Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the λ-tangent formalism. Some material from pensieve://Projects/UEA.

In[∘]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Prolog

In[∘]:=
```
BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
    algebras and PBW algebras. It is in the public domain, available
    at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
    Dror Bar-Natan is committed to support it within
    reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
  Sort@{"**", ∈, $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
    UB, UProducts, USimp, UU, $Basis, $PBWRule, δ, adPower, adExp},
  "."];
Begin["`Private`"];
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ,
  SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ, ∈, $Basis, $k, $PBWRule}.
```

Utilities

In[∘]:=
```
δ_{i_,j_} := If[i == j, 1, 0];
```

In[∘]:=
```
SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

In[∘]:=
```
$k = 1;
```

In[∘]:=
```
CF[ℰ_] := Expand[ℰ] /. {∈^(k_.) /; k > $k → 0};
```

Implementing general universal enveloping algebras

In[∘]:=
```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

In[∘]:=
```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_{i_}[1] := U_i[];
UU_{i_}[x_^p_] := UU_i @@ Table[x, {p}];
UU_{i_}[ℰ_] := ℰ /. {
    U[xs__] :→ U_i[xs],
    x_ /; MemberQ[$Basis, x] :→ U_i[x]
   };
UU_{i_}[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

In[∘]:=
```
m_{s__}[0] = 0;
m_{s__}[x_Plus] := m_s /@ x;
m_{s__}[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

In[∘]:=
```
m_{i_,j_→k_}[c_. U_{i_}[x___] U_{j_}[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_{i_}[] U_{j_}[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_{i_}[xx___, x_] U_{j_}[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

In[∘]:=
```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

In[∘]:=
```
Supp[ℰ_] := Union@Cases[{ℰ}, U_{i_}[___] :→ i, ∞];
```

In[∘]:=
```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

In[∘]:=
```
End[]; EndPackage[];
```

Predefined Algebras

sl(2)

In[∘]:=
```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

In[∘]:=
```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
   NilQ[e] = NilQ[f] = True; NilQ[h] = False;
  );
```

$gl_{n,\epsilon}$

In[∘]:=
```
Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

UEA`SetAlgebra knows the ϵ-nilpotent algebra $gl_{n,\epsilon}$.

In[∘]:=
```
SetAlgebra[gl_{n_,ϵ_}] := (
   $Basis = Flatten@{
      Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
      Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
     };
   NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
   $PBWRule = Thread[$Basis → Range@Length@$Basis];
   B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
   B[y_{i_,j_}, y_{k_,l_}] := CF[ϵ δ_{j,k} y_{i,l} - ϵ δ_{l,i} y_{k,j}];
   B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :→ If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
  );
```

In[∘]:=
```
SetAlgebra[gl_{2,ϵ}];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

Out[∘]=

$\{y_{1,2} \to 1, x_{1,1} \to 2, x_{1,2} \to 3, x_{2,2} \to 4\}$

Out[∘]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \to 0 & \{y_{1,2}, x_{1,1}\} \to y_{1,2} & \{y_{1,2}, x_{1,2}\} \to -\epsilon\, x_{1,1} + \epsilon\, x_{2,2} & \{y_{1,2}, x_{2,2}\} \to -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \to -y_{1,2} & \{x_{1,1}, x_{1,1}\} \to 0 & \{x_{1,1}, x_{1,2}\} \to x_{1,2} & \{x_{1,1}, x_{2,2}\} \to 0 \\ \{x_{1,2}, y_{1,2}\} \to \epsilon\, x_{1,1} - \epsilon\, x_{2,2} & \{x_{1,2}, x_{1,1}\} \to -x_{1,2} & \{x_{1,2}, x_{1,2}\} \to 0 & \{x_{1,2}, x_{2,2}\} \to x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \to y_{1,2} & \{x_{2,2}, x_{1,1}\} \to 0 & \{x_{2,2}, x_{1,2}\} \to -x_{1,2} & \{x_{2,2}, x_{2,2}\} \to 0 \end{pmatrix}$$

In[∘]:=
```
Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_, 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];
```

In[∘]:=
```
x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis
```

Out[∘]=

$y_{1,2}\, \eta 0_{1,2} + x_{1,1}\, \xi 0_{1,1} + x_{1,2}\, \xi 0_{1,2} + x_{2,2}\, \xi 0_{2,2}$

Out[∘]=

$y_{1,2}\, \eta 1_{1,2} + x_{1,1}\, \xi 1_{1,1} + x_{1,2}\, \xi 1_{1,2} + x_{2,2}\, \xi 1_{2,2}$

In[∘]:=
```
adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   CF[$Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]].
     Table[Coefficient[y, b], {b, $Basis}]]
  ]
```

In[∘]:=
```
SSQ[ξθ1,1 x1,1]
```
Out[∘]=

True

In[∘]:=
```
adExp[ξθ1,1 x1,1][x1]
```
Out[∘]=

$e^{\xi\theta_{1,1}} y_{1,2}\, \eta 1_{1,2} + x_{1,1}\, \xi 1_{1,1} + e^{-\xi\theta_{1,1}} x_{1,2}\, \xi 1_{1,2} + x_{2,2}\, \xi 1_{2,2}$

In[∘]:=
```
adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

In[∘]:=
```
adExp[ξθ1,2 x1,2][x1]
```
Out[∘]=

$y_{1,2}\, \eta 1_{1,2} - \in x_{1,1}\, \eta 1_{1,2}\, \xi\theta_{1,2} + \in x_{2,2}\, \eta 1_{1,2}\, \xi\theta_{1,2} - \in x_{1,2}\, \eta 1_{1,2}\, \xi\theta_{1,2}^2 + x_{1,1}\, \xi 1_{1,1} + x_{1,2}\, \xi\theta_{1,2}\, \xi 1_{1,1} + x_{1,2}\, \xi 1_{1,2} + x_{2,2}\, \xi 1_{2,2} - x_{1,2}\, \xi\theta_{1,2}\, \xi 1_{2,2}$

In[∘]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λ x;
λTangent[xs_List] := λTangent @@ xs
```

In[∘]:=
```
$Basis ($Basis /. {x → ξ, y → η})
```
Out[∘]=

$\{y_{1,2}\, \eta_{1,2},\ x_{1,1}\, \xi_{1,1},\ x_{1,2}\, \xi_{1,2},\ x_{2,2}\, \xi_{2,2}\}$

In[∘]:=
```
λTangent@(λ $Basis ($Basis /. {x → ξ, y → η}))
```
Out[∘]=

$e^{\lambda \xi_{1,1} - \lambda \xi_{2,2}} y_{1,2}\, \eta_{1,2} + x_{1,1}\, \xi_{1,1} + e^{\lambda \xi_{2,2}} x_{1,2}\, \xi_{1,2} - e^{\lambda \xi_{1,1}} \in \lambda\, x_{1,1}\, \eta_{1,2}\, \xi_{1,2} + e^{\lambda \xi_{1,1}} \in \lambda\, x_{2,2}\, \eta_{1,2}\, \xi_{1,2} + e^{\lambda \xi_{2,2}} \lambda\, x_{1,2}\, \xi_{1,1}\, \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2\, x_{1,2}\, \eta_{1,2}\, \xi_{1,2}^2 + x_{2,2}\, \xi_{2,2}$

In[∘]:=
```
$Basis
```
Out[∘]=

$\{y_{1,2},\ x_{1,1},\ x_{1,2},\ x_{2,2}\}$

In[∘]:=
```
lhs =
  Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})] // λTangent
```

Out[∘]=

$$
\begin{aligned}
&e^{\lambda\xi1_{1,1}-\lambda\xi1_{2,2}+\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,y_{1,2}\,\eta1_{1,2}+e^{\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,y_{1,2}\,\eta2_{1,2}+x_{1,1}\,\xi1_{1,1}-\\
&e^{\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,\lambda\,y_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}+e^{\lambda\xi1_{2,2}-\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,x_{1,2}\,\xi1_{1,2}-e^{\lambda\xi1_{1,1}}\,\epsilon\,\lambda\,x_{1,1}\,\eta1_{1,2}\,\xi1_{1,2}+\\
&e^{\lambda\xi1_{1,1}}\,\epsilon\,\lambda\,x_{2,2}\,\eta1_{1,2}\,\xi1_{1,2}+e^{\lambda\xi1_{2,2}}\,\epsilon\,\lambda\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\xi1_{2,2}}\,\epsilon\,\lambda\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,2}+\\
&2\,e^{\lambda\xi1_{1,1}+\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,y_{1,2}\,\eta1_{1,2}\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\xi1_{2,2}+\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,y_{1,2}\,\eta2_{1,2}^2\,\xi1_{1,2}+\\
&e^{\lambda\xi1_{2,2}-\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}+e^{\lambda\xi1_{2,2}}\,\epsilon\,\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}-e^{\lambda\xi1_{2,2}}\,\epsilon\,\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}-\\
&e^{\lambda\xi1_{2,2}+\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,\epsilon\,\lambda^3\,y_{1,2}\,\eta2_{1,2}^2\,\xi1_{1,1}\,\xi1_{1,2}-e^{\lambda\xi1_{1,1}+\lambda\xi1_{2,2}-\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi1_{1,2}^2+x_{2,2}\,\xi1_{2,2}+\\
&e^{\lambda\xi2_{1,1}-\lambda\xi2_{2,2}}\,\lambda\,y_{1,2}\,\eta2_{1,2}\,\xi1_{2,2}+x_{1,1}\,\xi2_{1,1}+e^{\lambda\xi2_{2,2}}\,x_{1,2}\,\xi2_{1,2}-e^{\lambda\xi1_{1,1}-\lambda\xi1_{2,2}+\lambda\xi2_{1,1}}\,\epsilon\,\lambda\,x_{1,1}\,\eta1_{1,2}\,\xi2_{1,2}+\\
&e^{\lambda\xi1_{1,1}-\lambda\xi1_{2,2}+\lambda\xi2_{1,1}}\,\epsilon\,\lambda\,x_{2,2}\,\eta1_{1,2}\,\xi2_{1,2}-e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda\,x_{1,1}\,\eta2_{1,2}\,\xi2_{1,2}+e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda\,x_{2,2}\,\eta2_{1,2}\,\xi2_{1,2}+\\
&e^{\lambda\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}+e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}-e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}-\\
&2\,e^{\lambda\xi1_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+2\,e^{\lambda\xi1_{2,2}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+\\
&2\,e^{\lambda\xi1_{2,2}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}\,\xi2_{1,2}-e^{\lambda\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}-\\
&e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}+e^{\lambda\xi2_{1,1}}\,\epsilon\,\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}+e^{\lambda\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi2_{1,1}\,\xi2_{1,2}-\\
&e^{\lambda\xi1_{1,1}-\lambda\xi1_{2,2}+\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi2_{1,2}^2-e^{\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^2\,x_{1,2}\,\eta2_{1,2}\,\xi2_{1,2}^2+\\
&e^{\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}^2-e^{\lambda\xi2_{1,1}+\lambda\xi2_{2,2}}\,\epsilon\,\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}^2+x_{2,2}\,\xi2_{2,2}
\end{aligned}
$$

In[∘]:=
```
rhs = λTangent@
   ($Basis ($Basis /. {x_{α_,α_} :→ CF[ξ1_{α,α} + ξ2_{α,α} + ϵ f_{α,α}[λ]], x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]}))
```

Out[∘]=

$$
\begin{aligned}
&\epsilon\,x_{1,1}\,f_{1,1}'[\lambda]+e^{\xi1_{2,2}+\xi2_{2,2}}\,\epsilon\,x_{1,2}\,f_{1,2}[\lambda]\,f_{1,1}'[\lambda]+\\
&e^{\xi1_{2,2}+\xi2_{2,2}}\,x_{1,2}\,f_{1,2}'[\lambda]+e^{\xi1_{2,2}+\xi2_{2,2}}\,\epsilon\,x_{1,2}\,f_{2,2}[\lambda]\,f_{1,2}'[\lambda]+\epsilon\,x_{2,2}\,f_{2,2}'[\lambda]+\\
&e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,y_{1,2}\,g_{1,2}'[\lambda]+e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,\epsilon\,y_{1,2}\,f_{1,1}[\lambda]\,g_{1,2}'[\lambda]-\\
&e^{\xi1_{1,1}+\xi2_{1,1}}\,\epsilon\,x_{1,1}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]+e^{\xi1_{1,1}+\xi2_{1,1}}\,\epsilon\,x_{2,2}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]-\\
&e^{\xi1_{1,1}+\xi1_{2,2}+\xi2_{1,1}+\xi2_{2,2}}\,\epsilon\,x_{1,2}\,f_{1,2}[\lambda]^2\,g_{1,2}'[\lambda]-e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,\epsilon\,y_{1,2}\,f_{2,2}[\lambda]\,g_{1,2}'[\lambda]
\end{aligned}
$$

In[∘]:=
```
SSBasis = Select[$Basis, SSQ]
```

Out[∘]=

$\{x_{1,1},\ x_{2,2}\}$

In[∘]:=
```
eqns = (Coefficient[lhs - rhs, #] == 0) & /@ $Basis
```

Out[∘]=

$$\Big\{ e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}}\, \eta1_{1,2} + e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}}\, \eta2_{1,2} - e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}}\, \lambda\, \eta2_{1,2}\, \xi1_{1,1} + 2\, e^{\lambda \xi1_{1,1}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \in \lambda^2\, \eta1_{1,2}\, \eta2_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \in \lambda^2\, \eta2_{1,2}^2\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}+\lambda \xi2_{1,1}-\lambda \xi2_{2,2}} \in \lambda^3\, \eta2_{1,2}^2\, \xi1_{1,1}\, \xi1_{1,2} + e^{\lambda \xi2_{1,1}-\lambda \xi2_{2,2}}\, \lambda\, \eta2_{1,2}\, \xi1_{2,2} - e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\, g_{1,2}'[\lambda] - e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}} \in f_{1,1}[\lambda]\, g_{1,2}'[\lambda] + e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}} \in f_{2,2}[\lambda]\, g_{1,2}'[\lambda] == 0,$$

$$\xi1_{1,1} - e^{\lambda \xi1_{1,1}} \in \lambda\, \eta1_{1,2}\, \xi1_{1,2} + e^{\lambda \xi1_{2,2}} \in \lambda\, \eta2_{1,2}\, \xi1_{1,2} + e^{\lambda \xi1_{2,2}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2} + \xi2_{1,1} - e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}} \in \lambda\, \eta1_{1,2}\, \xi2_{1,2} - e^{\lambda \xi2_{1,1}} \in \lambda\, \eta2_{1,2}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2} - e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2} - \in f_{1,1}'[\lambda] + e^{\xi1_{1,1}+\xi2_{1,1}} \in f_{1,2}[\lambda]\, g_{1,2}'[\lambda] == 0,$$

$$e^{\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}}\, \xi1_{1,2} + e^{\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}}\, \lambda\, \xi1_{1,1}\, \xi1_{1,2} - e^{\lambda \xi1_{1,1}+\lambda \xi1_{2,2}-\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \in \lambda^2\, \eta1_{1,2}\, \xi1_{1,2}^2 + e^{\lambda \xi2_{2,2}}\, \xi2_{1,2} + e^{\lambda \xi2_{2,2}}\, \lambda\, \xi1_{1,1}\, \xi2_{1,2} - 2\, e^{\lambda \xi1_{1,1}+\lambda \xi2_{2,2}} \in \lambda^2\, \eta1_{1,2}\, \xi1_{1,2}\, \xi2_{1,2} + 2\, e^{\lambda \xi1_{2,2}+\lambda \xi2_{2,2}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,2}\, \xi2_{1,2} + 2\, e^{\lambda \xi1_{2,2}+\lambda \xi2_{2,2}} \in \lambda^3\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2}\, \xi2_{1,2} - e^{\lambda \xi2_{2,2}}\, \lambda\, \xi1_{2,2}\, \xi2_{1,2} + e^{\lambda \xi2_{2,2}}\, \lambda\, \xi2_{1,1}\, \xi2_{1,2} - e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \in \lambda^2\, \eta1_{1,2}\, \xi2_{1,2}^2 - e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \in \lambda^2\, \eta2_{1,2}\, \xi2_{1,2}^2 + e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \in \lambda^3\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2}^2 - e^{\lambda \xi2_{1,1}+\lambda \xi2_{2,2}} \in \lambda^3\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2}^2 - e^{\xi1_{2,2}+\xi2_{2,2}} \in f_{1,2}[\lambda]\, f_{1,1}'[\lambda] - e^{\xi1_{2,2}+\xi2_{2,2}}\, f_{1,2}'[\lambda] - e^{\xi1_{2,2}+\xi2_{2,2}} \in f_{2,2}[\lambda]\, f_{1,2}'[\lambda] + e^{\xi1_{1,1}+\xi1_{2,2}+\xi2_{1,1}+\xi2_{2,2}} \in f_{1,2}[\lambda]^2\, g_{1,2}'[\lambda] == 0,$$

$$e^{\lambda \xi1_{1,1}} \in \lambda\, \eta1_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}} \in \lambda\, \eta2_{1,2}\, \xi1_{1,2} - e^{\lambda \xi1_{2,2}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,1}\, \xi1_{1,2} + \xi1_{2,2} + e^{\lambda \xi1_{1,1}-\lambda \xi1_{2,2}+\lambda \xi2_{1,1}} \in \lambda\, \eta1_{1,2}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \in \lambda\, \eta2_{1,2}\, \xi2_{1,2} - e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,1}\, \xi2_{1,2} + e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{2,2}\, \xi2_{1,2} + \xi2_{2,2} - \in f_{2,2}'[\lambda] - e^{\xi1_{1,1}+\xi2_{1,1}} \in f_{1,2}[\lambda]\, g_{1,2}'[\lambda] == 0 \Big\}$$

In[∘]:=
```
Simplify[eqns]
```

Out[∘]=

$$\Big\{ e^{-((1+\lambda)(\xi1_{2,2}+\xi2_{2,2}))} \Big(e^{\lambda \xi1_{1,1}+\xi1_{2,2}+\lambda \xi2_{1,1}+\xi2_{2,2}}\, \eta1_{1,2} \left(1 + 2\, e^{\lambda \xi1_{2,2}} \in \lambda^2\, \eta2_{1,2}\, \xi1_{1,2}\right) + e^{\lambda \xi1_{2,2}} \Big(-e^{(1+\lambda)\xi1_{2,2}+\lambda \xi2_{1,1}+\xi2_{2,2}} \in \lambda^2\, \eta2_{1,2}^2\, (1+\lambda \xi1_{1,1})\, \xi1_{1,2} + e^{\xi1_{2,2}+\lambda \xi2_{1,1}+\xi2_{2,2}}\, \eta2_{1,2}\, (1 - \lambda \xi1_{1,1} + \lambda \xi1_{2,2}) - e^{\xi1_{1,1}+\xi2_{1,1}+\lambda \xi2_{2,2}}\, (1 + \in f_{1,1}[\lambda] - \in f_{2,2}[\lambda])\, g_{1,2}'[\lambda] \Big) \Big) == 0,$$

$$\xi2_{1,1} + \xi1_{1,1} \left(1 + \in \lambda^2\, \eta2_{1,2} \left(e^{\lambda \xi1_{2,2}}\, \xi1_{1,2} + e^{\lambda \xi2_{1,1}}\, \xi2_{1,2}\right)\right) + e^{\xi1_{1,1}+\xi2_{1,1}} \in f_{1,2}[\lambda]\, g_{1,2}'[\lambda] == \in \Big(e^{\lambda(\xi1_{1,1}-\xi1_{2,2})}\, \lambda\, \eta1_{1,2} \left(e^{\lambda \xi1_{2,2}}\, \xi1_{1,2} + e^{\lambda \xi2_{1,1}}\, \xi2_{1,2}\right) + \lambda\, \eta2_{1,2} \left(-e^{\lambda \xi1_{2,2}}\, \xi1_{1,2} + e^{\lambda \xi2_{1,1}}\, (1+\lambda \xi1_{2,2})\, \xi2_{1,2}\right) + f_{1,1}'[\lambda] \Big),$$

$$e^{-\lambda(\xi1_{2,2}+\xi2_{1,1})} \Big(-e^{\lambda(\xi1_{1,1}+2\xi1_{2,2}+\xi2_{2,2})} \in \lambda^2\, \eta1_{1,2}\, \xi1_{1,2}^2 + e^{\lambda(\xi1_{2,2}+\xi2_{2,2})}\, \xi1_{1,2} \Big(-2\, e^{\lambda(\xi1_{1,1}+\xi2_{1,1})} \in \lambda^2\, \eta1_{1,2}\, \xi2_{1,2} + e^{\lambda \xi1_{2,2}} \left(1 + 2\, e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi2_{1,2}\right) + e^{\lambda \xi1_{2,2}}\, \lambda\, \xi1_{1,1} \left(1 + 2\, e^{\lambda \xi2_{1,1}} \in \lambda^2\, \eta2_{1,2}\, \xi2_{1,2}\right) \Big) - e^{\lambda \xi2_{1,1}} \Big(e^{\lambda(\xi1_{2,2}+\xi2_{2,2})}\, (-1 - \lambda \xi1_{1,1} + \lambda \xi1_{2,2} - \lambda \xi2_{1,1})\, \xi2_{1,2} + e^{\lambda(\xi2_{1,1}+\xi2_{2,2})} \in \lambda^2 \left(e^{\lambda \xi1_{1,1}}\, \eta1_{1,2} + e^{\lambda \xi1_{2,2}}\, \eta2_{1,2}\, (1 - \lambda \xi1_{1,1} + \lambda \xi1_{2,2})\right) \xi2_{1,2}^2 + e^{(1+\lambda)\xi1_{2,2}+\xi2_{2,2}} \left(\in f_{1,2}[\lambda]\, f_{1,1}'[\lambda] + (1 + \in f_{2,2}[\lambda])\, f_{1,2}'[\lambda] - e^{\xi1_{1,1}+\xi2_{1,1}} \in f_{1,2}[\lambda]^2\, g_{1,2}'[\lambda]\right) \Big) \Big) == 0,$$

$$\xi1_{2,2} + \in \lambda\, \eta1_{1,2} \left(e^{\lambda \xi1_{1,1}}\, \xi1_{1,2} + e^{\lambda(\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1})}\, \xi2_{1,2}\right) + \xi2_{2,2} == \in \Big(\lambda\, \eta2_{1,2} \left(e^{\lambda \xi1_{2,2}}\, (1 + \lambda \xi1_{1,1})\, \xi1_{1,2} + e^{\lambda \xi2_{1,1}}\, (-1 + \lambda \xi1_{1,1} - \lambda \xi1_{2,2})\, \xi2_{1,2}\right) + f_{2,2}'[\lambda] + e^{\xi1_{1,1}+\xi2_{1,1}}\, f_{1,2}[\lambda]\, g_{1,2}'[\lambda] \Big) \Big\}$$

In[∘]:=
```
Length@eqns
```

Out[∘]=

4

In[∘]:=
```
unknowns = $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}
```

Out[∘]=

$$\{g_{1,2}[\lambda],\ f_{1,1}[\lambda],\ f_{1,2}[\lambda],\ f_{2,2}[\lambda]\}$$

In[∘]:=
```
$Basis /. {x_αβ__ :→ f_αβ[0] == 0, y_αβ__ :→ g_αβ[0] == 0}
```
Out[∘]=

$$\{g_{1,2}[0] == 0,\ f_{1,1}[0] == 0,\ f_{1,2}[0] == 0,\ f_{2,2}[0] == 0\}$$

In[∘]:=
```
{sol1} = DSolve[
  Join[
   (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
   $Basis /. {x_αβ__ :→ f_αβ[0] == 0, y_αβ__ :→ g_αβ[0] == 0}
  ],
  $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]},
  λ
 ]
```

```
Set: Lists {sol1} and
   DSolve[{e^(Times[≪2≫]+Times[≪3≫]+Times[≪2≫]+Times[≪3≫]) η1_{1,2} + e^(Times[≪2≫]+Times[≪3≫]) η2_{1,2} − e^(Times[≪2≫]+Times[≪3≫]) λ η2_{1,2} ξ1_{1,1} + 2
       e^(Times[≪2≫]+Times[≪2≫]+Times[≪3≫]) ϵ λ^2 η1_{1,2} η2_{1,2} ξ1_{1,2} − e^(Times[≪2≫]+Times[≪2≫]+Times[≪3≫]) ϵ λ^2 η2^2_{1,2} ξ1_{1,2} −
     e^(Times[≪2≫]+Times[≪2≫]+Times[≪3≫]) ϵ λ^3 η2^2_{1,2} ξ1_{1,1} ξ1_{1,2} + e^(Times[≪2≫]+Times[≪3≫]) λ η2_{1,2} ξ1_{2,2} −
     e^(Subscript[≪3≫]+Times[≪2≫]+Subscript[≪3≫]+Times[≪2≫]) g_{1,2}′[λ] − e^(Subscript[≪3≫]+Times[≪2≫]+Subscript[≪3≫]+Times[≪2≫]) ϵ f_{1,1}[λ] g_{1,2}′[λ] +
     e^(Subscript[≪3≫]+Times[≪2≫]+Subscript[≪3≫]+Times[≪2≫]) ϵ f_{2,2}[λ] g_{1,2}′[λ] == 0, ≪6≫, f_{2,2}[0] == 0}, ≪1≫, λ] are
   not the same shape.
```

Out[∘]=

$$\text{DSolve}\Big[\Big\{e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\eta1_{1,2}+e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\eta2_{1,2}-e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\lambda\,\eta2_{1,2}\,\xi1_{1,1}+2\,e^{\lambda\,\xi1_{1,1}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta1_{1,2}\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta2_{1,2}^2\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^3\,\eta2_{1,2}^2\,\xi1_{1,1}\,\xi1_{1,2}+e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\lambda\,\eta2_{1,2}\,\xi1_{2,2}-e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,g_{1,2}'[\lambda]-e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,\epsilon\,f_{1,1}[\lambda]\,g_{1,2}'[\lambda]+e^{\xi1_{1,1}-\xi1_{2,2}+\xi2_{1,1}-\xi2_{2,2}}\,\epsilon\,f_{2,2}[\lambda]\,g_{1,2}'[\lambda]==0,$$

$$\xi1_{1,1}-e^{\lambda\,\xi1_{1,1}}\,\epsilon\,\lambda\,\eta1_{1,2}\,\xi1_{1,2}+e^{\lambda\,\xi1_{2,2}}\,\epsilon\,\lambda\,\eta2_{1,2}\,\xi1_{1,2}+e^{\lambda\,\xi1_{2,2}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}+\xi2_{1,1}-e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda\,\eta1_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda\,\eta2_{1,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}-e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}-\epsilon\,f_{1,1}'[\lambda]+e^{\xi1_{1,1}+\xi2_{1,1}}\,\epsilon\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]==0,$$

$$e^{\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\xi1_{1,2}+e^{\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\lambda\,\xi1_{1,1}\,\xi1_{1,2}-e^{\lambda\,\xi1_{1,1}+\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta1_{1,2}\,\xi1_{1,2}^2+e^{\lambda\,\xi2_{2,2}}\,\xi2_{1,2}+e^{\lambda\,\xi2_{2,2}}\,\lambda\,\xi1_{1,1}\,\xi2_{1,2}-2\,e^{\lambda\,\xi1_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta1_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+2\,e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+2\,e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^3\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi2_{2,2}}\,\lambda\,\xi1_{2,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{2,2}}\,\lambda\,\xi2_{1,1}\,\xi2_{1,2}-e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta1_{1,2}\,\xi2_{1,2}^2-e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi2_{1,2}^2+e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^3\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}^2-e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\epsilon\,\lambda^3\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}^2-e^{\xi1_{2,2}+\xi2_{2,2}}\,\epsilon\,f_{1,2}[\lambda]\,f_{1,1}'[\lambda]-e^{\xi1_{2,2}+\xi2_{2,2}}\,f_{1,2}'[\lambda]-e^{\xi1_{2,2}+\xi2_{2,2}}\,\epsilon\,f_{2,2}[\lambda]\,f_{1,2}'[\lambda]+e^{\xi1_{1,1}+\xi1_{2,2}+\xi2_{1,1}+\xi2_{2,2}}\,\epsilon\,f_{1,2}[\lambda]^2\,g_{1,2}'[\lambda]==0,$$

$$e^{\lambda\,\xi1_{1,1}}\,\epsilon\,\lambda\,\eta1_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}}\,\epsilon\,\lambda\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}+\xi1_{2,2}+e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda\,\eta1_{1,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda\,\eta2_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\,\epsilon\,\lambda^2\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}+\xi2_{2,2}-\epsilon\,f_{2,2}'[\lambda]-e^{\xi1_{1,1}+\xi2_{1,1}}\,\epsilon\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]==0,$$

$$g_{1,2}[0]==0,\ f_{1,1}[0]==0,\ f_{1,2}[0]==0,\ f_{2,2}[0]==0\Big\},$$

$$\{g_{1,2}[\lambda],\ f_{1,1}[\lambda],\ f_{1,2}[\lambda],\ f_{2,2}[\lambda]\},$$

$$\lambda\Big]$$

In[∘]:=
```
$k
```
Out[∘]=

1

In[ ]:= $\mathbf{sol1\ /.\ \lambda \to 1}$

Out[ ]=

sol1

In[ ]:=

$$\mathbf{osol} = \mathbb{E}_{\{1,2\}\to\{2\}}\Big[x_{1,1}[2]\,\xi_{1,1}[1] + x_{1,1}[2]\,\xi_{1,1}[2] + e^{-\xi_{1,1}[2]}\,x_{1,2}[2]\,\xi_{1,2}[1] + e^{-\xi_{2,2}[1]}\,x_{1,2}[2]\,\xi_{1,2}[2] +$$
$$e^{-\xi_{1,1}[2]}\,x_{1,3}[2]\,\xi_{1,3}[1] + e^{-\xi_{3,3}[1]}\,x_{1,3}[2]\,\xi_{1,3}[2] + e^{-\xi_{1,1}[2]}\,x_{1,4}[2]\,\xi_{1,4}[1] +$$
$$e^{-\xi_{4,4}[1]}\,x_{1,4}[2]\,\xi_{1,4}[2] + e^{-\xi_{1,1}[2]}\,x_{1,5}[2]\,\xi_{1,5}[1] + e^{-\xi_{5,5}[1]}\,x_{1,5}[2]\,\xi_{1,5}[2] +$$
$$x_{2,2}[2]\,\xi_{2,2}[1] + x_{2,2}[2]\,\xi_{2,2}[2] + e^{-\xi_{2,2}[2]}\,x_{2,3}[2]\,\xi_{2,3}[1] - x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] +$$
$$e^{-\xi_{3,3}[1]}\,x_{2,3}[2]\,\xi_{2,3}[2] + e^{-\xi_{2,2}[2]}\,x_{2,4}[2]\,\xi_{2,4}[1] - x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] +$$
$$e^{-\xi_{4,4}[1]}\,x_{2,4}[2]\,\xi_{2,4}[2] + e^{-\xi_{2,2}[2]}\,x_{2,5}[2]\,\xi_{2,5}[1] - x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] +$$
$$e^{-\xi_{5,5}[1]}\,x_{2,5}[2]\,\xi_{2,5}[2] + x_{3,3}[2]\,\xi_{3,3}[1] + x_{3,3}[2]\,\xi_{3,3}[2] + e^{-\xi_{3,3}[2]}\,x_{3,4}[2]\,\xi_{3,4}[1] -$$
$$x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] - x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] + e^{-\xi_{4,4}[1]}\,x_{3,4}[2]\,\xi_{3,4}[2] +$$
$$e^{-\xi_{3,3}[2]}\,x_{3,5}[2]\,\xi_{3,5}[1] - x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] - x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] +$$
$$e^{-\xi_{5,5}[1]}\,x_{3,5}[2]\,\xi_{3,5}[2] + x_{4,4}[2]\,\xi_{4,4}[1] + x_{4,4}[2]\,\xi_{4,4}[2] + e^{-\xi_{4,4}[2]}\,x_{4,5}[2]\,\xi_{4,5}[1] -$$
$$x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] - x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] - x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] +$$
$$e^{-\xi_{5,5}[1]}\,x_{4,5}[2]\,\xi_{4,5}[2] + x_{5,5}[2]\,\xi_{5,5}[1] + x_{5,5}[2]\,\xi_{5,5}[2],\ 0,\ 0\Big][\![1]\!]\ /.\ \xi_{i\_,i\_}[\_] \to 0$$

Out[ ]=

$$x_{1,2}[2]\,\xi_{1,2}[1] + x_{1,2}[2]\,\xi_{1,2}[2] + x_{1,3}[2]\,\xi_{1,3}[1] + x_{1,3}[2]\,\xi_{1,3}[2] +$$
$$x_{1,4}[2]\,\xi_{1,4}[1] + x_{1,4}[2]\,\xi_{1,4}[2] + x_{1,5}[2]\,\xi_{1,5}[1] + x_{1,5}[2]\,\xi_{1,5}[2] + x_{2,3}[2]\,\xi_{2,3}[1] -$$
$$x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] + x_{2,3}[2]\,\xi_{2,3}[2] + x_{2,4}[2]\,\xi_{2,4}[1] - x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] +$$
$$x_{2,4}[2]\,\xi_{2,4}[2] + x_{2,5}[2]\,\xi_{2,5}[1] - x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] + x_{2,5}[2]\,\xi_{2,5}[2] + x_{3,4}[2]\,\xi_{3,4}[1] -$$
$$x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] - x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] + x_{3,4}[2]\,\xi_{3,4}[2] + x_{3,5}[2]\,\xi_{3,5}[1] -$$
$$x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] - x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] + x_{3,5}[2]\,\xi_{3,5}[2] + x_{4,5}[2]\,\xi_{4,5}[1] -$$
$$x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] - x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] - x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] + x_{4,5}[2]\,\xi_{4,5}[2]$$

In[ ]:=

$$\mathbf{Expand@Total}[\mathbf{sol1}\ /.\ \{\xi_{\alpha\beta\_\_} \mapsto \xi_{\alpha\beta}[1],\ \eta_{\alpha\beta\_\_} \mapsto \xi_{\alpha\beta}[2],\ \mathbf{Rule} \to \mathbf{Times},\ \lambda \to 1,\ f_{\alpha\beta\_\_}[\_] \mapsto x_{\alpha\beta}[2]\}] -$$
$$\mathbf{osol}$$

Out[ ]=

$$\mathrm{Total}[\mathrm{sol1}] - x_{1,2}[2]\,\xi_{1,2}[1] - x_{1,2}[2]\,\xi_{1,2}[2] - x_{1,3}[2]\,\xi_{1,3}[1] - x_{1,3}[2]\,\xi_{1,3}[2] -$$
$$x_{1,4}[2]\,\xi_{1,4}[1] - x_{1,4}[2]\,\xi_{1,4}[2] - x_{1,5}[2]\,\xi_{1,5}[1] - x_{1,5}[2]\,\xi_{1,5}[2] - x_{2,3}[2]\,\xi_{2,3}[1] +$$
$$x_{1,3}[2]\,\xi_{1,2}[2]\,\xi_{2,3}[1] - x_{2,3}[2]\,\xi_{2,3}[2] - x_{2,4}[2]\,\xi_{2,4}[1] + x_{1,4}[2]\,\xi_{1,2}[2]\,\xi_{2,4}[1] -$$
$$x_{2,4}[2]\,\xi_{2,4}[2] - x_{2,5}[2]\,\xi_{2,5}[1] + x_{1,5}[2]\,\xi_{1,2}[2]\,\xi_{2,5}[1] - x_{2,5}[2]\,\xi_{2,5}[2] - x_{3,4}[2]\,\xi_{3,4}[1] +$$
$$x_{1,4}[2]\,\xi_{1,3}[2]\,\xi_{3,4}[1] + x_{2,4}[2]\,\xi_{2,3}[2]\,\xi_{3,4}[1] - x_{3,4}[2]\,\xi_{3,4}[2] - x_{3,5}[2]\,\xi_{3,5}[1] +$$
$$x_{1,5}[2]\,\xi_{1,3}[2]\,\xi_{3,5}[1] + x_{2,5}[2]\,\xi_{2,3}[2]\,\xi_{3,5}[1] - x_{3,5}[2]\,\xi_{3,5}[2] - x_{4,5}[2]\,\xi_{4,5}[1] +$$
$$x_{1,5}[2]\,\xi_{1,4}[2]\,\xi_{4,5}[1] + x_{2,5}[2]\,\xi_{2,4}[2]\,\xi_{4,5}[1] + x_{3,5}[2]\,\xi_{3,4}[2]\,\xi_{4,5}[1] - x_{4,5}[2]\,\xi_{4,5}[2]$$