

Hidden Utilities and T_EX

Headers

pdf

Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

pdf

Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

pdf

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

pdf

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

pdf

This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

pdf

```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

pdf

```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ $\epsilon$ _, op_] := Collect[Normal@Series[_ $\epsilon$ _, { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_ $\epsilon$ _] := SS[_ $\epsilon$ _, Together];
```

pdf

```
In[ ]:= CCF[_ $\epsilon$ _] := PP_CCF@Expand[Expand[_ $\epsilon$ _] /.  $e^p \rightarrow e^{\text{Factor}[p]}$ ];
CF[_ $\epsilon$ _] := PP_CF@If[$k ==  $\infty$ ,
  CCF[_ $\epsilon$ _,
  CCF[Normal@Series[_ $\epsilon$ _ /.  $q \rightarrow e^{\hbar \epsilon}$ , { $\epsilon$ , 0, $k}]]
];
```

pdf

```
In[ ]:= x[i_, j_] := x[<>i<>j]; a[i_] := a[<>i]; A[i_] := A[<>i];
ξ[i_, j_] := ξ[<>i<>j]; α[i_] := α[<>i]; ℳ[i_] := ℳ[<>i];
```

pdf

```
In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

pdf

The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

pdf

```
In[ ]:= (* q=eh γ ∈; Ai[n]=eh ∈ ai; Bi[n]=eh γ bi *)
```

pdf

```
In[ ]:= $n = 5; $k = ∞; $E := {$n, $k}
```

pdf

```
In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; A[i][_]],
  Table[As[i, 1, $n - 1],
    a[i]* = α[i]; α[i]* = a[i]; a[i]]
}
(#0 = U0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[ ]:= #* & /@ $EPBWGens
```

Time: 0.00 seconds Out[]:=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34,
 ξ35, ξ45, A1[_]*, A2[_]*, A3[_]*, A4[_]*, α1, α2, α3, α4}
```

pdf

```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{\hbar^k \epsilon^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]]
  }],
  U[____], F
] /. F → CF]

```

pdf

```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

pdf

```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

pdf

```

In[*]:= Si[_] := UCF@Module[{i1},  $\mathcal{E}$  /. {
  Ui[f_, fs____] ⇒ mi1,i→i[Si[f] Si1[Ui1[fs]]]
}];

```

pdf

The $\mathbb{A}$ -algebra

pdf

```
In[*]:= RR_k[ai_, aj_] /;  $\tau[ai] = \tau[aj] = "a" := U_k[aj, ai];$ 
RR_k[ai_, Aj_[n_]] /;  $\tau[ai] = "a" \wedge \tau[Aj] = "A" := U_k[Aj[n], ai];$ 
RR_k[Ai_[m_], Aj_[n_]] /;  $\tau[Ai] = \tau[Aj] = "A" := U_k[Aj[n], Ai[m]];$ 
RR_k[am_, xij_] /;  $\tau[am] = "a" \wedge \tau[xij] = "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@am$ ;
   $U_k[xij, am] + \gamma(\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij] / 2$ 
];
RR_k[Am_[n_], xij_] /;  $\tau[Am] = "A" \wedge \tau[xij] = "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@Am$ ;  $q^{-n}(\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij, Am[n]]$ 
]
```

```
In[*]:= RR_0[a[1], x[1, 2]]
```

Time: 0.01 seconds Out[*]=

```
 $\gamma U_0[x12] + U_0[x12, a1]$ 
```

pdf

We take the reduction rules RR for xij's from Yamane page

509:

$$\begin{array}{lll} \text{(I)} & i < m < j < n, & \text{(II)} \quad i < m < n < j, & \text{(III)} \quad i < m < j = n, \\ & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \\ \text{(IV)} & i < m < j < n, & \text{(V)} \quad i < j = m < n, & \text{(VI)} \quad i < j < m < n, \\ & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$

$$\begin{array}{ll} q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

pdf

```
In[*]:= RR_k[xmn_, xij_] /;  $\tau[xij] = \tau[xmn] = "x" := Module[{i, j, m, n},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m, n} =  $\mathcal{L}@xmn$ ;
  Which[
     $(i == m \wedge j < n) \vee (i < m < j \wedge n == j), q^{-2} U_k[xij, xmn], (* \text{IVIII} *)$ 
     $(i < m < j \wedge n < j) \vee (j < m), U_k[xij, xmn], (* \text{II} \vee \text{VI} *)$ 
     $i < m < j \wedge n > j, U_k[xij, xmn] + (q^{-2} - q^2) U_k[x[i, n], x[m, j]], (* \text{IV} *)$ 
     $j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* \text{V} *)$ 
  ]
]
```

```

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.03 seconds Out|=
-q U2[x24] U3[x12] + q^2 U2[x23, x34] U3[x12]

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.06 seconds Out|=
q^2 U0[x14] - q^3 U0[x12, x24] - q^3 U0[x13, x34] + q^4 U0[x12, x23, x34]

In[*]:= Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
Time: 40.53 seconds Out|=
{}

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}

Out[*]=
{}

Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
  ]]
]
» {x12, x13, x23, a1, a2}

Out[*]=
{}

```

pdf

The Co-Product, Co-Associativity, and the Hopf Axiom

pdf

In[]:=

```

Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1→k[mj, j1→j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]]
]

```

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]
```

Time: 0.08 seconds Out[]=

$$\begin{aligned}
U_i[x_{12}] &\rightarrow U_j[A_1[-1]] U_k[x_{12}] + U_j[x_{12}] U_k[A_1[1]] \\
U_i[x_{13}] &\rightarrow U_j[A_1[-1], A_2[-1]] U_k[x_{13}] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{12}, A_2[-1]] U_k[x_{23}, A_1[1]] + U_j[x_{13}] U_k[A_1[1], A_2[1]] \\
U_i[x_{14}] &\rightarrow \\
&\quad U_j[A_1[-1], A_2[-1], A_3[-1]] U_k[x_{14}] + \left(-\frac{1}{q^2} + q^2\right) U_j[x_{12}, A_2[-1], A_3[-1]] U_k[x_{24}, A_1[1]] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{13}, A_3[-1]] U_k[x_{34}, A_1[1], A_2[1]] + U_j[x_{14}] U_k[A_1[1], A_2[1], A_3[1]] \\
U_i[x_{15}] &\rightarrow U_j[A_1[-1], A_2[-1], A_3[-1], A_4[-1]] U_k[x_{15}] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{12}, A_2[-1], A_3[-1], A_4[-1]] U_k[x_{25}, A_1[1]] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{13}, A_3[-1], A_4[-1]] U_k[x_{35}, A_1[1], A_2[1]] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{14}, A_4[-1]] U_k[x_{45}, A_1[1], A_2[1], A_3[1]] + \\
&\quad U_j[x_{15}] U_k[A_1[1], A_2[1], A_3[1], A_4[1]] \\
U_i[x_{23}] &\rightarrow U_j[A_2[-1]] U_k[x_{23}] + U_j[x_{23}] U_k[A_2[1]] \\
U_i[x_{24}] &\rightarrow U_j[A_2[-1], A_3[-1]] U_k[x_{24}] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{23}, A_3[-1]] U_k[x_{34}, A_2[1]] + U_j[x_{24}] U_k[A_2[1], A_3[1]] \\
U_i[x_{25}] &\rightarrow \\
&\quad U_j[A_2[-1], A_3[-1], A_4[-1]] U_k[x_{25}] + \left(-\frac{1}{q^2} + q^2\right) U_j[x_{23}, A_3[-1], A_4[-1]] U_k[x_{35}, A_2[1]] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{24}, A_4[-1]] U_k[x_{45}, A_2[1], A_3[1]] + U_j[x_{25}] U_k[A_2[1], A_3[1], A_4[1]] \\
U_i[x_{34}] &\rightarrow U_j[A_3[-1]] U_k[x_{34}] + U_j[x_{34}] U_k[A_3[1]] \\
U_i[x_{35}] &\rightarrow U_j[A_3[-1], A_4[-1]] U_k[x_{35}] + \\
&\quad \left(-\frac{1}{q^2} + q^2\right) U_j[x_{34}, A_4[-1]] U_k[x_{45}, A_3[1]] + U_j[x_{35}] U_k[A_3[1], A_4[1]] \\
U_i[x_{45}] &\rightarrow U_j[A_4[-1]] U_k[x_{45}] + U_j[x_{45}] U_k[A_4[1]] \\
U_i[a_1] &\rightarrow U_j[a_1] U_k[] + U_j[] U_k[a_1] \\
U_i[a_2] &\rightarrow U_j[a_2] U_k[] + U_j[] U_k[a_2] \\
U_i[a_3] &\rightarrow U_j[a_3] U_k[] + U_j[] U_k[a_3] \\
U_i[a_4] &\rightarrow U_j[a_4] U_k[] + U_j[] U_k[a_4]
\end{aligned}$$

```
In[*]:= Short[{ $\Delta_{4,5}$ [a2],  $\Delta_{7,9}$ [A2[77]],  $\Delta_{1,2}$ [x12],  $\Delta_{1,2}$ [x15]}, 8]
```

```
Time: 0.02 seconds Out[ ]=
```

$$\begin{aligned} & \left\{ U_4[a2] U_5[] + U_4[] U_5[a2], U_7[A2[77]] U_9[A2[77]], \right. \\ & U_1[A1[-1]] U_2[x12] + U_1[x12] U_2[A1[1]], U_1[A1[-1], A2[-1], A3[-1], A4[-1]] U_2[x15] + \\ & \left(-\frac{1}{q^2} + q^2 \right) U_1[x12, A2[-1], A3[-1], A4[-1]] U_2[x25, A1[1]] + \\ & \left(-\frac{1}{q^2} + q^2 \right) U_1[x13, A3[-1], A4[-1]] U_2[x35, A1[1], A2[1]] + \\ & \left(-\frac{1}{q^2} + q^2 \right) U_1[x14, A4[-1]] U_2[x45, A1[1], A2[1], A3[1]] + \\ & \left. U_1[x15] U_2[A1[1], A2[1], A3[1], A4[1]] \right\} \end{aligned}$$

```
In[*]:=  $\Delta_{1 \rightarrow 1, 2}$ [U1[x13, a2] U0[x34]]
```

```
Time: 0.03 seconds Out[ ]=
```

$$\begin{aligned} & U_0[x34] U_1[A1[-1], A2[-1], a2] U_2[x13] + U_0[x34] U_1[A1[-1], A2[-1]] U_2[x13, a2] + \\ & \left(-\frac{1}{q^2} + q^2 \right) U_0[x34] U_1[x12, A2[-1], a2] U_2[x23, A1[1]] + U_0[x34] U_1[x13, a2] U_2[A1[1], A2[1]] + \\ & \left(-\frac{1}{q^2} + q^2 \right) U_0[x34] U_1[x12, A2[-1]] U_2[x23, A1[1], a2] + U_0[x34] U_1[x13] U_2[A1[1], A2[1], a2] \end{aligned}$$

```
In[*]:= Module[{Bas = $PBWGens, P},
```

```
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P //  $\Delta_{1 \rightarrow 1, 2}$  //  $\Delta_{2 \rightarrow 2, 3}$ ) == (P //  $\Delta_{1 \rightarrow 1, 3}$  //  $\Delta_{1 \rightarrow 1, 2}$ )
  ]]
]
```

```
>> {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A1[1], A4[-1], A2[1], A1[-1], A1[2], A2[-2], A2[0], A4[2], A2[-1], A1[-2]}
```

```
Out[*]=
```

```
{ }
```

```
In[*]:= Module[{Bas = $PBWGens, P},
```

```
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> (P //  $m_{1, 2 \rightarrow 1}$  //  $\Delta_{1 \rightarrow 1, 2}$ ) == (P //  $\Delta_{2 \rightarrow 3, 4}$  //  $\Delta_{1 \rightarrow 1, 2}$  //  $m_{1, 3 \rightarrow 1}$  //  $m_{2, 4 \rightarrow 2}$ )
  ]]
]
```

```
>> {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A4[2], A2[0], A4[-1], A2[-2], A4[2], A1[0], A2[-1], A2[-2], A1[0], A4[1]}
```

```
Out[*]=
```

```
{ }
```

Is Δ a quantization of the co-bracket?

```
In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P → UCF[( $\Delta_{1 \rightarrow 1, 2}[P] - \Delta_{1 \rightarrow 2, 1}[P]$ ) // .
      { $q^{n-}$  →  $1 + n \hbar / 2$ ,  $U_i[xs\_]$ ,  $A[n\_]$ ,  $as\_]$  →  $U_i[xs] + n \hbar U_i[xs, a@@L[A], as] / 2$ }
      ] /.  $\hbar^n \rightarrow 0$  ;  $n > 1 \rightarrow 0$ 
    ]
  ]
```

Time: 0.08 seconds Out[]:=

```
U1[x12] →  $\hbar U_1[x12] U_2[a1] - \hbar U_1[a1] U_2[x12]$ 
U1[x13] →  $\hbar U_1[x13] U_2[a1] - 2 \hbar U_1[x23] U_2[x12] - \hbar U_1[a1] U_2[x13] + 2 \hbar U_1[x12] U_2[x23]$ 
U1[x14] →  $\hbar U_1[x14] U_2[a1] - 2 \hbar U_1[x24] U_2[x12] -$ 
   $2 \hbar U_1[x34] U_2[x13] - \hbar U_1[a1] U_2[x14] + 2 \hbar U_1[x12] U_2[x24] + 2 \hbar U_1[x13] U_2[x34]$ 
U1[x15] →  $\hbar U_1[x15] U_2[a1] - 2 \hbar U_1[x25] U_2[x12] - 2 \hbar U_1[x35] U_2[x13] - 2 \hbar U_1[x45] U_2[x14] -$ 
   $\hbar U_1[a1] U_2[x15] + 2 \hbar U_1[x12] U_2[x25] + 2 \hbar U_1[x13] U_2[x35] + 2 \hbar U_1[x14] U_2[x45]$ 
U1[x23] →  $\hbar U_1[x23] U_2[a2] - \hbar U_1[a2] U_2[x23]$ 
U1[x24] →  $\hbar U_1[x24] U_2[a2] - 2 \hbar U_1[x34] U_2[x23] - \hbar U_1[a2] U_2[x24] + 2 \hbar U_1[x23] U_2[x34]$ 
U1[x25] →  $\hbar U_1[x25] U_2[a2] - 2 \hbar U_1[x35] U_2[x23] -$ 
   $2 \hbar U_1[x45] U_2[x24] - \hbar U_1[a2] U_2[x25] + 2 \hbar U_1[x23] U_2[x35] + 2 \hbar U_1[x24] U_2[x45]$ 
U1[x34] →  $\hbar U_1[x34] U_2[a3] - \hbar U_1[a3] U_2[x34]$ 
U1[x35] →  $\hbar U_1[x35] U_2[a3] - 2 \hbar U_1[x45] U_2[x34] - \hbar U_1[a3] U_2[x35] + 2 \hbar U_1[x34] U_2[x45]$ 
U1[x45] →  $\hbar U_1[x45] U_2[a4] - \hbar U_1[a4] U_2[x45]$ 
U1[a1] → 0
U1[a2] → 0
U1[a3] → 0
U1[a4] → 0
```

pdf

The Antipode

pdf

```
In[ ]:= Do[As[i, 1, $n - 1],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n_]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj-[x[i, n]] = mj1, j → j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

```

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→2,3 // S2 // m2,3→1)
  ]]
]

» {x12, x13, x14, x15, x23, x24, x25, x34, x35, x45, a1, a2, a3, a4,
  A3[-1], A2[0], A4[2], A4[-2], A4[2], A4[-2], A2[-1], A2[0], A2[2], A3[1]}

```

Out[*]=

```

{x12 -> 0, x13 -> 0, x14 -> 0, x15 -> 0, x23 -> 0, x24 -> 0, x25 -> 0, x34 -> 0, x35 -> 0, x45 -> 0,
a1 -> 0, a2 -> 0, a3 -> 0, a4 -> 0, A3[-1] -> U1[], A2[0] -> U1[], A4[2] -> U1[], A4[-2] -> U1[],
A4[2] -> U1[], A4[-2] -> U1[], A2[-1] -> U1[], A2[0] -> U1[], A2[2] -> U1[], A3[1] -> U1[]}

```

pdf

The Generating Functions for the $\mathbb{A}$ operations

```

In[*]:= BeginProfile[]

```

Time: 0.11 seconds Out[]=

```

ProfileRoot

```

```

In[*]:= $k = 1;

```

pdf

NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

pdf

```

In[*]:= Supp[ $\mathcal{E}$ _] := Union@Cases[{ $\mathcal{E}$ }, Ui[_] -> i,  $\infty$ ];

```

```

In[*]:= Supp[2  $\hbar$  U1[x45] U2[x34]]

```

Time: 0.01 seconds Out[]=

```

{1, 2}

```

pdf

```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_  $\boxtimes$  y_ := Module[{i, is = Supp[x]  $\cap$  Supp[y],  $\sigma$ , z},
  z = x; Do[z = mi $\rightarrow$  $\sigma$ i[z], {i, is}];
  z = Expand[y z]; Do[z = m $\sigma$ i, i $\rightarrow$ i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x  $\boxtimes$  y - y  $\boxtimes$  x];

```

```

In[*]:= (2  $\hbar$ 1 U1[x45] U2[x34])  $\boxtimes$  (2  $\hbar$ 2 U1[x45] U2[x34])

```

Time: 0.03 seconds Out[]=

```

4  $\hbar$ 1  $\hbar$ 2 U1[x45, x45] U2[x34, x34]

```

pdf

```
In[*]:= (*adPower[x_,k_,Env_] [c_*y_QU] :=UCF[c adPower[x,k,Env] [y]] ;
adPower[x_,k_,Env_] [y_Plus] := adPower[x,k,Env] /@ y;*)
adPower[_ , 0, Env_] [y_] := y;
(*adPower[x_,k_,Env_] [0]=0;*)
adPower[x_, k_, Env_] [y_] :=
  adPower[x, k, Env] [y] = UCF@UB[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];
```

```
In[*]:= Table[As[k, 0, 5], adPower[U_0[x12], k] [U_0[x23]]]
```

Time: 0.06 seconds Out:=

$$\{U_0[x23], (-1 - \gamma \in \hbar) U_0[x13] + 2 \gamma \in \hbar U_0[x12, x23], 0, 0, 0, 0\}$$

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U_i /@ Bas], P → Δi→j,k[P]]
]
```

Time: 0.08 seconds Out:=

$$\begin{aligned} U_i[x12] &\rightarrow U_j[x12] U_k[] + \epsilon \hbar U_j[x12] U_k[a1] + U_j[] U_k[x12] + \epsilon \hbar U_j[a1] U_k[x12] \\ U_i[x13] &\rightarrow U_j[x13] U_k[] + \epsilon \hbar U_j[x13] U_k[a1] + \epsilon \hbar U_j[x13] U_k[a2] + \\ &U_j[] U_k[x13] + \epsilon \hbar U_j[a1] U_k[x13] + \epsilon \hbar U_j[a2] U_k[x13] + 4 \gamma \in \hbar U_j[x12] U_k[x23] \\ U_i[x23] &\rightarrow U_j[x23] U_k[] + \epsilon \hbar U_j[x23] U_k[a2] + U_j[] U_k[x23] + \epsilon \hbar U_j[a2] U_k[x23] \\ U_i[a1] &\rightarrow U_j[a1] U_k[] + U_j[] U_k[a1] \\ U_i[a2] &\rightarrow U_j[a2] U_k[] + U_j[] U_k[a2] \end{aligned}$$

pdf

```
In[*]:= Ad[0] = Identity;
```

pdf

```
In[*]:= Ad[c_. U_i[xjk_] (EUs : U_[] ...)] [y_] /; τ[xjk] == "x" := PPAd@Module[{k = 0, s = 0, t},
  While[(t = adPower[U_i[xjk], k] [y]) != 0,
    s += c^k t / k!;
    ++k
  ];
  UCF[s ⊗ EUs]
]
```

```
In[*]:= Ad[ε ħ U_j[x12] U_k[]] [U_j[x23] U_k[a4]]
```

Time: 0.09 seconds Out:=

$$-\epsilon \hbar U_j[x13] U_k[a4] + U_j[x23] U_k[a4]$$

```
In[*]:= Ad[7 ħ U_j[x12] U_k[]] [U_j[x23] U_k[a4]]
```

Time: 0.08 seconds Out:=

$$(-7 \hbar - 7 \gamma \in \hbar^2) U_j[x13] U_k[a4] + U_j[x23] U_k[a4] + 14 \gamma \in \hbar^2 U_j[x12, x23] U_k[a4]$$

pdf

```
In[*]:= Wtaj[_] := Wtaj[x] = UB[U_0[aj], U_0[x]] / U_0[x]
```

```
In[ ]:= Table[Wta[j] [x23], {j, 1, $n}]
```

Time: 0.08 seconds Out[]=

$$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}, 0, 0\right\}$$

pdf

```
In[ ]:= Ad[c-. Ui[aj-] (EUs : U-[...] [y-] /; τ[aj] == "a" :=  
(y /. Ui[gs---] => Ui[gs] ec Total[Wtaj/e{gs}])
```

```
In[ ]:= Ad[π ± Uj[a1]] [Uj[x23] Uk[a4]]
```

Time: 0.03 seconds Out[]=

$$e^{-\frac{1}{2} \pm \pi \gamma} U_j[x23] U_k[a4]$$

pdf

```
In[ ]:= TTλ[ ] = 0; (* Translated Tangent *)  
TTλ[xs---, x-] := PPTT@UCF[Ad[x] [TTλ[xs]] + ∂λ x];  
TTλ[xs_List] := TTλ@@xs
```

```
In[ ]:= TTλ[λ ξ12 U0[x12] U1[ ], λ ξ23 U0[x23] U1[ ]]
```

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[x12] U_1[] + \lambda \xi_{12} \xi_{23} U_0[x13] U_1[] + \xi_{23} U_0[x23] U_1[]$$

```
In[ ]:= $PBWGens
```

Time: 0.00 seconds Out[]=

$$\{x12, x13, x23, a1, a2\}$$

```
In[ ]:= Join[  
Cases[$PBWGens, g- => Which[  
τ[g] === "x", λ (g*)i Uk[g],  
τ[g] === "a", (g*)i Uk[g]  
]],  
Cases[$PBWGens, g- => Which[  
τ[g] === "x", λ (g*)j Uk[g],  
τ[g] === "a", (g*)j Uk[g]  
]]  
]
```

Time: 0.03 seconds Out[]=

$$\{\lambda \xi_{12_i} U_k[x12], \lambda \xi_{13_i} U_k[x13], \lambda \xi_{23_i} U_k[x23], \alpha_{1_i} U_k[a1], \alpha_{2_i} U_k[a2], \\ \lambda \xi_{12_j} U_k[x12], \lambda \xi_{13_j} U_k[x13], \lambda \xi_{23_j} U_k[x23], \alpha_{1_j} U_k[a1], \alpha_{2_j} U_k[a2]\}$$

```
In[ ]:= Oi[poly-] := Total[  
CoefficientRules[poly, $PBWGens] /.  
(p- → c-) => c Ui@@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]  
]
```

In[*]:= $0_i [x_{12}^2 x_{13} + 7 x_{23}^{17}]$

Time: 0.03 seconds Out[=

$U_i [x_{12}, x_{12}, x_{13}] +$
 $7 U_i [x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$

pdf

In[*]:= **Seepage_i** [{}, c_] = c;
Seepage_i [ξs_List, P_] := Module[{lx},
 lx = \$PBWGens[[Length@ξs]];
 CF@Collect[P, lx, (Ad[Last[ξs] U_i[lx]] [0_i[Seepage_i[Most@ξs, #]]] /. U_i → Times) &]
]

pdf

In[*]:= **Invert_i** [ε_] := Module[{A, B, β, j}, (* ε=A U[] +ε B *)
 A = Coefficient[ε, U_i[]] /. ε → 0;
 B = UCF[(ε - A U_i[])];
 UCF[
 A⁻¹ (U_i[] + Sum[As[j, 1, \$k], (-1)^j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]

{ \$k = 1, \$n = 3 }

Time: 0.01 seconds Out[=

{ 2, 3 }

pdf

In[*]:= **sol** [kk_, nn_] := PP^{sol} [kk, nn] @Block[{ \$k = kk, \$n = nn }, Module[{ (*TTL, *) },
 ES[s_] [ε_] := (EchoLabel[s] [Short[ε, 5]]); ε);
 TTL = ES["TTL"] @TT_λ@
 Join[
 Cases[\$PBWGens, g_ => Which[
 τ[g] == "x", λ (g*)_i U_k[g],
 τ[g] == "a", (g*)_i U_k[g]
]],
 Cases[\$PBWGens, g_ => Which[
 τ[g] == "x", λ (g*)_j U_k[g],
 τ[g] == "a", (g*)_j U_k[g]
]]
];
 mons[-1] = {};
 mons[p_] := mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], U_k[____], ∞];
 ff = Table[
 Sum[ε^p f_{p,u}[λ], {p, 0, \$k}],
 {u, \$PBWGens}
] /. f_{p,u}[λ] /; τ[u] == "a" => If[p == 0, (u*)_i + (u*)_j, 0];
 fs = Flatten[Table[

```

fk, Times@@m,
{k, 0, $k}, {m, mons[k]}
]];
gg = Sum[
ep fp, Times@@u [λ] Times @@ u,
{p, 0, $k}, {u, Complement[mons[p], Uk /@ $PBWGens]}
];
G = Series[egg, {ε, 0, $k}];

```

$$\begin{aligned}
L_\lambda &= \mathbb{C}\langle \lambda_{i_1} x \rangle \mathbb{C}\langle \lambda_{i_2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i(x) \rangle \mathbb{C}\langle \text{Seepage}(f_i, A) \rangle &= \mathbb{C}\langle e^{f_i} A \rangle \\
\mathbb{T}(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(f_i, B). \\
R_\lambda &= \mathbb{C}\langle f + g \rangle, \quad f = f_i x; \quad g = \sum_{h \neq i} g_h x^h & e^f = F \quad e^g = G \quad R_\lambda = \mathbb{C}\langle FG \rangle \quad \partial_\lambda F = F \partial_\lambda f \\
\mathbb{T}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
\end{aligned}$$

```

Si[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
SG = ES["SG"] @ Sk[G];
TTR = ES["TTR"] @ UCF[Invertk[SG] ⊗ Sk[(∂λ(ff.$PBWGens) G + ∂λG)]];
diff = ES["diff"] @ UCF[TTL - TTR];
eqns = ES["eqns"] @ Flatten[Table[
Coefficient[Coefficient[diff, m], ε, k] == 0,
{k, 0, $k}, {m, mons[k]}
]];
inits = (#[0] == 0) & /@ fs;
ES["DSolve"] [
DSolve[eqns ∪ inits, fs, λ] /. F_Function => ReplacePart[F, 2 -> CF[F[[2]]]]
]]

```

In[*]:= sol[2, 2]

```

» TTL (eγ (α1i+α1j) ξ12i + eγ α1j ξ12j) Uk[x12]
» SG Uk[]
» TTR Uk[x12] (eγ (α1i+α1j) f0,x12'[λ] + eγ (α1i+α1j) ∈ f1,x12'[λ] + eγ (α1i+α1j) ∈2 f2,x12'[λ])
» diff
Uk[x12] (eγ (α1i+α1j) ξ12i + eγ α1j ξ12j - eγ (α1i+α1j) f0,x12'[λ] - eγ (α1i+α1j) ∈ f1,x12'[λ] - eγ (α1i+α1j) ∈2 f2,x12'[λ])
» eqns {eγ (α1i+α1j) ξ12i + eγ α1j ξ12j - eγ (α1i+α1j) f0,x12'[λ] == 0,
-eγ (α1i+α1j) f1,x12'[λ] == 0, -eγ (α1i+α1j) f2,x12'[λ] == 0}
» DSolve
{{f0,x12 -> Function[{λ}, λ ξ12i + e-γ α1i λ ξ12j], f1,x12 -> Function[{λ}, 0], f2,x12 -> Function[{λ}, 0]}}

```

Out[*]=

```

{{f0,x12 -> Function[{λ}, λ ξ12i + e-γ α1i λ ξ12j],
f1,x12 -> Function[{λ}, 0], f2,x12 -> Function[{λ}, 0]}}

```

In[*]:= sol[1, 3]

» TTL $\left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} \right) U_k[x_{12}] +$
 $\ll 5 \gg + \left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1-2\alpha_1-3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_i} + \right.$
 $\left. 2 e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2+3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_j} + 2 e^{\frac{3}{2}\gamma\alpha_2} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_j} \right) U_k[x_{13}, x_{23}]$

» SG $U_k[] + e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \in U_k[x_{12}, x_{13}] f_{1,x_{12}x_{13}}[\lambda] +$
 $U_k[x_{13}, x_{13}] \left(e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{\emptyset,x_{23}}[\lambda] f_{1,x_{12}x_{13}}[\lambda] + e^{\gamma(\alpha_1+\alpha_1+\ll 1 \gg+\alpha_2)} \in f_{1,\ll 3 \gg^2}[\lambda] \right) +$
 $e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in U_k[x_{12}, x_{23}] f_{1,x_{12}x_{23}}[\lambda] +$
 $U_k[x_{13}, x_{23}] \left(e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{\emptyset,x_{23}}[\lambda] f_{1,x_{12}x_{23}}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{1,x_{13}x_{23}}[\lambda] \right)$

» TTR $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}'}[\lambda] + e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}'}[\lambda] \right) + \ll 5 \gg + U_k[x_{13}, x_{23}]$
 $\left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \hbar f_{\emptyset,x_{23}}[\lambda] f_{\emptyset,x_{13}'}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{\emptyset,x_{23}}[\lambda] f_{1,x_{12}x_{23}'}[\lambda] + e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{1,x_{13}x_{23}'}[\lambda] \right)$

» diff $U_k[x_{12}] \left(e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - \right.$
 $\left. e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \in f_{1,x_{12}'}[\lambda] \right) +$
 $U_k[x_{13}] \left(e^{\ll 1 \gg} \xi_{13_i} + e^{\ll 1 \gg} \ll 1 \gg + \ll 22 \gg \right) + \ll 3 \gg + U_k[x_{12}, x_{23}] \left(\ll 1 \gg + \ll 8 \gg \right) +$
 $U_k[x_{13}, x_{23}] \left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1-2\ll 2 \gg_i-3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_i} + \ll 8 \gg \right)$

» eqns

$$\left\{ \begin{aligned} & e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} \xi_{12_i} + e^{\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_j} - e^{\frac{1}{2}\gamma(2\alpha_1+2\alpha_1-\alpha_2-\alpha_2)} f_{\emptyset,x_{12}'}[\lambda] = 0, e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \xi_{13_i} + \\ & e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \xi_{13_j} + e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \lambda \xi_{12_i} \xi_{23_i} - \ll 1 \gg + \ll 1 \gg + e^{\frac{1}{2}\gamma(\ll 1 \gg)} \lambda \xi_{12_j} \xi_{23_j} - \\ & e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} f_{\emptyset,x_{23}}[\lambda] f_{\emptyset,x_{12}'}[\lambda] - e^{\frac{1}{2}\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} f_{\emptyset,x_{13}'}[\lambda] = 0, \ll 6 \gg, \ll 1 \gg, \ll 1 \gg \end{aligned} \right\}$$

» DSolve $\left\{ \left\{ f_{\emptyset,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$
 $f_{\emptyset,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2)} \lambda \xi_{23_j}],$
 $f_{\emptyset,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{13_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{23_i}],$
 $\ll 5 \gg, f_{1,x_{13}^2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0] \left. \right\}$

Out[]=

$$\left\{ \left\{ f_{\emptyset,x_{12}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{12_i} + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda \xi_{12_j}], \right. \right.$$

$$f_{\emptyset,x_{23}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{23_i} + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2)} \lambda \xi_{23_j}],$$

$$f_{\emptyset,x_{13}} \rightarrow \text{Function}[\{\lambda\}, \lambda \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{13_j} - e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \lambda^2 \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{23}} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{13}} \rightarrow \text{Function}[\{\lambda\}, -e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{12}x_{23}} \rightarrow \text{Function}[\{\lambda\}, 2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{23_i}],$$

$$f_{1,x_{12}x_{13}} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \gamma \lambda^2 \hbar \xi_{12_j} \xi_{13_i}],$$

$$f_{1,x_{13}x_{23}} \rightarrow \text{Function}[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{13_j} \xi_{23_i}],$$

$$f_{1,x_{13}^2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x_{12}} \rightarrow \text{Function}[\{\lambda\}, 0] \left. \right\}$$

```
In[*]:= sol[1, 4]
```

```
Time: 560.02 seconds Out[=
```

```
$Aborted
```