
```

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```
In[•]:= UCF[ℰ_] := PP_UCF@Module[{F, k}, Expand@Collect[
          ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
            U_i_[l___, AB_[n_], r___] :→
             Sum[As[k, 0, $k], ((-2)^k ℏ^k ϵ^k n^k)/k! U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
           }],
         U_[___], F
        ] /. F → CF]
```

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```
In[•]:= m_s__[0] = 0;
        m_s__[x_Plus] := UCF[m_s /@ x];
        m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
        m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
        m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
        m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
          UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
        m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
           c U_k[xx, x, y, yy],
           ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
          ];
        m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :→ i, ∞];
```

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```
In[•]:= Unprotect[NonCommutativeMultiply];
        NonCommutativeMultiply[x_] := x;
        x_ ⊠ y_ := PP_NCM@ Module[{i, is = Supp[x] ⋂ Supp[y], σ, z, k},
            z = x; Do[z = m_i→σ@i[z], {i, is}];
            z = Expand[y z];
            If[$k < ∞, z = z /. ϵ^k_. /; k > $k :→ 0];
            Do[z = m_σ@i,i→i[z], {i, is}]; z];
        UB[x_, y_] := PP_UB@UCF[x ⊠ y - y ⊠ x];
```

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```
In[•]:= Δ_i_→j_,k_[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
              U_i[] → U_j[] U_k[],
              U_i[f_, fs___] :→ m_k,k1→k@ m_j,j1→j[Δ_j,k[f] Δ_i→j1,k1[U_i[fs]]]
             }];
        Δ_i_→j_,k_,l__[ℰ_] := ℰ // Δ_i→j,k // Δ_k→k,l
```

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```
In[∘]:= S_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
          U_i[f_, fs___] ⧴ m_{i1,i→i}[S_i[f] S_i1[U_i1[fs]]]
        }];
```

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The 𝔸-algebra

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```
In[∘]:= RR_k_[ai_, aj_] /; τ[ai] == τ[aj] == "a" := U_k[aj, ai];
RR_k_[ai_, Aj_[n_]] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := U_k[Aj[n], ai];
RR_k_[Aj_[n_], ai_] /;  τ[ai] == "a" ∧ τ[Aj] == "A"  := U_k[ai, Aj[n]];
RR_k_[Ai_[m_], Aj_[n_]] /;  τ[Ai] == τ[Aj] == "A"  := U_k[Aj[n], Ai[m]];
RR_k_[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = ι@xij; {m} = ι@am;
    U_k[xij, am] + γ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[xij] / 2
  ];
RR_k_[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@Am; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[xij, Am[n]]
  ]
```

```
In[∘]:= RR_0[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[∘]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

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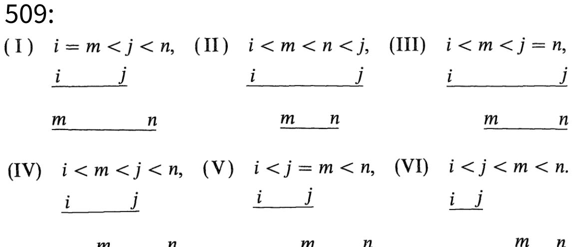
We take the reduction rules RR for xij's from Yamane page 509:



$$q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 \qquad \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$
$$[e_{ij},\ e_{mn}] = 0 \qquad \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$
$$[e_{ij},\ e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} \qquad \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{IV})}.$$
$$q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} \qquad \text{if } ((i,j),\ (m,n)) \in C_{(\mathrm{V})}.$$

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In[∘]:=
```
RR_k_[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[xij, xmn], (* I∨III *)
   (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
   i < m < j ∧ n > j, U_k[xij, xmn] + (q^-2 - q^2) U_k[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
 ]
```

In[∘]:=
```
U1[x34] U2[x23] U3[x12] // m1,2→2
```

Time: 0.06 seconds Out[∘]=

$-q\,U_2[x24]\,U_3[x12] + q^2\,U_2[x23, x34]\,U_3[x12]$

In[∘]:=
```
U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
```

Time: 0.05 seconds Out[∘]=

$q^2\,U_0[x14] - q^3\,U_0[x12, x24] - q^3\,U_0[x13, x34] + q^4\,U_0[x12, x23, x34]$

In[∘]:=
```
Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
    ]]
 ]]
```

Time: 1.05 seconds Out[∘]=

{}

In[∘]:=
```
Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
     P = U1[X] U2[Y] U3[Z];
     {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
    ]]
 ]]
```

» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[∘]=

{}

```
Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
   ]]
]
```

» {x12, x13, x23, a1, a2}

Out[∘]=

{}

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[∘]:=

```
Do[As[i, 1, $n - 1],
  (* Warning: these initialization fail if $n is increased later on *)
  Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
   ]]
]
```

In[∘]:=
```
Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U_i /@ Bas], P → Δ_{i→j,k}[P]]
 ]
```

Time: 0.06 seconds Out[∘]=

$U_i[x12] \to U_j[A1[-1]]\,U_k[x12] + U_j[x12]\,U_k[A1[1]]$

$U_i[x13] \to U_j[A1[-1], A2[-1]]\,U_k[x13] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1]]\,U_k[x23, A1[1]] + U_j[x13]\,U_k[A1[1], A2[1]]$

$U_i[x14] \to U_j[A1[-1], A2[-1], A3[-1]]\,U_k[x14] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1], A3[-1]]\,U_k[x24, A1[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x13, A3[-1]]\,U_k[x34, A1[1], A2[1]] + U_j[x14]\,U_k[A1[1], A2[1], A3[1]]$

$U_i[x15] \to U_j[A1[-1], A2[-1], A3[-1], A4[-1]]\,U_k[x15] + \left(-\frac{1}{q^2}+q^2\right) U_j[x12, A2[-1], A3[-1], A4[-1]]\,U_k[x25, A1[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x13, A3[-1], A4[-1]]\,U_k[x35, A1[1], A2[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x14, A4[-1]]\,U_k[x45, A1[1], A2[1], A3[1]] + U_j[x15]\,U_k[A1[1], A2[1], A3[1], A4[1]]$

$U_i[x23] \to U_j[A2[-1]]\,U_k[x23] + U_j[x23]\,U_k[A2[1]]$

$U_i[x24] \to U_j[A2[-1], A3[-1]]\,U_k[x24] + \left(-\frac{1}{q^2}+q^2\right) U_j[x23, A3[-1]]\,U_k[x34, A2[1]] + U_j[x24]\,U_k[A2[1], A3[1]]$

$U_i[x25] \to U_j[A2[-1], A3[-1], A4[-1]]\,U_k[x25] + \left(-\frac{1}{q^2}+q^2\right) U_j[x23, A3[-1], A4[-1]]\,U_k[x35, A2[1]] + \left(-\frac{1}{q^2}+q^2\right) U_j[x24, A4[-1]]\,U_k[x45, A2[1], A3[1]] + U_j[x25]\,U_k[A2[1], A3[1], A4[1]]$

$U_i[x34] \to U_j[A3[-1]]\,U_k[x34] + U_j[x34]\,U_k[A3[1]]$

$U_i[x35] \to U_j[A3[-1], A4[-1]]\,U_k[x35] + \left(-\frac{1}{q^2}+q^2\right) U_j[x34, A4[-1]]\,U_k[x45, A3[1]] + U_j[x35]\,U_k[A3[1], A4[1]]$

$U_i[x45] \to U_j[A4[-1]]\,U_k[x45] + U_j[x45]\,U_k[A4[1]]$

$U_i[a1] \to U_j[a1]\,U_k[] + U_j[]\,U_k[a1]$

$U_i[a2] \to U_j[a2]\,U_k[] + U_j[]\,U_k[a2]$

$U_i[a3] \to U_j[a3]\,U_k[] + U_j[]\,U_k[a3]$

$U_i[a4] \to U_j[a4]\,U_k[] + U_j[]\,U_k[a4]$

In[∘]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U_1[X];
    X → (P // Δ_{1→1,2} // Δ_{2→2,3}) == (P // Δ_{1→1,3} // Δ_{1→1,2})
   ]]
]
```

» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}

Out[∘]=

{}

In[∘]:=
```
Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas],
     P = U1[X] U2[Y];
     {X, Y} → UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
    ]]
]
```

» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}

Out[∘]=

{}

Is Δ a quantization of the co-bracket?

In[∘]:=
```
$k = 1
```

Time: 0.02 seconds Out[∘]=

1

In[∘]:=
```
Column@Module[{Bas = $PBWGens, P},
   Table[As[P, U1 /@ Bas],
    P → UCF[(Δ1→1,2[P] - Δ1→2,1[P]) //.
        {q^n_· :> 1 + n ħ / 2, U_i_[xs___, A_[n_], as___] :> U_i[xs] + n ħ U_i[xs, a @@ ι[A], as] / 2}
      ] /. ħ^n_ /; n > 1 :> 0
   ]
  ]
```

Time: 0.05 seconds Out[∘]=

$U_1[x12] \to -4 \epsilon \hbar U_1[x12] U_2[a1] + 4 \epsilon \hbar U_1[a1] U_2[x12]$

$U_1[x13] \to -4 \epsilon \hbar U_1[x13] U_2[a1] - 4 \epsilon \hbar U_1[x13] U_2[a2] - 4 \gamma \epsilon \hbar U_1[x23] U_2[x12] +$
$\quad 4 \epsilon \hbar U_1[a1] U_2[x13] + 4 \epsilon \hbar U_1[a2] U_2[x13] + 4 \gamma \epsilon \hbar U_1[x12] U_2[x23]$

$U_1[x23] \to -4 \epsilon \hbar U_1[x23] U_2[a2] + 4 \epsilon \hbar U_1[a2] U_2[x23]$

$U_1[a1] \to 0$

$U_1[a2] \to 0$

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The Antipode

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In[∘]:=
```
Do[As[i, 1, $n - 1],
  S_j_[a[i]] = -U_j[a[i]];
  S_j_[A[i][n_]] = U_j[A[i][-n]];
  S_j_[x[i, i + 1]] = -q^-2 U_j[x[i, i + 1]] (*?*)
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 S_j_[x[i, n]] = m_{j1,j→j}[
   q S_j[x[i, n - 1]] S_j1[x[n - 1, n]] - q^-1 S_j[x[n - 1, n]] S_j1[x[i, n - 1]]
  ]
]
```

In[∘]:=
```
$k = 2
```

Time: 0.00 seconds Out[∘]=

2

In[∘]:=
```
Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U_1[X];
    X → (P // Δ_{1→2,3} // S_2 // m_{2,3→1})
   ]]
]
```

» {x12, x13, x23, a1, a2, A1[-2], A2[-1], A1[1], A2[-1], A1[-2], A2[1], A1[2], A2[0], A2[2], A2[0]}

Out[∘]=

{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A1[-2] → U_1[],
A2[-1] → U_1[], A1[1] → U_1[], A2[-1] → U_1[], A1[-2] → U_1[],
A2[1] → U_1[], A1[2] → U_1[], A2[0] → U_1[], A2[2] → U_1[], A2[0] → U_1[]}

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The Generating Functions for the 𝔸 operations

In[∘]:=
```
BeginProfile[]
```

Time: 0.05 seconds Out[∘]=

ProfileRoot

In[∘]:=
```
$k = 1;
```

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Ad, TT, Seepage

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```
In[∘]:= PotQ[0] = True;
PotQ[c_. U_i_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
In[∘]:= Ad[x_][y_] /; PotQ[x] ∧ (Head[y] =!= Plus) := PP_AdPoty[ⅇ^(UB[y,x]/y) y];
Ad[x_][y_Plus] /; PotQ[x] := PP_AdPotPlus[Ad[x] /@ y];
```

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```
In[∘]:= Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PP_AdNil@Module[{k = 1, s = y, t = y},
    While[(t = UB[t, x]) =!= 0, s += t / k!; ++k];
    UCF[s]
   ]
```

```
In[∘]:= $k = 2
```

Time: 0.00 seconds Out[∘]=

2

```
In[∘]:= Ad[ϵ ℏ U_j[x12] U_k[]][U_j[x23] U_k[a4]]
```

Time: 0.06 seconds Out[∘]=

$$\left(-\epsilon\,\hbar-\gamma\,\epsilon^2\,\hbar^2\right) U_j[x13]\,U_k[a4]+U_j[x23]\,U_k[a4]+2\,\gamma\,\epsilon^2\,\hbar^2\,U_j[x12,\,x23]\,U_k[a4]$$

```
In[∘]:= PotQ[7 ℏ U_j[x12] U_k[a3]]
```

Time: 0.02 seconds Out[∘]=

$$\mathrm{PotQ}[7\,\hbar\,U_j[x12]\,U_k[a3]]$$

```
In[∘]:= Ad[7 ℏ U_j[x12] U_k[a3]][U_j[x23] U_k[a4]]
```

Time: 0.09 seconds Out[∘]=

$$U_j[x23]\,U_k[a4]+\left(-7\,\hbar-7\,\gamma\,\epsilon\,\hbar^2-\frac{7}{2}\,\gamma^2\,\epsilon^2\,\hbar^3\right) U_j[x13]\,U_k[a3,\,a4]+$$
$$\left(14\,\gamma\,\epsilon\,\hbar^2+14\,\gamma^2\,\epsilon^2\,\hbar^3\right) U_j[x12,\,x23]\,U_k[a3,\,a4]-98\,\gamma^2\,\epsilon^2\,\hbar^4\,U_j[x12,\,x13]\,U_k[a3,\,a3,\,a4]+$$
$$98\,\gamma^2\,\epsilon^2\,\hbar^4\,U_j[x12,\,x12,\,x23]\,U_k[a3,\,a3,\,a4]-\frac{686}{3}\,\gamma^2\,\epsilon^2\,\hbar^5\,U_j[x12,\,x12,\,x13]\,U_k[a3,\,a3,\,a3,\,a4]$$

```
In[∘]:= Ad[π ⅈ U_j[a1]][U_j[x23] U_k[a4]]
```

Time: 0.05 seconds Out[∘]=

$$e^{\frac{i\pi\gamma}{2}}\,U_j[x23]\,U_k[a4]$$

pdf

```
In[∘]:= TT_λ_[] = 0; (* Translated Tangent *)
TT_λ_[xs___, x_] := PP_TT@UCF[Ad[x][TT_λ[xs]] + ∂_λ x];
TT_λ_[xs_List] := TT_λ @@ xs
```

In[∘]:= $TT_\lambda[\lambda\,\xi 12\,U_0[x12]\,U_1[],\ \lambda\,\xi 23\,U_0[x23]\,U_1[]]$

Time: 0.08 seconds Out[∘]=

$\xi 12\,U_0[x12]\,U_1[] + \lambda\,\xi 12\,\xi 23\,U_0[x13]\,U_1[] + \xi 23\,U_0[x23]\,U_1[]$

pdf

In[∘]:=
```
𝕆i_[poly_] := Total[
   CoefficientRules[poly, #i & /@ $PBWGens] /.
    (p_ → c_) ⧴ c Ui @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
  ];
𝕆i_,is__[poly_] := 𝕆i[𝕆is[poly]];
ℙi_[ℰ_] := ℰ /. Ui[xs___] ⧴ Times @@ (#i & /@ {xs});
ℙi_,is__[poly_] := ℙi[ℙis[poly]];
```

In[∘]:= $\mathbb{O}_i\left[x12_i^2\,x13_i + 7\,x23_i^{17}\right]$

Time: 0.06 seconds Out[∘]=

$U_i[x12, x12, x13] +$
$7\,U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$

In[∘]:= $\mathbb{O}_{i,j}\left[x12_i^2\,x13_i\,x24_j + 7\,x23_i^{17}\,x25_j^3\right]$

Time: 0.03 seconds Out[∘]=

$U_i[x12, x12, x13]\,U_j[x24] +$
$7\,U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]$
$U_j[x25, x25, x25]$

In[∘]:= $\mathbb{P}_{i,j}@\mathbb{O}_{i,j}\left[x12_i^2\,x13_i\,x24_j + 7\,x23_i^{17}\,x25_j^3\right]$

Time: 0.06 seconds Out[∘]=

$x12_i^2\,x13_i\,x24_j + 7\,x23_i^{17}\,x25_j^3$

pdf

In[∘]:=
```
Seepagei_[{}, c_] = c;
Seepagei_[ξs_List, P_] := Module[{lx},
   lx = $PBWGens⟦Length@ξs⟧;
   CF@Collect[P, lxi, ℙi[Ad[Last[ξs] Ui[lx]][𝕆i[Seepagei[Most@ξs, #]]]]
      &]
  ];
Seepagei_,is_[ξs_List, ξss__List, P_] := Seepagei[ξs, Seepageis[ξss, P]]
```

In[∘]:= $k = 1

Time: 0.00 seconds Out[∘]=

1

In[∘]:= $PBWGens*

Time: 0.01 seconds Out[∘]=

$\{\xi 12, \xi 13, \xi 14, \xi 15, \xi 23, \xi 24, \xi 25, \xi 34, \xi 35, \xi 45, \alpha 1, \alpha 2, \alpha 3, \alpha 4\}$

In[∘]:= Seepage$_i$[$PBWGens*, $x35_i$]

Time: 0.16 seconds Out[∘]=

$e^{\frac{1}{2}(\alpha 2-\alpha 3-\alpha 4)\gamma}\,x35_i + 2\,e^{\frac{1}{2}(\alpha 2-3\,\alpha 4)\gamma}\,\gamma\,\epsilon\,\xi 45\,\hbar\,x35_i\,x45_i$

In[∘]:=
```
Seepage_{i,j}[#_i & /@ $PBWGens*, #_j & /@ $PBWGens*, x35_i x45_j]
```

Time: 0.25 seconds Out[∘]=

$$e^{\frac{1}{2}\gamma(\alpha 2_i-\alpha 3_i+\alpha 3_j-\alpha 4_i-2\alpha 4_j)}\, x35_i\, x45_j + 2\, e^{-\frac{1}{2}\gamma(-\alpha 2_i-\alpha 3_j+3\alpha 4_i+2\alpha 4_j)}\, \gamma\, \epsilon\, \hbar\, x35_i\, x45_i\, x45_j\, \xi 45_i$$

pdf

In[∘]:=
```
Invert_{is__}[ℰ_] := Module[{A, B, U1, j}, (* ℰ=A U[]+∈ B *)
  U1 = Times @@ (U_#[] & /@ {is});
  A = Coefficient[ℰ, U1] /. ϵ → 0;
  B = UCF[(ℰ - A U1)];
  UCF[
   A^-1 (U1 + Sum[As[j, 1, $k], (-1)^j NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
 ]
```

In[∘]:=
```
$k = 3;
Invert_{i,j}[7 U_i[] U_j[] + ϵ U_i[x12] U_j[x23]]
```

Time: 0.09 seconds Out[∘]=

$$\frac{1}{7} U_i[\,]\, U_j[\,] - \frac{1}{49} \epsilon\, U_i[x12]\, U_j[x23] + \frac{1}{343} \epsilon^2 U_i[x12, x12]\, U_j[x23, x23] - \frac{\epsilon^3 U_i[x12, x12, x12]\, U_j[x23, x23, x23]}{2401}$$

```
{$k = 1, $n = 3}
```

Time: 0.01 seconds Out[∘]=

{2, 3}

pdf

In[∘]:=
```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
 PP_TriangularDSolve@Module[{ss = {}, reqns = eqns, rinits = inits, rfs = fs, s, p},
   While[rfs =!= {},
    {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
    {{s}} = DSolve[reqns⟦p⟧ ∧ rinits⟦p⟧, rfs⟦p⟧, λ];
    AppendTo[ss, s = s /. F_Function :→ ReplacePart[F, 2 → CF[F⟦2⟧]]];
    reqns = Delete[reqns, p] /. s /. e_ == 0 :→ CF[e] == 0;
    rinits = Delete[rinits, p];
    rfs = Delete[rfs, p];
   ];
   ss
  ]
```

pdf

$L_\lambda = \mathbb{OE}(\lambda \xi_{i,1} x)\, \mathbb{OE}(\lambda \xi_{i,2} x)$

Seepage is defined by: $\underbrace{\mathbb{OE}(f_i x_i)}_{\mathbb{O}f} \mathbb{O}(\text{Seepage}(\{f_i\}, A)) = \mathbb{O}(e^{f_i x_i} A)$

$TT(L_\lambda) = L^{-1} \partial_\lambda L_\lambda$

$S(B) = \text{Seepage}(\{f_i\}, B)$.

$R_\lambda = \mathbb{OE}(f + g)$, $f = f_i x_i$ $g = \sum_{\substack{w \\ |w|\neq 1}} g_w w$ $e^f = F$ $e^g = G$ $R_\lambda = \mathbb{O}(FG)$ $\partial_\lambda F = F \partial_\lambda f$

$TT(R_\lambda) = R_\lambda^{-1} (\partial_\lambda R_\lambda) = \mathbb{O}(FG)^{-1} \mathbb{O}(\partial_\lambda F G) = S(G)^{-1} \mathbb{O}(F)^{-1} \mathbb{O}(F(\partial_\lambda f) G + \partial_\lambda G)) = S(G)^{-1} S((\partial_\lambda f) G + \partial_\lambda G)$

pdf

In[ ]:=
```
amt_{i_,j_→k_} := PP_{"amt"[$k,$n]}@Module[{(*TTL, *)},
  ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
  TTL = ES["TTL"]@PP_{"TTL"}@TT_λ@
     Join[l1 = Table[If[τ[g] === "x", λ, 1] (g^*)_i U_k[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], U_k[___], ∞];
  ff = Table[
     Sum[ϵ^p f_{p,u_k}[λ], {p, 0, $k}],
     {u, $PBWGens}
    ] /. f_{p_,u_k}[λ] /; τ[u] == "a" :→ If[p == 0, (u^*)_i + (u^*)_j, 0];
  fs = Flatten[Table[
     f_{p,ℙ_k[m]},
     {p, 0, $k}, {m, mons[p]}
    ]];
  gg = Sum[
    ϵ^p f_{p,ℙ_k[u]}[λ] ℙ_k[u],
    {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
   ];
  G = Series[e^gg, {ϵ, 0, $k}];
  S[B_] := UCF[SS@O_k[Seepage_k[ff, SS@UCF@B]]];
  SG = ES["SG"]@PP_{"SG"}@S[G];
  TTR = ES["TTR"]@PP_{"TTR"}@UCF[Invert_k[SG] ⊠ S[(∂_λ (ff.(#_k & /@ $PBWGens)) G + ∂_λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
     Coefficient[Coefficient[diff, m], ϵ, p] == 0,
     {p, 0, $k}, {m, mons[p]}
    ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  𝔼[Total[fs] /. f_{p_,u_} :→ ϵ^p u (f_{p,u} /. ss)[λ]] /. λ | ℏ | γ → 1
 ]
```

In[ ]:=
```
Block[{$k = 1, $n = 3}, amt_{i,j→k}]
```

» TTL $\left(e^{-\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i + e^{-\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right) U_k[x12] + \ll 5 \gg + (\ll 1 \gg)\, U_k[x13, x23]$

» *SG* $U_k[\,] + \ll 3 \gg + U_k[x13, x23]\left(e^{-\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\epsilon\, f_{0,x23_k}[\lambda]\, f_{1,x12_k\, x23_k}[\lambda] + e^{-\frac{3}{2}\gamma(\ll 1 \gg)}\,\epsilon\, f_{1,\ll 1 \gg}[\lambda]\right)$

» TTR

$U_k[x12]\left(e^{-\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\, f_{0,\ll 3 \gg_k}{}'[\lambda] + e^{-\frac{1}{2}\gamma(\ll 1 \gg)}\,\epsilon\, f_{1\,\ll 1 \gg\,\ll 1 \gg}{}'[\lambda]\right) + \ll 5 \gg + U_k[x13, x23]\,(\ll 1 \gg)$

Out[∘]=

$$\begin{aligned}
&\mathbb{E}\Big[x12_k\left(\xi12_i + e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\right) - 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x13_k\,\xi12_j\,\xi13_i - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x13_k\,\xi12_j\,\xi23_i + \\
&\quad 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x23_k\,\xi12_j\,\xi23_i - 2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\epsilon\, x13_k\, x23_k\,\xi13_j\,\xi23_i + \\
&\quad x13_k\left(\xi13_i + e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i\right) + x23_k\left(\xi23_i + e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i)}\,\xi23_j\right)\Big]
\end{aligned}$$

In[∘]:= **Block[{$k = 1, $n = 4}, amt$_{i,j\to k}$]**

» TTL $\left(e^{-\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i + e^{-\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right) U_k[x12] + \ll 21 \gg + (\ll 1 \gg)\, U_k[x24, x34]$

» *SG*

$U_k[\,] + \ll 16 \gg + U_k[x24, x34]\left(e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\,\alpha3_i-3\,\alpha3_j)}\,\epsilon\, f_{0,\ll 3 \gg_k}[\lambda]\, f_{1,x23_k\,\ll 3 \gg_k}[\lambda] + e^{\ll 1 \gg}\,\epsilon\, f_{1\,\ll 1 \gg\,\ll 1 \gg}[\lambda]\right)$

» TTR $U_k[x12]\left(e^{-\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\, f_{0,\ll 3 \gg_k}{}'[\lambda] + e^{-\frac{1}{2}\gamma(\ll 1 \gg)}\,\epsilon\, f_{1\,\ll 1 \gg\,\ll 1 \gg}{}'[\lambda]\right) +$

$\ll 21 \gg + U_k[x24, x34]\,(\ll 1 \gg)$

Out[∘]=

$$\begin{aligned}
&\mathbb{E}\Big[x12_k\left(\xi12_i + e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\right) - 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x13_k\,\xi12_j\,\xi13_i - \\
&\quad 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x14_k\,\xi12_j\,\xi14_i - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x13_k\,\xi12_j\,\xi23_i + \\
&\quad 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x23_k\,\xi12_j\,\xi23_i - 2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\epsilon\, x13_k\, x23_k\,\xi13_j\,\xi23_i + \\
&\quad x13_k\left(\xi13_i + e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\xi13_j - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i\right) + \\
&\quad x23_k\left(\xi23_i + e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i-\alpha3_i)}\,\xi23_j\right) + 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x24_k\,\xi12_j\,\xi24_i - \\
&\quad 4\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\epsilon\, x14_k\, x23_k\,\xi13_j\,\xi24_i - 2\,e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i-\alpha3_i)}\,\epsilon\, x23_k\, x24_k\,\xi23_j\,\xi24_i + \epsilon\, x13_k\, x14_k \\
&\qquad\left(-2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\xi13_j\,\xi14_i + 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi14_i\,\xi23_i + 2\,e^{\frac{1}{2}(3\alpha1_i-\alpha3_i)}\,\xi12_j\,\xi13_j\,\xi24_i\right) + \\
&\quad 2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\epsilon\, x13_k\, x34_k\,\xi13_j\,\xi34_i - 2\,e^{\frac{1}{2}(\alpha1_i+\alpha3_i)}\,\epsilon\, x14_k\, x34_k\,\xi14_j\,\xi34_i - \\
&\quad e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i-\alpha3_i)}\,\epsilon\, x24_k\,\xi23_j\,\xi34_i + 2\,e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i-\alpha3_i)}\,\epsilon\, x23_k\, x34_k\,\xi23_j\,\xi34_i - \\
&\quad 2\,e^{\frac{1}{2}(-\alpha1_i+\alpha2_i+\alpha3_i)}\,\epsilon\, x24_k\, x34_k\,\xi24_j\,\xi34_i + \epsilon\, x14_k\left(-e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi24_i - e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\xi13_j\,\xi34_i\right) + \\
&\quad x14_k\left(\xi14_i + e^{\frac{1}{2}(\alpha1_i+\alpha3_i)}\,\xi14_j - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi24_i - e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\xi13_j\,\xi34_i\right) + \\
&\quad x24_k\left(\xi24_i + e^{\frac{1}{2}(-\alpha1_i+\alpha2_i+\alpha3_i)}\,\xi24_j - e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i-\alpha3_i)}\,\xi23_j\,\xi34_i\right) + \\
&\quad \epsilon\, x13_k\, x24_k\left(-2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i\,\xi24_i - 2\,e^{\frac{1}{2}(3\alpha2_i-2\alpha3_i)}\,\xi13_j\,\xi23_j\,\xi34_i\right) + \\
&\quad \epsilon\, x14_k\, x24_k\left(-2\,e^{\frac{1}{2}(\alpha1_i+\alpha3_i)}\,\xi14_j\,\xi24_i + 2\,e^{\alpha2_i}\,\xi14_j\,\xi23_j\,\xi34_i + 2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i-\alpha3_i)}\,\xi13_j\,\xi24_i\,\xi34_i\right) + \\
&\quad x34_k\left(\xi34_i + e^{\frac{1}{2}(-\alpha2_i+2\alpha3_i)}\,\xi34_j\right)\Big]
\end{aligned}$$

In[ ]:= **Block[{$k = 2, $n = 3}, $amt_{i,j\to k}$]**

» TTL $\left(e^{-\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i + e^{-\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right) U_k[x12] + \ll 14 \gg$

» $\mathcal{SG}$ $U_k[\,] + \ll 20 \gg +$

$U_k[x13, x23, x23]\left(2\, e^{\frac{1}{2}\gamma(\ll 1 \gg)}\,\gamma\,\epsilon^2\,\hbar\, f_{0,x23_k}[\lambda]\, f_{1,x13_k\, x23_k}[\lambda] + e^{\ll 1 \gg} \ll 3 \gg + e^{\ll 1 \gg}\,\epsilon^2\, f_{2,\ll 1 \gg}[\lambda]\right)$

» TTR $\ll 1 \gg$

Out[ ]=

$$
\begin{aligned}
&\mathbb{E}\Big[x12_k\left(\xi12_i + e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\right) - 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x13_k\, \xi12_j\, \xi13_i - \\
&\quad e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x13_k\, \xi12_j\, \xi23_i - \frac{1}{2}\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon^2\, x13_k\, \xi12_j\, \xi23_i + 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon\, x12_k\, x23_k\, \xi12_j\, \xi23_i + \\
&\quad 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon^2\, x12_k\, x23_k\, \xi12_j\, \xi23_i + 2\,e^{2\alpha1_i-\alpha2_i}\,\epsilon^2\, x12_k^2\, x23_k\, \xi12_j^2\, \xi23_i - \\
&\quad 2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\epsilon\, x13_k\, x23_k\, \xi13_j\, \xi23_i + 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon^2\, x12_k\, x23_k^2\, \xi12_j\, \xi23_i^2 + e^{2\alpha1_i-\alpha2_i}\,\epsilon^2\, x13_k^2\, \xi12_j^2\, \xi23_i^2 - \\
&\quad \frac{2}{9}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,\epsilon^2\, x13_k^3\, \xi12_j^3\, \xi23_i^3 + x13_k\left(\xi13_i + e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j - e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i\right) + \\
&\quad \epsilon^2\, x12_k\, x13_k\left(2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i - 2\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi23_i\right) + \\
&\quad \epsilon^2\, x12_k^2\, x13_k\left(2\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi13_i - \frac{2}{3}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j^3\,\xi23_i\right) + \\
&\quad \epsilon^2\, x12_k\, x13_k\, x23_k\left(-4\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i\,\xi23_i - 4\,e^{\frac{3\alpha1_i}{2}}\,\xi12_j\,\xi13_j\,\xi23_i\right) + \\
&\quad \epsilon^2\, x13_k\, x23_k\left(2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j\,\xi23_i - 2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i^2\right) + \\
&\quad \epsilon^2\, x12_k\, x13_k^2\left(2\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi13_i^2 + \frac{2}{3}\,e^{\frac{3}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j^3\,\xi23_i^2\right) + \\
&\quad \epsilon^2\, x13_k\, x23_k^2\left(2\,e^{\frac{1}{2}(\alpha1_i+\alpha2_i)}\,\xi13_j\,\xi23_i^2 - \frac{2}{3}\,e^{\frac{1}{2}(2\alpha1_i-\alpha2_i)}\,\xi12_j\,\xi23_i^3\right) + \\
&\quad \epsilon^2\, x13_k^2\, x23_k\left(2\,e^{\alpha1_i+\alpha2_i}\,\xi13_j^2\,\xi23_i + \frac{2}{3}\,e^{2\alpha1_i-\alpha2_i}\,\xi12_j^2\,\xi23_i^3\right) + x23_k\left(\xi23_i + e^{\frac{1}{2}(-\alpha1_i+2\alpha2_i)}\,\xi23_j\right)\Big]
\end{aligned}
$$

pdf

```
In[∘]:= aΔt_{i_→j_,k_} := PP"aΔt"[$k,$n] @Module[{(*TTL, *)},
    ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 2]]; ℰ);
    TTL =
     ES["TTL"]@TT_λ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)_i Δ_{i→j,k}[U_i[g]]], {g, $PBWGens}];
    mons[-1] = {};
    mons[p_] := mons[p - 1] ⋃
      Union@Cases[Coefficient[TTL, ϵ, p], c_. U_j[x___] U_k[y___] :→ U_j[x] U_k[y], ∞];
    ffj = Table[
       Sum[ϵ^p f_{p,u_j}[λ], {p, 0, $k}],
       {u, $PBWGens}
      ] /. f_{p_,u_j}[λ] /; τ[u] == "a" :→ If[p == 0, (u*)_i, 0];
    ffk = ffj /. j → k;
    fs = Flatten[Table[
       f_{p,ℙ_{j,k}[m]},
       {p, 0, $k}, {m, mons[p]}
      ]];
    gg = Sum[
      ϵ^p f_{p,ℙ_{j,k}[u]}[λ] ℙ_{j,k}[u],
      {p, 0, $k}, {u, Complement[mons[p], U_j[] (U_k /@ $PBWGens) ⋃ U_k[] (U_j /@ $PBWGens)]}
     ];
    G = Series[e^gg, {ϵ, 0, $k}];
    𝒮[B_] := UCF[SS@𝕆_{j,k}[Seepage_{j,k}[ffj, ffk, SS@UCF@B]]];
    𝒮G = ES["𝒮G"]@PP"𝒮G"@𝒮[G];
    TTR = ES["TTR"]@PP"TTR"@UCF[
       Invert_{j,k}[𝒮G] ⊠ 𝒮[(∂_λ (ffj.(#_j & /@ $PBWGens) + ffk.(#_k & /@ $PBWGens)) G + ∂_λ G)]];
    diff = UCF[TTL - TTR];
    eqns = Flatten[Table[
       Coefficient[Coefficient[diff, m], ϵ, p] == 0,
       {p, 0, $k}, {m, mons[p]}
      ]];
    inits = (#[0] == 0) & /@ fs;
    ss = TriangularDSolve[eqns, inits, fs, λ, CF];
    𝔼[Total[fs] /. f_{p_,u_} :→ ϵ^p u (f_{p,u} /. ss)[λ]] /. λ | ℏ | γ → 1
   ]
```

```
In[∘]:= Block[{$k = 1, $n = 3}, aΔt_{i→j,k}]
```

» TTL $e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\xi 12_i\,U_j[x12]\,U_k[\,] + \ll 32\gg + 2\,e^{-\frac{3\ll 1\gg\ll 1\gg\ll 1\gg}{2}}\,\gamma\,\epsilon\,\lambda\,\hbar\,\xi 13_i\,\xi 23_i\,U_j[\,]\,U_k[x13, x23]$

» SG $U_j[\,]\,U_k[\,] + \ll 1\gg + \ll 20\gg + \ll 1\gg +$
$U_j[\,]\,U_k[x13, x23]\left(e^{-\frac{3\gamma\alpha 2_i}{2}}\,\epsilon\,f_{0,x23_k}[\lambda]\,f_{1,x12_k\,x23_k}[\lambda] + e^{-\frac{\ll 1\gg}{2}}\,\epsilon\,f_{1,\ll 1\gg\,\ll 1\gg}[\lambda]\right)$

» TTR $\ll 1\gg$

Out[∘]=

$$\mathbb{E}[x12_j\,\xi 12_i - 2\,\epsilon\,a1_k\,x12_j\,\xi 12_i + x12_k\,\xi 12_i + 2\,\epsilon\,a1_j\,x12_k\,\xi 12_i + x13_j\,\xi 13_i - 2\,\epsilon\,a1_k\,x13_j\,\xi 13_i - 2\,\epsilon\,a2_k\,x13_j\,\xi 13_i + x13_k\,\xi 13_i + 2\,\epsilon\,a1_j\,x13_k\,\xi 13_i + 2\,\epsilon\,a2_j\,x13_k\,\xi 13_i + \epsilon\,x12_k\,x13_j\,\xi 12_i\,\xi 13_i - \epsilon\,x12_j\,x13_k\,\xi 12_i\,\xi 13_i + x23_j\,\xi 23_i - 2\,\epsilon\,a2_k\,x23_j\,\xi 23_i + x23_k\,\xi 23_i + 2\,\epsilon\,a2_j\,x23_k\,\xi 23_i - \epsilon\,x12_k\,x23_j\,\xi 12_i\,\xi 23_i + \epsilon\,x13_k\,x23_j\,\xi 13_i\,\xi 23_i - \epsilon\,x13_j\,x23_k\,\xi 13_i\,\xi 23_i + \epsilon\,x12_j\,x23_k\,(4\,\xi 13_i + \xi 12_i\,\xi 23_i)]$$

In[∘]:=

$$\mathbb{E}[x12_j\,\xi 12_i - 2\,\epsilon\,a1_k\,x12_j\,\xi 12_i + x12_k\,\xi 12_i + 2\,\epsilon\,a1_j\,x12_k\,\xi 12_i + x13_j\,\xi 13_i - 2\,\epsilon\,a1_k\,x13_j\,\xi 13_i - 2\,\epsilon\,a2_k\,x13_j\,\xi 13_i + x13_k\,\xi 13_i + 2\,\epsilon\,a1_j\,x13_k\,\xi 13_i + 2\,\epsilon\,a2_j\,x13_k\,\xi 13_i + \epsilon\,x12_k\,x13_j\,\xi 12_i\,\xi 13_i - \epsilon\,x12_j\,x13_k\,\xi 12_i\,\xi 13_i + x23_j\,\xi 23_i - 2\,\epsilon\,a2_k\,x23_j\,\xi 23_i + x23_k\,\xi 23_i + 2\,\epsilon\,a2_j\,x23_k\,\xi 23_i - \epsilon\,x12_k\,x23_j\,\xi 12_i\,\xi 23_i + \epsilon\,x13_k\,x23_j\,\xi 13_i\,\xi 23_i - \epsilon\,x13_j\,x23_k\,\xi 13_i\,\xi 23_i + \epsilon\,x12_j\,x23_k\,(4\,\xi 13_i + \xi 12_i\,\xi 23_i)]\ /.\ \epsilon \to 0$$

Time: 0.06 seconds Out[∘]=

$$\mathbb{E}[x12_j\,\xi 12_i + x12_k\,\xi 12_i + x13_j\,\xi 13_i + x13_k\,\xi 13_i + x23_j\,\xi 23_i + x23_k\,\xi 23_i]$$