

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

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This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ $\epsilon$ _, op_] := Collect[Normal@Series[_ $\epsilon$ _, { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_ $\epsilon$ _] := SS[_ $\epsilon$ _, Together];
```

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```
In[ ]:= CCF[_ $\epsilon$ _] := PP_CCF@Expand[Expand[_ $\epsilon$ _] /.  $e^p \rightarrow e^{\text{Factor}[p]}$ ];
CF[_ $\epsilon$ _] := PP_CF@If[$k ==  $\infty$ ,
  CCF[_ $\epsilon$ _,
  CCF[Normal@Series[_ $\epsilon$ _ /.  $q \rightarrow e^{\hbar \epsilon}$ , { $\epsilon$ , 0, $k}]]
];
```

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```
In[ ]:= x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
```

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```
In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[ ]:= (* q=eh γ ∈; Ai[n]=eh ∈ ai; Bi[n]=eh γ bi *)
```

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```
In[ ]:= $n = 3; $k = ∞; $E := {$n, $k}
```

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```
In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; A[i][_]],
  Table[As[i, 1, $n - 1],
    a[i]* = α[i]; α[i]* = a[i]; a[i]]
}
(#0 = U0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[ ]:= #* & /@ $EPBWGens
```

Time: 0.00 seconds Out[]:=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34,
 ξ35, ξ45, A1[_]*, A2[_]*, A3[_]*, A4[_]*, α1, α2, α3, α4}
```

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```
In[*]:= UCF[ $\mathcal{E}$ _] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence @@ Table[a @@ L[AB], k], r]
  }],
  U[____], F
] /. F → CF]
```

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```
In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}$ _] := ms[UCF[ $\mathcal{E}$ ]];
```

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```
In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@ mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l
```

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```
In[*]:= Si[_] := UCF@Module[{i1},  $\mathcal{E}$  /. {
  Ui[f_, fs____] ⇒ mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];
```

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The $\mathbb{A}$ -algebra

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```
In[ ]:= RR_k[ai_, aj_] /;  $\tau[ai] == \tau[aj] == "a" := U_k[aj, ai];$ 
RR_k[ai_, Aj_[n_]] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[Aj[n], ai];$ 
RR_k[Ai_[m_], Aj_[n_]] /;  $\tau[Ai] == \tau[Aj] == "A" := U_k[Aj[n], Ai[m]];$ 
RR_k[am_, xij_] /;  $\tau[am] == "a" \wedge \tau[xij] == "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@am$ ;
   $U_k[xij, am] + \gamma(\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij] / 2$ 
];
RR_k[Am_[n_], xij_] /;  $\tau[Am] == "A" \wedge \tau[xij] == "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@Am$ ;  $q^{-n}(\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij, Am[n]]$ 
]
```

```
In[ ]:= RR_0[a[1], x[1, 2]]
```

Time: 0.01 seconds Out[]:=

```
 $\gamma U_0[x12] + U_0[x12, a1]$ 
```

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We take the reduction rules RR for xij's from Yamane page

509:

- (I) $i < m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,
 $\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$
 (IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$,
 $\frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n} \quad \frac{i \quad j}{m \quad n}$

$$\begin{array}{ll}
 q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\
 [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\
 [e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\
 q^2e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}.
 \end{array}$$

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```
In[ ]:= RR_k[xmn_, xij_] /;  $\tau[xij] == \tau[xmn] == "x" := Module[{i, j, m, n},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m, n} =  $\mathcal{L}@xmn$ ;
  Which[
    ( $i == m \wedge j < n$ )  $\vee$  ( $i < m < j \wedge n == j$ ),  $q^{-2} U_k[xij, xmn]$ , (* IVIII *)
    ( $i < m < j \wedge n < j$ )  $\vee$  ( $j < m$ ),  $U_k[xij, xmn]$ , (* IIVVI *)
     $i < m < j \wedge n > j$ ,  $U_k[xij, xmn] + (q^{-2} - q^2) U_k[x[i, n], x[m, j]]$ , (* IV *)
     $j == m$ ,  $q^2 U_k[xij, xmn] - q U_k[x[i, n]]$  (* V *)
  ]
]
```

```

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.03 seconds Out|=
-q U2[x24] U3[x12] + q^2 U2[x23, x34] U3[x12]

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.06 seconds Out|=
q^2 U0[x14] - q^3 U0[x12, x24] - q^3 U0[x13, x34] + q^4 U0[x12, x23, x34]

In[*]:= Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
Time: 40.53 seconds Out|=
{}

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}

Out[*]=
{}

Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
  ]]
]
» {x12, x13, x23, a1, a2}

Out[*]=
{}

```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[]:=

```

Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1→k[mj, j1→j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]]
]

```

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@Bas], P → Δi→j,k[P]]
]
```

Time: 0.12 seconds Out[]=

$$\begin{aligned}
U_i[x_{12}] &\rightarrow U_j[x_{12}] U_k[] - 2 \in \hbar U_j[x_{12}] U_k[a_1] + U_j[] U_k[x_{12}] + \\
&2 \in \hbar U_j[a_1] U_k[x_{12}] + 2 \epsilon^2 \hbar^2 U_j[a_1, a_1] U_k[x_{12}] + 2 \epsilon^2 \hbar^2 U_j[x_{12}] U_k[a_1, a_1] \\
U_i[x_{13}] &\rightarrow U_j[x_{13}] U_k[] - 2 \in \hbar U_j[x_{13}] U_k[a_1] - \\
&2 \in \hbar U_j[x_{13}] U_k[a_2] + U_j[] U_k[x_{13}] + 2 \in \hbar U_j[a_1] U_k[x_{13}] + 2 \in \hbar U_j[a_2] U_k[x_{13}] + \\
&2 \epsilon^2 \hbar^2 U_j[a_1, a_1] U_k[x_{13}] + 4 \epsilon^2 \hbar^2 U_j[a_1, a_2] U_k[x_{13}] + 2 \epsilon^2 \hbar^2 U_j[a_2, a_2] U_k[x_{13}] + \\
&4 \gamma \in \hbar U_j[x_{12}] U_k[x_{23}] + 8 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, a_2] U_k[x_{23}] + 2 \epsilon^2 \hbar^2 U_j[x_{13}] U_k[a_1, a_1] + \\
&4 \epsilon^2 \hbar^2 U_j[x_{13}] U_k[a_1, a_2] + 2 \epsilon^2 \hbar^2 U_j[x_{13}] U_k[a_2, a_2] - 8 \gamma \epsilon^2 \hbar^2 U_j[x_{12}] U_k[x_{23}, a_1] \\
U_i[x_{14}] &\rightarrow U_j[x_{14}] U_k[] - 2 \in \hbar U_j[x_{14}] U_k[a_1] - 2 \in \hbar U_j[x_{14}] U_k[a_2] - \\
&2 \in \hbar U_j[x_{14}] U_k[a_3] + U_j[] U_k[x_{14}] + 2 \in \hbar U_j[a_1] U_k[x_{14}] + 2 \in \hbar U_j[a_2] U_k[x_{14}] + \\
&2 \in \hbar U_j[a_3] U_k[x_{14}] + 2 \epsilon^2 \hbar^2 U_j[a_1, a_1] U_k[x_{14}] + 4 \epsilon^2 \hbar^2 U_j[a_1, a_2] U_k[x_{14}] + \\
&4 \epsilon^2 \hbar^2 U_j[a_1, a_3] U_k[x_{14}] + 2 \epsilon^2 \hbar^2 U_j[a_2, a_2] U_k[x_{14}] + 4 \epsilon^2 \hbar^2 U_j[a_2, a_3] U_k[x_{14}] + \\
&2 \epsilon^2 \hbar^2 U_j[a_3, a_3] U_k[x_{14}] + 4 \gamma \in \hbar U_j[x_{12}] U_k[x_{24}] + 8 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, a_2] U_k[x_{24}] + \\
&8 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, a_3] U_k[x_{24}] + 4 \gamma \in \hbar U_j[x_{13}] U_k[x_{34}] + 8 \gamma \epsilon^2 \hbar^2 U_j[x_{13}, a_3] U_k[x_{34}] + \\
&2 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_1, a_1] + 4 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_1, a_2] + 4 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_1, a_3] + \\
&2 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_2, a_2] + 4 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_2, a_3] + 2 \epsilon^2 \hbar^2 U_j[x_{14}] U_k[a_3, a_3] - \\
&8 \gamma \epsilon^2 \hbar^2 U_j[x_{12}] U_k[x_{24}, a_1] - 8 \gamma \epsilon^2 \hbar^2 U_j[x_{13}] U_k[x_{34}, a_1] - 8 \gamma \epsilon^2 \hbar^2 U_j[x_{13}] U_k[x_{34}, a_2] \\
U_i[x_{23}] &\rightarrow U_j[x_{23}] U_k[] - 2 \in \hbar U_j[x_{23}] U_k[a_2] + U_j[] U_k[x_{23}] + \\
&2 \in \hbar U_j[a_2] U_k[x_{23}] + 2 \epsilon^2 \hbar^2 U_j[a_2, a_2] U_k[x_{23}] + 2 \epsilon^2 \hbar^2 U_j[x_{23}] U_k[a_2, a_2] \\
U_i[x_{24}] &\rightarrow U_j[x_{24}] U_k[] - 2 \in \hbar U_j[x_{24}] U_k[a_2] - \\
&2 \in \hbar U_j[x_{24}] U_k[a_3] + U_j[] U_k[x_{24}] + 2 \in \hbar U_j[a_2] U_k[x_{24}] + 2 \in \hbar U_j[a_3] U_k[x_{24}] + \\
&2 \epsilon^2 \hbar^2 U_j[a_2, a_2] U_k[x_{24}] + 4 \epsilon^2 \hbar^2 U_j[a_2, a_3] U_k[x_{24}] + 2 \epsilon^2 \hbar^2 U_j[a_3, a_3] U_k[x_{24}] + \\
&4 \gamma \in \hbar U_j[x_{23}] U_k[x_{34}] + 8 \gamma \epsilon^2 \hbar^2 U_j[x_{23}, a_3] U_k[x_{34}] + 2 \epsilon^2 \hbar^2 U_j[x_{24}] U_k[a_2, a_2] + \\
&4 \epsilon^2 \hbar^2 U_j[x_{24}] U_k[a_2, a_3] + 2 \epsilon^2 \hbar^2 U_j[x_{24}] U_k[a_3, a_3] - 8 \gamma \epsilon^2 \hbar^2 U_j[x_{23}] U_k[x_{34}, a_2] \\
U_i[x_{34}] &\rightarrow U_j[x_{34}] U_k[] - 2 \in \hbar U_j[x_{34}] U_k[a_3] + U_j[] U_k[x_{34}] + \\
&2 \in \hbar U_j[a_3] U_k[x_{34}] + 2 \epsilon^2 \hbar^2 U_j[a_3, a_3] U_k[x_{34}] + 2 \epsilon^2 \hbar^2 U_j[x_{34}] U_k[a_3, a_3] \\
U_i[a_1] &\rightarrow U_j[a_1] U_k[] + U_j[] U_k[a_1] \\
U_i[a_2] &\rightarrow U_j[a_2] U_k[] + U_j[] U_k[a_2] \\
U_i[a_3] &\rightarrow U_j[a_3] U_k[] + U_j[] U_k[a_3]
\end{aligned}$$

```
In[*]:= Short[{Δ4,5[a2], Δ7,9[A2[77]], Δ1,2[x12], Δ1,2[x15]}, 8]
```

Time: 0.02 seconds Out[]=

$$\begin{aligned}
&\{U_4[a_2] U_5[] + U_4[] U_5[a_2], U_7[A_2[77]] U_9[A_2[77]]\}, \\
&U_1[A_1[-1]] U_2[x_{12}] + U_1[x_{12}] U_2[A_1[1]], U_1[A_1[-1], A_2[-1], A_3[-1], A_4[-1]] U_2[x_{15}] + \\
&\left(-\frac{1}{q^2} + q^2\right) U_1[x_{12}, A_2[-1], A_3[-1], A_4[-1]] U_2[x_{25}, A_1[1]] + \\
&\left(-\frac{1}{q^2} + q^2\right) U_1[x_{13}, A_3[-1], A_4[-1]] U_2[x_{35}, A_1[1], A_2[1]] + \\
&\left(-\frac{1}{q^2} + q^2\right) U_1[x_{14}, A_4[-1]] U_2[x_{45}, A_1[1], A_2[1], A_3[1]] + \\
&U_1[x_{15}] U_2[A_1[1], A_2[1], A_3[1], A_4[1]]\}
\end{aligned}$$

```
In[*]:= Δ1→1,2[U1[x13, a2] U0[x34]]
```

```
Time: 0.03 seconds Out[ ]=
```

$$\begin{aligned} & U_0[x34] U_1[A1[-1], A2[-1], a2] U_2[x13] + U_0[x34] U_1[A1[-1], A2[-1]] U_2[x13, a2] + \\ & \left(-\frac{1}{q^2} + q^2\right) U_0[x34] U_1[x12, A2[-1], a2] U_2[x23, A1[1]] + U_0[x34] U_1[x13, a2] U_2[A1[1], A2[1]] + \\ & \left(-\frac{1}{q^2} + q^2\right) U_0[x34] U_1[x12, A2[-1]] U_2[x23, A1[1], a2] + U_0[x34] U_1[x13] U_2[A1[1], A2[1], a2] \end{aligned}$$

```
In[*]:= $k = 1
```

```
Time: 0.01 seconds Out[ ]=
```

```
1
```

```
In[*]:= Module[{Bas = $PBWGens, P},
```

```
Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
```

```
DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
```

```
P = U1[X];
```

```
X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
```

```
]]
```

```
]
```

```
>> {x12, x13, x14, x23, x24, x34, a1, a2, a3, A2[1], A2[0],  
A1[-2], A3[-2], A1[-2], A1[-2], A3[2], A3[-2], A2[-2], A1[2]}
```

```
Out[*]=
```

```
{ }
```

```
In[*]:= $k = 1; $n = 4
```

```
Time: 0.02 seconds Out[ ]=
```

```
4
```

```
In[*]:= Module[{Bas = $PBWGens, P},
```

```
Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
```

```
DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
```

```
P = U1[X] U2[Y];
```

```
{X, Y} → UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
```

```
]]
```

```
]
```

```
>> {x12, x13, x14, x23, x24, x34, a1, a2, a3, A3[0],  
A2[-1], A2[0], A2[-2], A1[1], A1[-2], A2[2], A3[1], A1[0], A3[-2]}
```

```
Out[*]=
```

```
{ }
```

Is Δ a quantization of the co-bracket?

```
In[*]:= $k = 1
```

```
Time: 0.01 seconds Out[ ]=
```

```
1
```

```

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 / @ Bas],
    P → UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- → 1 + n ħ / 2, Ui[xs____, A_-[n_], as____] → Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 → 0
  ]
]

```

Time: 0.09 seconds Out[]=

```

U1[x12] → -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] → -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 γ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 γ ∈ ħ U1[x12] U2[x23]
U1[x14] → -4 ∈ ħ U1[x14] U2[a1] - 4 ∈ ħ U1[x14] U2[a2] - 4 ∈ ħ U1[x14] U2[a3] -
  4 γ ∈ ħ U1[x24] U2[x12] - 4 γ ∈ ħ U1[x34] U2[x13] + 4 ∈ ħ U1[a1] U2[x14] +
  4 ∈ ħ U1[a2] U2[x14] + 4 ∈ ħ U1[a3] U2[x14] + 4 γ ∈ ħ U1[x12] U2[x24] + 4 γ ∈ ħ U1[x13] U2[x34]
U1[x23] → -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[x24] → -4 ∈ ħ U1[x24] U2[a2] - 4 ∈ ħ U1[x24] U2[a3] - 4 γ ∈ ħ U1[x34] U2[x23] +
  4 ∈ ħ U1[a2] U2[x24] + 4 ∈ ħ U1[a3] U2[x24] + 4 γ ∈ ħ U1[x23] U2[x34]
U1[x34] → -4 ∈ ħ U1[x34] U2[a3] + 4 ∈ ħ U1[a3] U2[x34]
U1[a1] → 0
U1[a2] → 0
U1[a3] → 0

```

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The Antipode

pdf

```

In[*]:= Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1,j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

```

In[*]:= \$k = 2

Time: 0.00 seconds Out[]=

2

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P //  $\Delta_{1 \rightarrow 2, 3}$  // S2 // m2, 3 \rightarrow 1)
  ]
]

```

» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A2[0],
A1[0], A2[0], A2[1], A2[0], A3[0], A3[-1], A1[1], A1[-2], A3[0]}

Out[]:=

```

{x12 -> 0, x13 -> 0, x14 -> 0, x23 -> 0, x24 -> 0, x34 -> 0, a1 -> 0, a2 -> 0,
a3 -> 0, A2[0] -> U1[], A1[0] -> U1[], A2[0] -> U1[], A2[1] -> U1[], A2[0] -> U1[],
A3[0] -> U1[], A3[-1] -> U1[], A1[1] -> U1[], A1[-2] -> U1[], A3[0] -> U1[]}

```

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The Generating Functions for the $\mathbb{A}$ operations

```

In[ ]:= BeginProfile[]

```

Time: 0.30 seconds Out[]:=
ProfileRoot

```

In[ ]:= $k = 1;

```

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NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

pdf

```

In[ ]:= Supp[ $\mathcal{E}$ _] := Union@Cases[{ $\mathcal{E}$ }, Ui[_] -> i,  $\infty$ ];

```

```

In[ ]:= Supp[2  $\hbar$  U1[x45] U2[x34]]

```

Time: 0.01 seconds Out[]:=
{1, 2}

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```

In[ ]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_  $\boxtimes$  y_ := Module[{i, is = Supp[x]  $\cap$  Supp[y],  $\sigma$ , z},
  z = x; Do[z = mi  $\rightarrow$   $\sigma$  @ i[z], {i, is}];
  z = Expand[y z]; Do[z = m $\sigma$  @ i, i  $\rightarrow$  i[z], {i, is}]; z;
UB[x_, y_] := PPUB@UCF[x  $\boxtimes$  y - y  $\boxtimes$  x];

```

```

In[ ]:= (2  $\hbar$ 1 U1[x45] U2[x34])  $\boxtimes$  (2  $\hbar$ 2 U1[x45] U2[x34])

```

Time: 0.03 seconds Out[]:=
4 $\hbar$ ₁ $\hbar$ ₂ U₁[x45, x45] U₂[x34, x34]

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```
In[*]:= (*adPower[x_,k_,Env_] [c_*y_QU] :=UCF[c adPower[x,k,Env] [y]]];
adPower[x_,k_,Env_] [y_Plus] := adPower[x,k,Env] /@ y;*)
adPower[_ , 0, Env_] [y_] := y;
(*adPower[x_,k_,Env_] [0]=0;*)
adPower[x_, k_, Env_] [y_] :=
  adPower[x, k, Env] [y] = UCF@UB[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];
```

```
In[*]:= Table[As[k, 0, 5], adPower[U_0[x12], k] [U_0[x23]]]
```

Time: 0.06 seconds Out:=

$$\{U_0[x23], (-1 - \gamma \in \hbar) U_0[x13] + 2 \gamma \in \hbar U_0[x12, x23], 0, 0, 0, 0\}$$

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U_i /@ Bas], P → Δi→j,k[P]]
]
```

Time: 0.08 seconds Out:=

$$\begin{aligned} U_i[x12] &\rightarrow U_j[x12] U_k[] + \epsilon \hbar U_j[x12] U_k[a1] + U_j[] U_k[x12] + \epsilon \hbar U_j[a1] U_k[x12] \\ U_i[x13] &\rightarrow U_j[x13] U_k[] + \epsilon \hbar U_j[x13] U_k[a1] + \epsilon \hbar U_j[x13] U_k[a2] + \\ &U_j[] U_k[x13] + \epsilon \hbar U_j[a1] U_k[x13] + \epsilon \hbar U_j[a2] U_k[x13] + 4 \gamma \in \hbar U_j[x12] U_k[x23] \\ U_i[x23] &\rightarrow U_j[x23] U_k[] + \epsilon \hbar U_j[x23] U_k[a2] + U_j[] U_k[x23] + \epsilon \hbar U_j[a2] U_k[x23] \\ U_i[a1] &\rightarrow U_j[a1] U_k[] + U_j[] U_k[a1] \\ U_i[a2] &\rightarrow U_j[a2] U_k[] + U_j[] U_k[a2] \end{aligned}$$

pdf

```
In[*]:= Ad[0] = Identity;
```

pdf

```
In[*]:= Ad[c_. U_i[xjk_] (EUs : U_[] ...)] [y_] /; τ[xjk] == "x" := PPAd@Module[{k = 0, s = 0, t},
  While[(t = adPower[U_i[xjk], k] [y]) != 0,
    s += c^k t / k!;
    ++k
  ];
  UCF[s ⊗ EUs]
]
```

```
In[*]:= Ad[ε ħ U_j[x12] U_k[]] [U_j[x23] U_k[a4]]
```

Time: 0.09 seconds Out:=

$$-\epsilon \hbar U_j[x13] U_k[a4] + U_j[x23] U_k[a4]$$

```
In[*]:= Ad[7 ħ U_j[x12] U_k[]] [U_j[x23] U_k[a4]]
```

Time: 0.08 seconds Out:=

$$(-7 \hbar - 7 \gamma \in \hbar^2) U_j[x13] U_k[a4] + U_j[x23] U_k[a4] + 14 \gamma \in \hbar^2 U_j[x12, x23] U_k[a4]$$

pdf

```
In[*]:= Wtaj[_] := Wtaj[x] = UB[U_0[aj], U_0[x]] / U_0[x]
```

```
In[ ]:= Table[Wta[j] [x23], {j, 1, $n}]
```

Time: 0.08 seconds Out[]=

$$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}, 0, 0\right\}$$

pdf

```
In[ ]:= Ad[c-. Ui[aj-] (EUs : U-[...] [y-] /; τ[aj] == "a" :=  
(y /. Ui[gs---] => Ui[gs] ec Total[Wtaj/e{gs}])
```

```
In[ ]:= Ad[π ± Uj[a1]] [Uj[x23] Uk[a4]]
```

Time: 0.03 seconds Out[]=

$$e^{-\frac{1}{2} \pm \pi \gamma} U_j[x23] U_k[a4]$$

pdf

```
In[ ]:= TTλ[ ] = 0; (* Translated Tangent *)  
TTλ[xs---, x-] := PPTT@UCF[Ad[x] [TTλ[xs]] + ∂λ x];  
TTλ[xs_List] := TTλ@@xs
```

```
In[ ]:= TTλ[λ ξ12 U0[x12] U1[ ], λ ξ23 U0[x23] U1[ ]]
```

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[x12] U_1[] + \lambda \xi_{12} \xi_{23} U_0[x13] U_1[] + \xi_{23} U_0[x23] U_1[]$$

```
In[ ]:= $PBWGens
```

Time: 0.00 seconds Out[]=

$$\{x12, x13, x23, a1, a2\}$$

```
In[ ]:= Join[  
Cases[$PBWGens, g- => Which[  
τ[g] === "x", λ (g*)i Uk[g],  
τ[g] === "a", (g*)i Uk[g]  
]],  
Cases[$PBWGens, g- => Which[  
τ[g] === "x", λ (g*)j Uk[g],  
τ[g] === "a", (g*)j Uk[g]  
]]  
]
```

Time: 0.03 seconds Out[]=

$$\{\lambda \xi_{12i} U_k[x12], \lambda \xi_{13i} U_k[x13], \lambda \xi_{23i} U_k[x23], \alpha_{1i} U_k[a1], \alpha_{2i} U_k[a2], \\ \lambda \xi_{12j} U_k[x12], \lambda \xi_{13j} U_k[x13], \lambda \xi_{23j} U_k[x23], \alpha_{1j} U_k[a1], \alpha_{2j} U_k[a2]\}$$

```
In[ ]:= Oi[poly-] := Total[  
CoefficientRules[poly, $PBWGens] /.  
(p- → c-) => c Ui@@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]  
]
```

In[*]:= $0_i [x_{12}^2 x_{13} + 7 x_{23}^{17}]$

Time: 0.03 seconds Out[=

$U_i [x_{12}, x_{12}, x_{13}] +$
 $7 U_i [x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$

pdf

In[*]:= **Seepage_i** [{}, c_] = c;
Seepage_i [ξs_List, P_] := Module[{lx},
 lx = \$PBWGens[[Length@ξs]];
 CF@Collect[P, lx, (Ad[Last[ξs] U_i[lx]] [0_i[Seepage_i[Most@ξs, #]]] /. U_i → Times) &]
]

pdf

In[*]:= **Invert_i** [ε_] := Module[{A, B, β, j}, (* ε=A U[] +ε B *)
 A = Coefficient[ε, U_i[]] /. ε → 0;
 B = UCF[(ε - A U_i[])];
 UCF[
 A⁻¹ (U_i[] + Sum[As[j, 1, \$k], (-1)^j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]

{ \$k = 1, \$n = 3 }

Time: 0.01 seconds Out[=

{ 2, 3 }

pdf

In[*]:= **sol** [kk_, nn_] := PP^{sol} [kk, nn] @Block[{ \$k = kk, \$n = nn }, Module[{ (*TTL, *) },
 ES[s_] [ε_] := (EchoLabel[s] [Short[ε, 5]] ; ε);
 TTL = ES["TTL"] @TT_λ@
 Join[
 Cases[\$PBWGens, g_ => Which[
 τ[g] == "x", λ (g*)_i U_k[g],
 τ[g] == "a", (g*)_i U_k[g]
]],
 Cases[\$PBWGens, g_ => Which[
 τ[g] == "x", λ (g*)_j U_k[g],
 τ[g] == "a", (g*)_j U_k[g]
]]
];
 mons[-1] = {};
 mons[p_] := mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], U_k[____], ∞];
 ff = Table[
 Sum[ε^p f_{p,u}[λ], {p, 0, \$k}],
 {u, \$PBWGens}
] /. f_{p,u}[λ] /; τ[u] == "a" => If[p == 0, (u*)_i + (u*)_j, 0];
 fs = Flatten[Table[

```

fk, Times@@m,
{k, 0, $k}, {m, mons[k]}
]];
gg = Sum[
ep fp, Times@@u [λ] Times @@ u,
{p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
];
G = Series[egg, {ε, 0, $k}];

```

$$\begin{aligned}
L_\lambda &= \mathbb{C}\langle \lambda_{i_1} x \rangle \mathbb{C}\langle \lambda_{i_2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i(x) \rangle \mathbb{C}\langle \text{Seepage}(f_i, A) \rangle &= \mathbb{C}\langle e^{f_i} A \rangle \\
\mathbb{T}(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(f_i, B). \\
R_\lambda &= \mathbb{C}\langle f + g \rangle, \quad f = f_i x; \quad g = \sum_{h \neq i} g_h x^h & e^f = F \quad e^g = G \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f \\
\mathbb{T}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
\end{aligned}$$

```

Si[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
SG = ES["SG"] @ Sk[G];
TTR = ES["TTR"] @ UCF[Invertk[SG] ⊗ Sk[(∂λ(ff.$PBWGens) G + ∂λG)]];
diff = ES["diff"] @ UCF[TTL - TTR];
eqns = ES["eqns"] @ Flatten[Table[
Coefficient[Coefficient[diff, m], ε, k] == 0,
{k, 0, $k}, {m, mons[k]}
]];
inits = (#[0] == 0) & /@ fs;
ES["DSolve"] [
DSolve[eqns ∪ inits, fs, λ] /. F_Function => ReplacePart[F, 2 -> CF[F[[2]]]]
]]

```

In[*]:= sol[2, 2]

```

» TTL (eγ (α1+α1j) ξ12i + eγ α1j ξ12j) Uk[x12]
» SG Uk[]
» TTR Uk[x12] (eγ (α1+α1j) f0,x12'[λ] + eγ (α1+α1j) ∈ f1,x12'[λ] + eγ (α1+α1j) ∈2 f2,x12'[λ])
» diff
Uk[x12] (eγ (α1+α1j) ξ12i + eγ α1j ξ12j - eγ (α1+α1j) f0,x12'[λ] - eγ (α1+α1j) ∈ f1,x12'[λ] - eγ (α1+α1j) ∈2 f2,x12'[λ])
» eqns {eγ (α1+α1j) ξ12i + eγ α1j ξ12j - eγ (α1+α1j) f0,x12'[λ] == 0,
-eγ (α1+α1j) f1,x12'[λ] == 0, -eγ (α1+α1j) f2,x12'[λ] == 0}
» DSolve
{{f0,x12 -> Function[{λ}, λ ξ12i + e-γ α1 λ ξ12j], f1,x12 -> Function[{λ}, 0], f2,x12 -> Function[{λ}, 0]}}

```

Out[*]=

```

{{f0,x12 -> Function[{λ}, λ ξ12i + e-γ α1 λ ξ12j],
f1,x12 -> Function[{λ}, 0], f2,x12 -> Function[{λ}, 0]}}

```

In[*]:= sol[1, 3]

» TTL $\left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} \right) U_k[x_{12}] +$
 $\ll 5 \gg + \left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1-2\alpha_1-3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_i} + \right.$
 $\left. 2 e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2+3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_j} + 2 e^{\frac{3}{2}\gamma\alpha_2} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_j} \right) U_k[x_{13}, x_{23}]$

» SG $U_k[] + e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \in U_k[x_{12}, x_{13}] f_{1,x_{12}x_{13}}[\lambda] +$
 $U_k[x_{13}, x_{13}] \left(e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{\theta,x_{23}}[\lambda] f_{1,x_{12}x_{13}}[\lambda] + e^{\gamma(\alpha_1+\alpha_1+\ll 1 \gg + \alpha_2)} \in f_{1,\ll 3 \gg^2}[\lambda] \right) +$
 $e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in U_k[x_{12}, x_{23}] f_{1,x_{12}x_{23}}[\lambda] +$
 $U_k[x_{13}, x_{23}] \left(e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{\theta,x_{23}}[\lambda] f_{1,x_{12}x_{23}}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{1,x_{13}x_{23}}[\lambda] \right)$

» TTR $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\theta,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}'}[\lambda] \right) + \ll 5 \gg + U_k[x_{13}, x_{23}]$
 $\left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \hbar f_{\theta,x_{23}}[\lambda] f_{\theta,x_{13}'}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{\theta,x_{23}}[\lambda] f_{1,x_{12}x_{23}'}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{1,x_{13}x_{23}'}[\lambda] \right)$

» diff $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - \right.$
 $\left. e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\theta,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}'}[\lambda] \right) +$
 $U_k[x_{13}] \left(e^{\ll 1 \gg} \xi_{13_i} + e^{\ll 1 \gg} \ll 1 \gg + \ll 22 \gg \right) + \ll 3 \gg + U_k[x_{12}, x_{23}] \left(\ll 1 \gg + \ll 8 \gg \right) +$
 $U_k[x_{13}, x_{23}] \left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1-2\ll 2 \gg_i-3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_i} + \ll 8 \gg \right)$

» eqns

$$\left\{ e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\theta,x_{12}'}[\lambda] = 0, e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \xi_{13_i} + \right.$$

$$e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \xi_{13_j} + e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \lambda \xi_{12_i} \xi_{23_i} - \ll 1 \gg + \ll 1 \gg + e^{\frac{1}{2}\gamma(\ll 1 \gg)} \lambda \xi_{12_j} \xi_{23_j} -$$

$$e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} f_{\theta,x_{23}}[\lambda] f_{\theta,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} f_{\theta,x_{13}'}[\lambda] = 0, \ll 6 \gg, \ll 1 \gg, \ll 1 \gg \left. \right\}$$

» DSolve $\left\{ \left\{ f_{\theta,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$
 $f_{\theta,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_1)} \lambda \xi_{23_j}],$
 $f_{\theta,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{13_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{23_i}],$
 $\ll 5 \gg, f_{1,x_{13}^2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0] \left. \right\}$

Out[]:=

$$\left\{ \left\{ f_{\theta,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$$

$$f_{\theta,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_1)} \lambda \xi_{23_j}],$$

$$f_{\theta,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{13_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{23}} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{13}} \rightarrow \text{Function}[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{12}x_{23}} \rightarrow \text{Function}[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{12}x_{13}} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{13_i}],$$

$$f_{1,x_{13}x_{23}} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{13_j} \xi_{23_i}],$$

$$f_{1,x_{13}^2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0] \left. \right\}$$

In[]:= sol[1, 4]

» **TTL** $\left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} \right) U_k[x_{12}] + \ll 21 \gg +$
 $\left(2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_i} \xi_{34_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-2\alpha_3-3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_j} \xi_{34_i} + \right.$
 $\left. 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-\alpha_2-\alpha_3-3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_i} \xi_{34_j} + 2 e^{-\frac{1}{2}\gamma(\alpha_1-3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_j} \xi_{34_j} \right) U_k[x_{24}, x_{34}]$

» **SG** $U_k[] + e^{\frac{1}{2}\gamma(3\alpha_1+3\alpha_1-\alpha_3-\alpha_3)} \in U_k[x_{12}, x_{13}] f_{1,x_{12}x_{13}}[\lambda] +$
 $\ll 14 \gg + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-\alpha_2-\alpha_2-\alpha_3-\alpha_3)} \in U_k[x_{23}, x_{34}] f_{1,x_{23}x_{34}}[\lambda] +$
 $U_k[x_{24}, x_{34}] \left(e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \in f_{\emptyset,x_{34}}[\lambda] f_{1,x_{23}x_{34}}[\lambda] + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \in f_{1,x_{24}x_{34}}[\lambda] \right)$

» **TTR** $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}}'[\lambda] + e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}}'[\lambda] \right) +$
 $\ll 21 \gg + U_k[x_{24}, x_{34}] \left(2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \gamma \in \hbar f_{\emptyset,x_{34}}[\lambda] f_{\emptyset,x_{24}}'[\lambda] + \right.$
 $\left. e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \in f_{\emptyset,x_{34}}[\lambda] f_{1,x_{23}x_{34}}'[\lambda] + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-3\alpha_3-3\alpha_3)} \in f_{1,x_{24}x_{34}}'[\lambda] \right)$

» **diff** $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - \right.$
 $\left. e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}}'[\lambda] - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}}'[\lambda] \right) +$
 $U_k[x_{13}] \left(e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2-\alpha_3-\alpha_3)} \xi_{13_i} + e^{\langle 1 \rangle} \xi_{13_j} + \ll 22 \gg \right) + U_k[x_{12}, x_{13}] (\ll 1 \gg) + \ll 17 \gg +$
 $\ll 1 \gg \ll 1 \gg + U_k[x_{23}, x_{34}] \left(-2 e^{\frac{1}{2}\gamma(\langle 6 \rangle + \alpha_3)} \gamma \in \lambda \hbar \xi_{23_i} \xi_{34_i} + \ll 8 \gg \right) + U_k[x_{24}, x_{34}] (\ll 1 \gg)$

» **eqns** $\left\{ e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}}'[\lambda] = 0, \right.$
 $e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2-\alpha_3-\alpha_3)} \xi_{13_i} + e^{\frac{1}{2}\gamma(\langle 1 \rangle)} \xi_{13_j} + \ll 9 \gg = 0,$
 $e^{\frac{1}{2}\gamma(\langle 1 \rangle)} \xi_{14_i} + \ll 27 \gg = 0, \langle 1 \rangle = 0, \langle 21 \rangle, \langle 1 \rangle = 0, \langle 1 \rangle = 0, \langle 1 \rangle = 0,$
 $2 e^{\frac{1}{2}\gamma(-\alpha_1-\alpha_1+3\alpha_3+3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_i} \xi_{34_i} - 2 e^{\frac{1}{2}\gamma(-\alpha_1-\alpha_2+2\alpha_3+3\alpha_3)} \gamma \in \lambda \hbar \xi_{24_j} \xi_{34_i} + \ll 8 \gg = 0 \}$

» **DSolve** $\left\{ \left\{ f_{\emptyset,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$
 $f_{\emptyset,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \lambda \xi_{23_j}], \ll 26 \gg, f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0] \}$

Out[]=

$\left\{ \left\{ f_{\emptyset,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$
 $f_{\emptyset,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \lambda \xi_{23_j}],$
 $f_{\emptyset,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \lambda \xi_{13_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{23_i}],$
 $f_{\emptyset,x_{34}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{34_i} + e^{\frac{1}{2}\gamma(\alpha_2-2\alpha_3)} \lambda \xi_{34_j}],$
 $f_{\emptyset,x_{24}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{24_i} + e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \lambda \xi_{24_j} - e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \lambda^2 \xi_{23_j} \xi_{34_i}],$
 $f_{\emptyset,x_{14}} \rightarrow \text{Function}[\{\lambda\},$
 $\lambda \xi_{14_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \lambda \xi_{14_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{24_i} - e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \lambda^2 \xi_{13_j} \xi_{34_i}],$
 $f_{1,x_{23}} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{13}} \rightarrow \text{Function}[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$
 $f_{1,x_{12}x_{23}} \rightarrow \text{Function}[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$
 $f_{1,x_{12}x_{13}} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{13_i}],$

$$\begin{aligned}
f_{1,x13 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{23i}\right], \\
f_{1,x13^2} &\rightarrow \text{Function}\left[\{\lambda\}, 0\right], f_{1,x12 \times 24} \rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i}\right], \\
f_{1,x12 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{14i}\right], \\
f_{1,x34} &\rightarrow \text{Function}\left[\{\lambda\}, 0\right], f_{1,x24} \rightarrow \text{Function}\left[\{\lambda\}, -e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x14} &\rightarrow \text{Function}\left[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{24i} - e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], \\
f_{1,x23 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{34i}\right], \\
f_{1,x23 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^2 \hbar \xi_{23j} \xi_{24i}\right], \\
f_{1,x14 \times 23} &\rightarrow \text{Function}\left[\{\lambda\}, -4 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{24i}\right], \\
f_{1,x13 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{34i}\right], f_{1,x13 \times 24} \rightarrow \\
&\text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{23i} \xi_{24i} - \frac{2}{3} e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{12i} \xi_{23j} \xi_{24i} + \right. \\
&\left. \frac{2}{3} e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^3 \hbar \xi_{12i} \xi_{23j} \xi_{24i} - 2 e^{-\frac{1}{2}\gamma(3\alpha_2-2\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{23j} \xi_{34i}\right], \\
f_{1,x13 \times 14} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{14i} + \right. \\
&2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{14i} \xi_{23i} + 2 e^{-\frac{1}{2}\gamma(3\alpha_1-\alpha_3)} \gamma \lambda^3 \hbar \xi_{12j} \xi_{13j} \xi_{24i} + \\
&\frac{1}{6} e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i}^2 \xi_{23j} \xi_{24i} - \frac{1}{6} e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i}^2 \xi_{23j} \xi_{24i} - \\
&e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^4 \hbar \xi_{12j} \xi_{13i} \xi_{23i} \xi_{34i} + e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \lambda^4 \hbar \xi_{12j} \xi_{13i} \xi_{23i} \xi_{34i} + \\
&\frac{1}{2} e^{\frac{1}{2}\gamma(\alpha_2-2\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{13i} \xi_{23i} \xi_{34j} - \frac{1}{2} e^{-\frac{1}{2}\gamma(-\alpha_2+2\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{13i} \xi_{23i} \xi_{34j} + \\
&\left. e^{\gamma(-\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^4 \hbar \xi_{12j} \xi_{13i} \xi_{23i} \xi_{34j} - e^{-\gamma(\alpha_1-\alpha_2+\alpha_3)} \gamma \lambda^4 \hbar \xi_{12j} \xi_{13i} \xi_{23i} \xi_{34j}\right], \\
f_{1,x24 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^2 \hbar \xi_{24j} \xi_{34i}\right], f_{1,x24^2} \rightarrow \text{Function}\left[\{\lambda\}, 0\right], \\
f_{1,x14 \times 34} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{34i}\right], \\
f_{1,x14 \times 24} &\rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_3)} \gamma \lambda^2 \hbar \xi_{14j} \xi_{24i} - \right. \\
&\frac{2}{3} e^{-\frac{1}{2}\gamma(-\alpha_1+2\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{14i} \xi_{23j} \xi_{34i} + \frac{2}{3} e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2+\alpha_3)} \gamma \lambda^3 \hbar \xi_{14i} \xi_{23j} \xi_{34i} + \\
&2 e^{-\gamma\alpha_2} \gamma \lambda^3 \hbar \xi_{14j} \xi_{23j} \xi_{34i} + 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{13j} \xi_{24i} \xi_{34i} + \\
&\frac{4}{3} e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^3 \hbar \xi_{13i} \xi_{24j} \xi_{34i} - \frac{4}{3} e^{-\frac{1}{2}\gamma(-\alpha_1+\alpha_2+\alpha_3)} \gamma \lambda^3 \hbar \xi_{13i} \xi_{24j} \xi_{34i} + \\
&\frac{1}{2} e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23i} \xi_{24j} \xi_{34i} - \frac{1}{2} e^{-\frac{1}{2}\gamma(-\alpha_1+\alpha_2+\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23i} \xi_{24j} \xi_{34i} + \\
&\frac{1}{2} e^{\frac{1}{2}\gamma(2\alpha_1-3\alpha_2)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23j} \xi_{24j} \xi_{34i} - \frac{1}{2} e^{-\frac{1}{2}\gamma(-2\alpha_1+3\alpha_2)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23j} \xi_{24j} \xi_{34i} - \\
&\left. \frac{1}{2} e^{\frac{1}{2}\gamma(\alpha_1-\alpha_2-\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23j} \xi_{24i} \xi_{34j} + \frac{1}{2} e^{-\frac{1}{2}\gamma(-\alpha_1+\alpha_2+\alpha_3)} \gamma \lambda^4 \hbar \xi_{12i} \xi_{23j} \xi_{24i} \xi_{34j} + \right.
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} e^{\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 - 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_j} \xi_{23_i} \xi_{24_j} \xi_{34_j} - \frac{1}{2} e^{-\frac{1}{2} \gamma (\alpha_1 - \alpha_2 + 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_j} \xi_{23_i} \xi_{24_j} \xi_{34_j} + \\
& \frac{1}{2} e^{\gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{23_j} \xi_{24_j} \xi_{34_j} - \frac{1}{2} e^{-\gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{23_j} \xi_{24_j} \xi_{34_j} + \\
& \frac{1}{2} e^{\frac{1}{2} \gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_i} \xi_{23_j} \xi_{34_i} \xi_{34_j} - \frac{1}{2} e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_i} \xi_{23_j} \xi_{34_i} \xi_{34_j} + \\
& \frac{1}{2} e^{\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 - 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_j} \xi_{23_i} \xi_{34_j^2} - \frac{1}{2} e^{-\frac{1}{2} \gamma (\alpha_1 - \alpha_2 + 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_j} \xi_{23_i} \xi_{34_j^2} - \\
& \frac{1}{2} e^{\frac{1}{2} \gamma (\alpha_1 - 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_i} \xi_{23_j} \xi_{34_j^2} + \frac{1}{2} e^{-\frac{1}{2} \gamma (-\alpha_1 + 3 \alpha_3)} \gamma \lambda^4 \hbar \xi_{13_i} \xi_{23_j} \xi_{34_j^2} \Big], \\
& f_{1,x14^2} \rightarrow \text{Function} \Big[\{\lambda\}, \frac{1}{6} e^{-\frac{1}{2} \gamma (-\alpha_1 + 2 \alpha_2 - \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{14_i} \xi_{23_j} \xi_{34_i} - \\
& \frac{1}{6} e^{\frac{1}{2} \gamma (\alpha_1 - 2 \alpha_2 + \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{14_i} \xi_{23_j} \xi_{34_i} - \frac{1}{3} e^{\frac{1}{2} \gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{13_i} \xi_{24_j} \xi_{34_i} + \\
& \frac{1}{3} e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{13_i} \xi_{24_j} \xi_{34_i} - \frac{1}{10} e^{\frac{1}{2} \gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_i} \xi_{24_j} \xi_{34_i} + \\
& \frac{1}{10} e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_i} \xi_{24_j} \xi_{34_i} - \frac{1}{10} e^{\frac{1}{2} \gamma (2 \alpha_1 - 3 \alpha_2)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_j} \xi_{34_i} + \\
& \frac{1}{10} e^{-\frac{1}{2} \gamma (-2 \alpha_1 + 3 \alpha_2)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_j} \xi_{34_i} + \frac{2}{5} e^{-\frac{1}{2} \gamma (2 \alpha_1 - \alpha_2)} \gamma \lambda^5 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} \xi_{34_i^2} - \\
& \frac{2}{5} e^{\frac{1}{2} \gamma (-2 \alpha_1 + \alpha_2)} \gamma \lambda^5 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} \xi_{34_i^2} - \frac{2}{3} e^{\frac{1}{2} \gamma (\alpha_2 - 2 \alpha_3)} \gamma \lambda^3 \hbar \xi_{13_i} \xi_{14_i} \xi_{34_j} + \\
& \frac{2}{3} e^{-\frac{1}{2} \gamma (-\alpha_2 + 2 \alpha_3)} \gamma \lambda^3 \hbar \xi_{13_i} \xi_{14_i} \xi_{34_j} - \frac{1}{2} e^{\frac{1}{2} \gamma (\alpha_2 - 2 \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{14_i} \xi_{23_i} \xi_{34_j} + \\
& \frac{1}{2} e^{-\frac{1}{2} \gamma (-\alpha_2 + 2 \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_i} \xi_{14_i} \xi_{23_i} \xi_{34_j} - e^{\gamma (-\alpha_1 + \alpha_2 - \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_j} \xi_{13_i} \xi_{24_i} \xi_{34_j} + \\
& e^{-\gamma (\alpha_1 - \alpha_2 + \alpha_3)} \gamma \lambda^4 \hbar \xi_{12_j} \xi_{13_i} \xi_{24_i} \xi_{34_j} - \frac{2}{5} e^{\gamma (-\alpha_1 + \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{12_j} \xi_{23_i} \xi_{24_i} \xi_{34_j} + \\
& \frac{2}{5} e^{-\gamma (\alpha_1 - \alpha_2 + \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{12_j} \xi_{23_i} \xi_{24_i} \xi_{34_j} + \frac{1}{10} e^{\frac{1}{2} \gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_i} \xi_{34_j} - \\
& \frac{1}{10} e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_i} \xi_{34_j} - \frac{1}{10} e^{\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 - 3 \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \\
& \xi_{12_j} \xi_{23_i} \xi_{24_j} \xi_{34_j} + \frac{1}{10} e^{-\frac{1}{2} \gamma (\alpha_1 - \alpha_2 + 3 \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{12_j} \xi_{23_i} \xi_{24_j} \xi_{34_j} - \\
& \frac{1}{10} e^{\gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_j} \xi_{34_j} + \frac{1}{10} e^{-\gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i^2} \xi_{23_j} \xi_{24_j} \xi_{34_j} - \\
& \frac{4}{5} e^{\frac{1}{2} \gamma (\alpha_2 - 2 \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_i} \xi_{34_i} \xi_{34_j} + \frac{4}{5} e^{-\frac{1}{2} \gamma (-\alpha_2 + 2 \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_i} \xi_{34_i} \xi_{34_j} - \\
& \frac{8}{5} e^{\gamma (-\alpha_1 + \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} \xi_{34_i} \xi_{34_j} + \frac{8}{5} e^{-\gamma (\alpha_1 - \alpha_2 + \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} \\
& \xi_{34_i} \xi_{34_j} - \frac{1}{10} e^{\frac{1}{2} \gamma (\alpha_1 - \alpha_2 - \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_j} \xi_{34_i} \xi_{34_j} + \frac{1}{10} e^{-\frac{1}{2} \gamma (-\alpha_1 + \alpha_2 + \alpha_3)} \\
& \gamma \lambda^5 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_j} \xi_{34_i} \xi_{34_j} - \frac{2}{5} e^{\frac{1}{2} \gamma (-2 \alpha_1 + 3 \alpha_2 - 4 \alpha_3)} \gamma \lambda^5 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} \xi_{34_j^2} +
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{5} e^{-\frac{1}{2} \gamma (2 \alpha 1_i - 3 \alpha 2_i + 4 \alpha 3_i)} \gamma \lambda^5 \hbar \xi 12_j \xi 13_i \xi 23_i \xi 34_j^2 - \frac{1}{10} e^{\frac{1}{2} \gamma (-\alpha 1_i + \alpha 2_i - 3 \alpha 3_i)} \gamma \lambda^5 \hbar \xi 12_i \xi 13_j \xi 23_i \xi 34_j^2 + \\
& \frac{1}{10} e^{-\frac{1}{2} \gamma (\alpha 1_i - \alpha 2_i + 3 \alpha 3_i)} \gamma \lambda^5 \hbar \xi 12_i \xi 13_j \xi 23_i \xi 34_j^2 + \frac{1}{10} e^{\frac{1}{2} \gamma (\alpha 1_i - 3 \alpha 3_i)} \gamma \lambda^5 \hbar \xi 12_i \xi 13_i \xi 23_j \xi 34_j^2 - \\
& \frac{1}{10} e^{-\frac{1}{2} \gamma (-\alpha 1_i + 3 \alpha 3_i)} \gamma \lambda^5 \hbar \xi 12_i \xi 13_i \xi 23_j \xi 34_j^2 \Big], f_{1,x12} \rightarrow \text{Function}[\{\lambda\}, 0] \Big\} \Big\}
\end{aligned}$$