

Pensieve header: The PBW multiplication tensor for \mathfrak{sl}_n using the nilpotent-only λ -tangent formalism. Some material from pensieve://Projects/glneps and from pensieve://Projects/UEA.

```
In[*]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Conventions

$x_{\alpha,\beta}$ is always with $\alpha \leq \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is always with $\beta \leq \alpha$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\beta,\alpha}$.

Variable declarations:

```
In[*]:= VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)_) [ _ ];
```

Greek - Latin duality:

```
In[*]:= {y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_{αβ})* := (u*)_{αβ};
((u : ((y | x | η | ξ) | (y | x | η | ξ)_) [i_])*) := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

```
In[*]:= {x_{1,2}, y_{2,3}, η_{1,2}[7]}*
Out[*]:= {ξ_{1,2}, η_{2,3}, y_{1,2}[7]}
```

Weights:

```
In[*]:= Wt[x_{α,β}] := {1, α - β}; Wt[y_{α,β}] := {0, α - β};
Wt[ξ_{α,β}] := {0, β - α}; Wt[η_{α,β}] := {1, β - α};
Wt[u_ [ _ ] ] := Wt[u];
```

The Engine

Canonical Forms:

```
In[*]:= LogReduce[ε_] :=
  ε /. c_ * Log[a_] => Log@Factor[a^c] /. Log[a_] + Log[b_] => Log@Factor[a b];
CF[ε_] := Expand@Collect[ε, Cases[ε, VarPattern, ∞], CCF] /. CCF -> Factor;
CF[ε_E] := CF /@ MapAt[LogReduce, ε, 1];
CF[ε : E_ [ ___ ] ] := CF /@ ε;
```

\mathbb{E} operations:

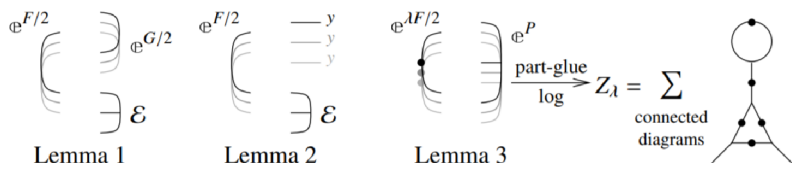
```
In[*]:=
 $\mathcal{E}_\mathbb{E}[\$] := \text{Length}[\mathcal{E}] - 1$ ;  $\mathbb{E}_\mathbb{E}[\mathcal{E}\_\_\_\_\_\_][\$] := \mathbb{E}[\mathcal{E}][\$]$ ;
 $\mathcal{E}_\mathbb{E}[k\_Integer] := \mathcal{E}[[k + 1]]$ ;  $\mathbb{E}_\mathbb{E}[\mathcal{E}\_\_\_\_\_\_][k\_Integer] := \{\mathcal{E}\}[[k + 1]]$ ;
 $\mathbb{E} /: \mathcal{E}_1 \mathbb{E} \equiv \mathcal{E}_2 \mathbb{E} := \text{Inner}[\text{CF}@\#1 == \text{CF}@\#2 \ \&, \mathcal{E}_1, \mathcal{E}_2, \text{And}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E}1s] \equiv \mathbb{E}[\mathcal{E}2s])$ ;
 $\mathbb{E} /: \mathcal{E}_1 \mathbb{E} * \mathcal{E}_2 \mathbb{E} := \mathbb{E}@\text{Table}[\text{CF}[\mathcal{E}1[\text{kk}] + \mathcal{E}2[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E}1[\$], \mathcal{E}2[\$]]\}]$ ;
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)}@\text{Table}[\text{CF}[\mathcal{E}1s] \mathbb{E}[\mathcal{E}2s]]$ ;

```

```
In[*]:=
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E}1s\_\_\_\_\_\_] // \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E}2s\_\_\_\_\_\_] := \text{Module}[\{\text{is} = r1 \cap d2, \text{lvs}, \text{llvs}\},$ 
   $\text{lvs} = \text{Cases}[\{\mathcal{E}1s\}, (x | y)\_\_\_\_\_\_[i\_] /; \text{MemberQ}[\text{is}, i], \infty] \cup$ 
   $\text{Cases}[\{\mathcal{E}2s\}, (\xi | \eta)\_\_\_\_\_\_[i\_] /; \text{MemberQ}[\text{is}, i], \infty]^*$ ;
   $\text{llvs} = \text{Map}[\$, \text{lvs}, \{2\}]$ ;
   $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])} @ @ (\text{Zip}[\text{lvs} \cup \text{llvs}^*][\{(\text{llvs})^* \cdot (\text{llvs})\}, \text{Times}$ 
   $\mathbb{E}[\mathcal{E}1s] /. \text{Thread}[\text{lvs} \rightarrow \text{llvs}],$ 
   $\mathbb{E}[\mathcal{E}2s] /. \text{Thread}[\text{lvs}^* \rightarrow \text{llvs}^*]$ 
   $1 \}\})$ 
 $]$ 

```

Zippping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.



Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ij} ’s and the y_{ij} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

```
In[*]:=
 $\text{Zip}_{vs\_}[\{\mathcal{F}\_\_, \mathcal{E}\_\_}] := \text{Module}[\{\text{dvs} = \text{Select}[\text{vs}, (\text{Wt}[\#] == \{0, 0\} \vee \text{Wt}[\#] == \{1, 0\}) \ \&]\},$ 
   $\{\mathcal{F}, \mathcal{E}\} // \text{Zip1}_{vs}(* // \text{Zip2}_{\text{Complement}[\text{vs}, \text{dvs}]}*) // \text{Zip2}_{\text{dvs}} //$ 
   $\text{Zip3}_{\text{Complement}[\text{vs}, \text{dvs}]} // \text{Zip3}_{\text{dvs}}$ 
 $]$  // Last

```

Getting rid of the quadratic.
Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E}_{e^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j}} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

```
In[*]:= Zip1_{ } = Identity;
Zip1_{vs_} @ {F_, E[Q_, P___]} := Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, D[u, v]F, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, D[u, v]Q, 0], {u, vs}, {v, vs}];
  {CF[(vs) . (F.Inverse[I - G.F]) . (vs) / 2],
   E[CF[Q - PowerExpand@Log[Det[I - G.F]] / 2 - vs.G.vs / 2], P]}
]
```

Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E}_{\oplus \sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2} \sum_{i, j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}_{|z_B \rightarrow z_B + F y_B} \right\rangle_B$.

```
In[*]:= Zip2_{ } = Identity;
Zip2_{vs_} @ {F_, E[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[D[u, v]F], 0], {u, vs}, {v, vs}];
  Y = Table[D[v, Q], {v, vs}] /. Table[v -> 0, {v, vs}];
  CF /@ {F, E[Q - Y.vs + Y.F.Y / 2, P]} /. Thread[vs -> vs + F.Y]}
]
```

Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i, j \in B} F_{ij} (\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

```

In[*]:= Zip3vs_@{F_, E_} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = E[[kk + 1]], {kk, 0, E@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂u,vF, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂uZ[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂vZ[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ E@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2(m+1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}]),
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
    F - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
    E @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, E@$}]]
  ] /. Table[v → 0, {v, vs}]]

```

Prolog

```
In[*]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

CF: Symbol CF appears in multiple contexts {UEA`, Global}; definitions in context UEA` may shadow or be shadowed by other definitions.

m: Symbol m appears in multiple contexts {UEA`, Global}; definitions in context UEA` may shadow or be shadowed by other definitions.

UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ , ϵ , \$Basis, \$k, \$PBWRule}.

Utilities

```
In[*]:=  $\delta_{i,j} := \text{If}[i == j, 1, 0];$ 
```

```
In[*]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And@@(SSQ/@(List@@x));
NilQ[x_Plus] := And@@(NilQ/@(List@@x));
```

```
In[*]:= $k = 1;
```

```
In[*]:= CF[ $\mathcal{E}_-$ ] := Expand[ $\mathcal{E} / . e^{L-} \rightarrow e^{\text{Expand}[L]}$ ] /. { $e^{k-} /; k > $k \rightarrow 0$ };
```

Implementing general universal enveloping algebras

```
In[ ]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_*x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_*x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[ ]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_P_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2->i};
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[ ]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i->j}[ε_] := ε /. U_i → U_j;
```

```
In[ ]:= m_{i,j->k}[c_. U_i[x__] U_j[]] := c U_k[x];
m_{i,j->k}[c_. U_i[] U_j[y__]] := c U_k[y];
m_{i,j->k}[c_. U_i[xx__, x_] U_j[y_, yy__]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_{i,j->i} U_j[yy] // Expand // m_{i,j->k}
  c // USimp
];
```

```
In[ ]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is__}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[is, d - j]}];
```

```
In[ ]:= Supp[ε_] := Union@Cases[ε, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_  $\otimes$  y_ := Module[{is = Supp[x]  $\cap$  Supp[y],  $\sigma$ , z},
  z = x; Do[z = mi $\rightarrow$  $\sigma$ ei[z], {i, is}];
  z = Expand[y z]; Do[z = m $\sigma$ ei, i $\rightarrow$ i[z], {i, is}]; z];
UB[x_, y_] := USimp[x  $\otimes$  y - y  $\otimes$  x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"s12\"."];
```

UEA`SetAlgebra knows "s12".

```
In[*]:= SetAlgebra["s12"] := (
  Print["In s12: <e,h,f>/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Algebra = "s12";
  $Basis = {e, h, f};
  $PBWRule = {e  $\rightarrow$  1, h  $\rightarrow$  2, f  $\rightarrow$  3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

$gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra gln, $\epsilon$ \"."];
```

UEA`SetAlgebra knows the ϵ -nilpotent algebra gl_{n, ϵ .}

```

In[*]:= SetAlgebra[gln,ε] := (
  $Algebra = gln,ε;
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  (*$Basis=Reverse@SortBy[$Basis,If[#[[1]]===x,{0,#[[3]]-#[[2]]},{1,#[[2]]-#[[3]]}&];*)
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[-ε δj,k yi,l + ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j / . xα,β => If[α ≤ β, ε xα,β, yβ,α]];
  DefRep[0] = Normal@SparseArray[{}, {n, n}];
  DefRep[ε_] := ε / . {
    xα,β => Normal@SparseArray[{α, β} → 1, {n, n}],
    yα,β => Normal@SparseArray[{β, α} → ε, {n, n}]
  }
);

```

```

In[*]:= adPower[x_, k_, Env_] [c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env] [y] = B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, {$Algebra, $k}] [y];

```

```

In[*]:= adExp[x_] [y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ε, 0, $k}]].
    Table[Coefficient[y, b], {b, $Basis}]]
]

```

```

In[*]:= adExp[x_] [y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k] [y] != 0,
    s += adPower[x, k] [y] / k!;
    ++k
  ];
  s
]

```

```

In[*]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x] [λTangent[xs]] + ∂λx;
λTangent[xs_List] := λTangent@@xs

```

```
In[*]:= TriangularDSolve[eqns_, funs_, iv_] := Module[{sol = {}, e, fun, der},
  MapThread[
    {e, fun} \[Mapsto] sol = Join[sol,
      Echo@MapAt[CF, First@DSolve[Echo@CF[e /. Derivative[1][f_][v_] \[Rrightarrow] der[f[v], v] /.
        sol /. der[f_, v_] \[Rrightarrow] \[D]_v f], fun, iv], {1, 2}]
    ],
  {eqns, funs}
];
sol
]
```

gl_{2,ε} Experiments

```
In[*]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} \[Rrightarrow] B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
  Table[{b1, b2} \[Rrightarrow] (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]=
{y1,2 \[Rrightarrow] 1, x1,1 \[Rrightarrow] 2, x1,2 \[Rrightarrow] 3, x2,2 \[Rrightarrow] 4}
```

```
Out[*]//MatrixForm=
(
  {y1,2, y1,2} \[Rrightarrow] 0           {y1,2, x1,1} \[Rrightarrow] y1,2   {y1,2, x1,2} \[Rrightarrow] -ε x1,1 + ε x2,2   {y1,2, x2,2} \[Rrightarrow] -y1,2
  {x1,1, y1,2} \[Rrightarrow] -y1,2       {x1,1, x1,1} \[Rrightarrow] 0           {x1,1, x1,2} \[Rrightarrow] x1,2           {x1,1, x2,2} \[Rrightarrow] 0
  {x1,2, y1,2} \[Rrightarrow] ε x1,1 - ε x2,2   {x1,2, x1,1} \[Rrightarrow] -x1,2       {x1,2, x1,2} \[Rrightarrow] 0           {x1,2, x2,2} \[Rrightarrow] x1,2
  {x2,2, y1,2} \[Rrightarrow] y1,2           {x2,2, x1,1} \[Rrightarrow] 0           {x2,2, x1,2} \[Rrightarrow] -x1,2       {x2,2, x2,2} \[Rrightarrow] 0
)
```

```
Out[*]//MatrixForm=
(
  {y1,2, y1,2} \[Rrightarrow] True   {y1,2, x1,1} \[Rrightarrow] True   {y1,2, x1,2} \[Rrightarrow] True   {y1,2, x2,2} \[Rrightarrow] True
  {x1,1, y1,2} \[Rrightarrow] True   {x1,1, x1,1} \[Rrightarrow] True   {x1,1, x1,2} \[Rrightarrow] True   {x1,1, x2,2} \[Rrightarrow] True
  {x1,2, y1,2} \[Rrightarrow] True   {x1,2, x1,1} \[Rrightarrow] True   {x1,2, x1,2} \[Rrightarrow] True   {x1,2, x2,2} \[Rrightarrow] True
  {x2,2, y1,2} \[Rrightarrow] True   {x2,2, x1,1} \[Rrightarrow] True   {x2,2, x1,2} \[Rrightarrow] True   {x2,2, x2,2} \[Rrightarrow] True
)
```

```
In[*]:= DeleteCases[Flatten[Table[{a, b, c} \[Rrightarrow] B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ \[Rrightarrow] 0]
```

```
Out[*]=
{}
```

```
In[*]:= lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]]
```

```
Out[*]=
e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} y_{1,2} η_{1,2} + e^{ξ_{2,1}-ξ_{2,2}} y_{1,2} η_{2,2} + e^{ξ_{1,2}-ξ_{2,1}+ξ_{2,2}} x_{1,2} ξ_{1,2} - e^{ξ_{1,1}} ∈ λ x_{1,1} η_{1,2} ξ_{1,2} +
e^{ξ_{1,1}} ∈ λ x_{2,2} η_{1,2} ξ_{1,2} + e^{ξ_{1,2}} ∈ λ x_{1,1} η_{2,2} ξ_{1,2} - e^{ξ_{1,2}} ∈ λ x_{2,2} η_{2,2} ξ_{1,2} +
2 e^{ξ_{1,1}+ξ_{2,1}-ξ_{2,2}} ∈ λ^2 y_{1,2} η_{1,2} η_{2,2} ξ_{1,2} - e^{ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} ∈ λ^2 y_{1,2} η_{2,2}^2 ξ_{1,2} -
e^{ξ_{1,1}+ξ_{1,2}-ξ_{2,1}+ξ_{2,2}} ∈ λ^2 x_{1,2} η_{1,2} ξ_{1,2}^2 + e^{ξ_{2,2}} x_{1,2} ξ_{2,2} - e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}} ∈ λ x_{1,1} η_{1,2} ξ_{2,2} +
e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}} ∈ λ x_{2,2} η_{1,2} ξ_{2,2} - e^{ξ_{2,1}} ∈ λ x_{1,1} η_{2,2} ξ_{2,2} + e^{ξ_{2,1}} ∈ λ x_{2,2} η_{2,2} ξ_{2,2} -
2 e^{ξ_{1,1}+ξ_{2,2}} ∈ λ^2 x_{1,2} η_{1,2} ξ_{1,2} ξ_{2,2} + 2 e^{ξ_{1,2}+ξ_{2,2}} ∈ λ^2 x_{1,2} η_{2,2} ξ_{1,2} ξ_{2,2} -
e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}+ξ_{2,2}} ∈ λ^2 x_{1,2} η_{1,2} ξ_{2,2}^2 - e^{ξ_{2,1}+ξ_{2,2}} ∈ λ^2 x_{1,2} η_{2,2} ξ_{2,2}^2
```

```
In[*]:= rhs = λTangent[
  Table[If[SSQ[b], b /. x_{α,α} → (ξ_{1,α} + ξ_{2,α} + e^{f_{α,α}[λ]}) x_{α,α},
    (b /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]}) b], {b, $Basis}]
]
```

```
Out[*]=
∈ x_{1,1} f_{1,1}'[λ] + e^{ξ_{1,2}+ξ_{2,2}} ∈ x_{1,2} f_{1,2}[λ] f_{1,1}'[λ] +
e^{ξ_{1,2}+ξ_{2,2}} x_{1,2} f_{1,2}'[λ] + e^{ξ_{1,2}+ξ_{2,2}} ∈ x_{1,2} f_{2,2}[λ] f_{1,2}'[λ] + ∈ x_{2,2} f_{2,2}'[λ] +
e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} y_{1,2} g_{1,2}'[λ] + e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} ∈ y_{1,2} f_{1,1}[λ] g_{1,2}'[λ] -
e^{ξ_{1,1}+ξ_{2,1}} ∈ x_{1,1} f_{1,2}[λ] g_{1,2}'[λ] + e^{ξ_{1,1}+ξ_{2,1}} ∈ x_{2,2} f_{1,2}[λ] g_{1,2}'[λ] -
e^{ξ_{1,1}+ξ_{1,2}+ξ_{2,1}+ξ_{2,2}} ∈ x_{1,2} f_{1,2}[λ]^2 g_{1,2}'[λ] - e^{ξ_{1,1}-ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} ∈ y_{1,2} f_{2,2}[λ] g_{1,2}'[λ]
```

```
In[*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis]
```

```
Out[*]=
{e^{-ξ_{1,2}+ξ_{2,1}-ξ_{2,2}} (-e^{ξ_{1,2}} η_{2,2} + e^{2ξ_{1,2}} ∈ λ^2 η_{2,2}^2 ξ_{1,2} -
e^{ξ_{1,1}} η_{1,2} (1 + 2 e^{ξ_{1,2}} ∈ λ^2 η_{2,2} ξ_{1,2}) + e^{ξ_{1,1}} (1 + ∈ f_{1,1}[λ] - ∈ f_{2,2}[λ]) g_{1,2}'[λ]) == 0,
∈ (λ η_{2,2} (e^{ξ_{1,2}} ξ_{1,2} - e^{ξ_{2,1}} ξ_{2,2}) - e^{ξ_{1,1}-ξ_{1,2}} λ η_{1,2} (e^{ξ_{1,2}} ξ_{1,2} + e^{ξ_{2,1}} ξ_{2,2}) - f_{1,1}'[λ] +
e^{ξ_{1,1}+ξ_{2,1}} f_{1,2}[λ] g_{1,2}'[λ]) == 0, e^{-ξ_{1,2}-ξ_{2,1}+ξ_{2,2}} (-e^{ξ_{1,1}+2ξ_{1,2}} ∈ λ^2 η_{1,2} ξ_{1,2}^2 +
e^{ξ_{1,2}} ξ_{1,2} (-2 e^{ξ_{1,1}+ξ_{2,1}} ∈ λ^2 η_{1,2} ξ_{2,2} + e^{ξ_{1,2}} (1 + 2 e^{ξ_{2,1}} ∈ λ^2 η_{2,2} ξ_{2,2})) +
e^{ξ_{2,1}} (e^{ξ_{1,2}} ξ_{2,2} - e^{ξ_{2,1}} ∈ λ^2 (e^{ξ_{1,1}} η_{1,2} + e^{ξ_{1,2}} η_{2,2}) ξ_{2,2}^2 +
e^{2ξ_{1,2}} (-∈ f_{1,2}[λ] f_{1,1}'[λ] - (1 + ∈ f_{2,2}[λ]) f_{1,2}'[λ] + e^{ξ_{1,1}+ξ_{2,1}} ∈ f_{1,2}[λ]^2 g_{1,2}'[λ])) ==
0, e^{ξ_{1,1}-ξ_{1,2}} ∈ λ η_{1,2} (e^{ξ_{1,2}} ξ_{1,2} + e^{ξ_{2,1}} ξ_{2,2}) ==
∈ (λ η_{2,2} (e^{ξ_{1,2}} ξ_{1,2} - e^{ξ_{2,1}} ξ_{2,2}) + f_{2,2}'[λ] + e^{ξ_{1,1}+ξ_{2,1}} f_{1,2}[λ] g_{1,2}'[λ])}
```

```
In[*]:= Length@eqns
```

```
Out[*]=
4
```

```
In[*]:= unknowns = $Basis /. {x_{αβ} → f_{αβ}[λ], y_{αβ} → g_{αβ}[λ]}
```

```
Out[*]=
{g_{1,2}[λ], f_{1,1}[λ], f_{1,2}[λ], f_{2,2}[λ]}
```

```

In[*]:= {sol0} = Block[{ $k = 0 }, Simplify@DSolve[
  And@@Join[
    CF /@ eqns,
    Select[$Basis, NilQ] /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
  ],
  Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]

Out[*]=
{{g1,2[λ] → λ (η1,2 + e-ε1,1+ε1,2 η2,2), f1,2[λ] → e-ε2,1 λ ξ1,2 + e-ε1,2 λ ξ2,2}}α,α → (ξ1,α + ξ2,α + e fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((a → b) → (a → e a + b)))
    ], {b, $Basis}
  ];
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]

Out[*]=
{{f1,1[λ] → eε1,2 λ2 η2,2 ξ1,2, f1,2[λ] → λ3 η2,2 ξ1,2 ξ2,2,
  f2,2[λ] → -eε1,2 λ2 η2,2 ξ1,2, g1,2[λ] → -e-ε1,1+2ε1,2 λ3 η2,22 ξ1,2}}{i,j}→{k}[CF@Sum[If[SSQ[b],
  b /. xα,α → (ξ1,α + ξ2,α) xα,α[k],
  b[k] (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) /. sol0],
  ], {b, $Basis}],
  CF@Sum[b[k] (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) /. sol1], {b, $Basis}
] /. {ξ1,αβ → ξαβ[i], η1,αβ → ηαβ[i], ξ2,αβ → ξαβ[j], η2,αβ → ηαβ[j], λ → 1};
cm[i, j → k]

Out[*]=
E{i,j}→{k}[y1,2[k] η1,2[i] + e-ε1,1[i]+ε2,2[i] y1,2[k] η1,2[j] + x1,1[k] ξ1,1[i] + x1,1[k] ξ1,1[j] +
  e-ε1,1[j] x1,2[k] ξ1,2[i] + e-ε2,2[i] x1,2[k] ξ1,2[j] + x2,2[k] ξ2,2[i] + x2,2[k] ξ2,2[j],
  eε2,2[i] x1,1[k] η1,2[j] ξ1,2[i] - eε2,2[i] x2,2[k] η1,2[j] ξ1,2[i] -
  e-ε1,1[i]+2ε2,2[i] y1,2[k] η1,2[j]2 ξ1,2[i] + x1,2[k] η1,2[j] ξ1,2[i] ξ1,2[j]]

```

In[*]:= **cm**[1, 2 → 2] // **cm**[2, 3 → 1]

Out[*]=

$$\begin{aligned}
& \mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + x_{1,1}[1] \xi_{1,1}[1] + x_{1,1}[1] \xi_{1,1}[2] + \\
& x_{1,1}[1] \xi_{1,1}[3] + e^{-\xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] + e^{-\xi_{1,1}[3] - \xi_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2]} x_{1,2}[1] \xi_{1,2}[3] + x_{2,2}[1] \xi_{2,2}[1] + x_{2,2}[1] \xi_{2,2}[2] + x_{2,2}[1] \xi_{2,2}[3], \\
& e^{\xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[2] \xi_{1,2}[1] - e^{\xi_{2,2}[1]} x_{2,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] - \\
& e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2]^2 \xi_{1,2}[1] + e^{-\xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} x_{1,1}[1] \eta_{1,2}[3] \xi_{1,2}[1] - \\
& e^{-\xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] - \\
& 2 e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + 2 \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[2] \eta_{1,2}[3] \xi_{1,2}[1] - \\
& e^{-\xi_{1,1}[1] - 2 \xi_{1,1}[2] + 2 \xi_{2,2}[1] + 2 \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[1] + e^{\xi_{2,2}[2]} x_{1,1}[1] \eta_{1,2}[3] \xi_{1,2}[2] - \\
& e^{\xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] - e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + 2 \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[2] + \\
& e^{-\xi_{1,1}[3]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[2] + e^{-\xi_{2,2}[2]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[3] + \\
& \left. e^{-\xi_{1,1}[2]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] \xi_{1,2}[3] + e^{-\xi_{2,2}[1]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] \xi_{1,2}[3] \right]
\end{aligned}$$

In[*]:= **Simplify**[(**cm**[1, 2 → 2] // **cm**[2, 3 → 1]) ≡ (**cm**[2, 3 → 2] // **cm**[1, 2 → 1])]]

Out[*]=

True

gl_{3,ε} Experiments

```
In[*]:= SetAlgebra[gl3,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
  Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
    {b1, $Basis}, {b2, $Basis}]
```

Out[*]=

1

Out[*]=

{y_{1,2} → 1, y_{1,3} → 2, y_{2,3} → 3, x_{1,1} → 4, x_{1,2} → 5, x_{2,2} → 6, x_{1,3} → 7, x_{2,3} → 8, x_{3,3} → 9}

Out[*]//MatrixForm=

$$\begin{pmatrix}
 \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, y_{1,3}\} \rightarrow 0 & \{y_{1,2}, y_{2,3}\} \rightarrow -\epsilon y_{1,3} & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} \\
 \{y_{1,3}, y_{1,2}\} \rightarrow 0 & \{y_{1,3}, y_{1,3}\} \rightarrow 0 & \{y_{1,3}, y_{2,3}\} \rightarrow 0 & \{y_{1,3}, x_{1,1}\} \rightarrow y_{1,3} \\
 \{y_{2,3}, y_{1,2}\} \rightarrow \epsilon y_{1,3} & \{y_{2,3}, y_{1,3}\} \rightarrow 0 & \{y_{2,3}, y_{2,3}\} \rightarrow 0 & \{y_{2,3}, x_{1,1}\} \rightarrow y_{2,3} \\
 \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, y_{1,3}\} \rightarrow -y_{1,3} & \{x_{1,1}, y_{2,3}\} \rightarrow 0 & \{x_{1,1}, x_{1,1}\} \rightarrow 0 \\
 \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, y_{1,3}\} \rightarrow -y_{2,3} & \{x_{1,2}, y_{2,3}\} \rightarrow 0 & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} \\
 \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, y_{1,3}\} \rightarrow 0 & \{x_{2,2}, y_{2,3}\} \rightarrow -y_{2,3} & \{x_{2,2}, x_{1,1}\} \rightarrow -x_{2,2} \\
 \{x_{1,3}, y_{1,2}\} \rightarrow -\epsilon x_{2,3} & \{x_{1,3}, y_{1,3}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{3,3} & \{x_{1,3}, y_{2,3}\} \rightarrow \epsilon x_{1,2} & \{x_{1,3}, x_{1,1}\} \rightarrow -\epsilon x_{1,3} \\
 \{x_{2,3}, y_{1,2}\} \rightarrow 0 & \{x_{2,3}, y_{1,3}\} \rightarrow y_{1,2} & \{x_{2,3}, y_{2,3}\} \rightarrow \epsilon x_{2,2} - \epsilon x_{3,3} & \{x_{2,3}, x_{1,1}\} \rightarrow -\epsilon x_{2,3} \\
 \{x_{3,3}, y_{1,2}\} \rightarrow 0 & \{x_{3,3}, y_{1,3}\} \rightarrow y_{1,3} & \{x_{3,3}, y_{2,3}\} \rightarrow y_{2,3} & \{x_{3,3}, x_{1,1}\} \rightarrow -\epsilon x_{3,3}
 \end{pmatrix}$$

Out[*]//MatrixForm=

$$\begin{pmatrix}
 \{y_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, y_{1,3}\} \rightarrow \text{True} & \{y_{1,2}, y_{2,3}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{y_{1,3}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,3}, y_{1,3}\} \rightarrow \text{True} & \{y_{1,3}, y_{2,3}\} \rightarrow \text{True} & \{y_{1,3}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,3}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{y_{2,3}, y_{1,2}\} \rightarrow \text{True} & \{y_{2,3}, y_{1,3}\} \rightarrow \text{True} & \{y_{2,3}, y_{2,3}\} \rightarrow \text{True} & \{y_{2,3}, x_{1,1}\} \rightarrow \text{True} & \{y_{2,3}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{1,1}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,1}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,2}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{2,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, y_{1,3}\} \rightarrow \text{True} & \{x_{2,2}, y_{2,3}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{1,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{1,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{1,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,3}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{2,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{2,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{2,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,3}, x_{1,2}\} \rightarrow \text{True} & \{ \\
 \{x_{3,3}, y_{1,2}\} \rightarrow \text{True} & \{x_{3,3}, y_{1,3}\} \rightarrow \text{True} & \{x_{3,3}, y_{2,3}\} \rightarrow \text{True} & \{x_{3,3}, x_{1,1}\} \rightarrow \text{True} & \{x_{3,3}, x_{1,2}\} \rightarrow \text{True} & \{
 \end{pmatrix}$$

```
In[*]:= DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
  {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
```

Out[*]=

{}

```

In[*]:= {sol0} = Block[{ $k = 0,
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. xα,α ⇒ (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
  ];
  Simplify@DSolve[
    And@@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
    ],
    Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
    λ
  ]
];

Out[*]= {
  {f1,2[λ] → e-ε2,1 λ ξ1,2 + e-ε2,2 λ ξ2,2, g2,3[λ] → λ (η1,3 + eε1,3 (e-ε1,2 η2,3 - λ η1,3 ξ1,2))},
  {f1,3[λ] → λ (e-ε2,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ε1,3 ξ2,3), f2,3[λ] → e-ε2,2 λ ξ1,2,3 + e-ε1,3 λ ξ2,3,
  {g1,2[λ] → λ (η1,2 + e-ε1,1+ε1,2 (η2,2 + eε1,3 λ η2,3 ξ1,2,3)), g1,3[λ] → λ (η1,3 + e-ε1,1+ε1,3 η2,3)}
};

In[*]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]];
rhs = λTangent[
  Echo@Table[If[SSQ[b],
    b /. xα,α ⇒ (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
    (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((v- → s-) ⇒ (v → ε v + s)))
  ], {b, $Basis}]
];
eqns = Transpose@{
  (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
  SBasis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
};
unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
(* {sol1} = DSolve[Flatten@eqns, unknowns, λ] *)
sol1 = TriangularDSolve[eqns, unknowns, λ];

```

- $$\begin{aligned} & \gg \{y_{1,2} (\lambda (\eta_{1,2} + e^{-\epsilon_{1,1} + \epsilon_{1,2}} (\eta_{2,2} + e^{\epsilon_{1,3}} \lambda \eta_{2,3} \xi_{1,2,3})) + e_{\mathbf{g}_{1,2}}[\lambda]), \\ & y_{1,3} (\lambda (\eta_{1,3} + e^{-\epsilon_{1,1} + \epsilon_{1,3}} \eta_{2,3})) + e_{\mathbf{g}_{1,3}}[\lambda]), \\ & y_{2,3} (\lambda (\eta_{2,3} + e^{\epsilon_{1,3}} (e^{-\epsilon_{1,2}} \eta_{2,3} - \lambda \eta_{2,3} \xi_{1,1,2}))) + e_{\mathbf{g}_{2,3}}[\lambda]), \\ & x_{1,1} (\xi_{1,1} + \xi_{2,1} + e_{\mathbf{f}_{1,1}}[\lambda]), x_{1,2} (e^{-\epsilon_{2,1}} \lambda \xi_{1,2} + e^{-\epsilon_{1,2}} \lambda \xi_{2,2} + e_{\mathbf{f}_{1,2}}[\lambda]), \\ & x_{2,2} (\xi_{1,2} + \xi_{2,2} + e_{\mathbf{f}_{2,2}}[\lambda]), x_{1,3} (\lambda (e^{-\epsilon_{2,1}} \xi_{1,3} - \lambda \xi_{1,2} \xi_{2,2} + e^{-\epsilon_{1,3}} \xi_{2,3})) + e_{\mathbf{f}_{1,3}}[\lambda]), \\ & x_{2,3} (e^{-\epsilon_{2,2}} \lambda \xi_{1,2} + e^{-\epsilon_{1,3}} \lambda \xi_{2,3} + e_{\mathbf{f}_{2,3}}[\lambda]), x_{3,3} (\xi_{1,3} + \xi_{2,3} + e_{\mathbf{f}_{3,3}}[\lambda]) \} \end{aligned}$$
- $$\gg \{2 e^{\epsilon_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} + 2 e^{\epsilon_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} + 3 e^{\epsilon_{1,2} + \epsilon_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{1,3} - e_{\mathbf{f}_{1,1}'}[\lambda] = \mathbf{0}, \mathbf{f}_{1,1}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{1,1}[\lambda] \rightarrow e^{\epsilon_{1,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + e^{\epsilon_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} + e^{\epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,3}\}$$
- $$\gg \{-2 e^{\epsilon_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} + 2 e^{\epsilon_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} - 3 e^{\epsilon_{1,2} + \epsilon_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{1,3} - e_{\mathbf{f}_{2,2}'}[\lambda] = \mathbf{0}, \mathbf{f}_{2,2}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{2,2}[\lambda] \rightarrow -e^{\epsilon_{1,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + e^{\epsilon_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} - e^{\epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,3}\}$$
- $$\gg \{-2 e^{\epsilon_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} - 2 e^{\epsilon_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} - e_{\mathbf{f}_{3,3}'}[\lambda] = \mathbf{0}, \mathbf{f}_{3,3}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{3,3}[\lambda] \rightarrow -e^{\epsilon_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} - e^{\epsilon_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3}\}$$
- $$\gg \{2 e^{\epsilon_{1,3} - \epsilon_{2,1} + \epsilon_{2,2}} \in \lambda \eta_{2,3} \xi_{1,3} - 3 e^{\epsilon_{1,2} + \epsilon_{1,3} - \epsilon_{2,1} + \epsilon_{2,2}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{1,3} + 3 e^{\epsilon_{1,2} + \epsilon_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \xi_{2,2} - 3 e^{\epsilon_{1,3} + \epsilon_{2,2}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{2,2} + 4 e^{\epsilon_{1,2} + \epsilon_{1,3} + \epsilon_{2,2}} \in \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,3} \xi_{2,2} - e^{\epsilon_{1,2} + \epsilon_{2,2}} \in \mathbf{f}_{1,2}'[\lambda] = \mathbf{0}, \mathbf{f}_{1,2}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{1,2}[\lambda] \rightarrow e^{-\epsilon_{1,2} + \epsilon_{1,3} - \epsilon_{2,1}} \lambda^2 \eta_{2,3} \xi_{1,3} - e^{\epsilon_{1,3} - \epsilon_{2,1}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,3} + \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2} - e^{-\epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{2,2} + e^{\epsilon_{1,3}} \lambda^4 \eta_{2,3} \xi_{1,2} \xi_{1,3} \xi_{2,2}\}$$
- $$\gg \{-2 e^{\epsilon_{1,3} - \epsilon_{2,2} + \epsilon_{2,3}} \in \lambda \eta_{2,2} \xi_{1,3} + 3 e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,3} \xi_{2,3} + 3 e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,3} \xi_{2,3} - e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \mathbf{f}_{2,3}'[\lambda] = \mathbf{0}, \mathbf{f}_{2,3}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{2,3}[\lambda] \rightarrow -e^{-\epsilon_{2,2}} \lambda^2 \eta_{2,2} \xi_{1,3} + \lambda^3 \eta_{2,3} \xi_{1,3} \xi_{2,3} + \lambda^3 \eta_{2,3} \xi_{1,3} \xi_{2,3}\}$$
- $$\gg \{3 e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \lambda^2 \eta_{2,2} \xi_{1,3} \xi_{2,3} + 3 e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,3} \xi_{2,3} + 3 e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \lambda^2 \eta_{2,3} \xi_{1,3} \xi_{2,3} - e^{\epsilon_{1,3} + \epsilon_{2,3}} \in \mathbf{f}_{1,3}'[\lambda] = \mathbf{0}, \mathbf{f}_{1,3}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{f}_{1,3}[\lambda] \rightarrow \lambda^3 \eta_{2,2} \xi_{1,3} \xi_{2,3} + \lambda^3 \eta_{2,3} \xi_{1,3} \xi_{2,3} + \lambda^3 \eta_{2,3} \xi_{1,3} \xi_{2,3}\}$$
- $$\gg \{2 e^{\epsilon_{1,2} - \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,3}} \in \lambda \eta_{1,2} \eta_{2,2} - 3 e^{\epsilon_{1,2} + \epsilon_{2,1} - \epsilon_{2,3}} \in \lambda^2 \eta_{1,2} \eta_{2,3} \xi_{1,2} - 3 e^{\epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,3}} \in \lambda^2 \eta_{1,3} \eta_{2,3} \xi_{1,3} - 4 e^{\epsilon_{1,2} + \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,3}} \in \lambda^3 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} - e^{\epsilon_{1,1} - \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,3}} \in \mathbf{g}_{1,3}'[\lambda] = \mathbf{0}, \mathbf{g}_{1,3}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{g}_{1,3}[\lambda] \rightarrow e^{-\epsilon_{1,1} + \epsilon_{1,2}} \lambda^2 \eta_{1,2} \eta_{2,2} - e^{-\epsilon_{1,1} + \epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,3} \xi_{1,2} - e^{-\epsilon_{1,1} + \epsilon_{1,2}} \lambda^3 \eta_{2,3}^2 \xi_{1,3} + e^{-\epsilon_{1,1} + \epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,3} \xi_{1,3} - e^{-\epsilon_{1,1} + \epsilon_{1,2} + \epsilon_{1,3}} \lambda^4 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3}\}$$
- $$\gg \{-3 e^{\epsilon_{1,2} + \epsilon_{2,1} - \epsilon_{2,2}} \in \lambda^2 \eta_{1,2}^2 \xi_{1,2} - 3 e^{\epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,2}} \in \lambda^2 \eta_{1,2} \eta_{2,3} \xi_{1,3} - 8 e^{\epsilon_{1,2} + \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,2}} \in \lambda^3 \eta_{1,2} \eta_{2,3} \xi_{1,2} \xi_{1,3} - 4 e^{2 \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,2}} \in \lambda^3 \eta_{1,3}^2 \xi_{1,3} \xi_{1,2} \xi_{1,3} - 5 e^{\epsilon_{1,2} + 2 \epsilon_{1,3} + \epsilon_{2,1} - \epsilon_{2,2}} \in \lambda^4 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} - e^{\epsilon_{1,1} - \epsilon_{1,2} + \epsilon_{2,1} - \epsilon_{2,2}} \in \mathbf{g}_{1,2}'[\lambda] = \mathbf{0}, \mathbf{g}_{1,2}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{g}_{1,2}[\lambda] \rightarrow -e^{-\epsilon_{1,1} + 2 \epsilon_{1,2}} \lambda^3 \eta_{1,2}^2 \xi_{1,2} - e^{-\epsilon_{1,1} + \epsilon_{1,2} + \epsilon_{1,3}} \lambda^3 \eta_{1,2} \eta_{2,3} \xi_{1,3} - 2 e^{-\epsilon_{1,1} + 2 \epsilon_{1,2} + \epsilon_{1,3}} \lambda^4 \eta_{1,2} \eta_{2,3} \xi_{1,2} \xi_{1,3} - e^{-\epsilon_{1,1} + \epsilon_{1,2} + 2 \epsilon_{1,3}} \lambda^4 \eta_{2,3}^2 \xi_{1,3} \xi_{1,2} \xi_{1,3} - e^{-\epsilon_{1,1} + 2 \epsilon_{1,2} + 2 \epsilon_{1,3}} \lambda^5 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3}\}$$
- $$\gg \{3 e^{\epsilon_{1,2} + \epsilon_{2,2} - \epsilon_{2,3}} \in \lambda^2 \eta_{2,2} \eta_{2,3} \xi_{1,2} - 3 e^{\epsilon_{1,3} + \epsilon_{2,2} - \epsilon_{2,3}} \in \lambda^2 \eta_{2,3} \eta_{2,3} \xi_{1,3} + 4 e^{\epsilon_{1,2} + \epsilon_{1,3} + \epsilon_{2,2} - \epsilon_{2,3}} \in \lambda^3 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} - 3 e^{\epsilon_{1,3} + \epsilon_{2,2} - \epsilon_{2,3}} \in \lambda^2 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} + 4 e^{\epsilon_{1,2} + \epsilon_{1,3} + \epsilon_{2,2} - \epsilon_{2,3}} \in \lambda^3 \eta_{2,3} \eta_{2,3} \xi_{1,2} \xi_{1,3} - e^{\epsilon_{1,2} - \epsilon_{1,3} + \epsilon_{2,2} - \epsilon_{2,3}} \in \mathbf{g}_{2,3}'[\lambda] = \mathbf{0}, \mathbf{g}_{2,3}[\mathbf{0}] = \mathbf{0}\}$$
- $$\gg \{\mathbf{g}_{2,3}[\lambda] \rightarrow e^{\epsilon_{1,3}} \lambda^3 \eta_{2,2} \eta_{2,3} \xi_{1,2} - e^{-\epsilon_{1,2} + 2 \epsilon_{1,3}} \lambda^3 \eta_{2,3} \eta_{2,3} \xi_{1,3} + e^{2 \epsilon_{1,3}} \lambda^4 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} - e^{-\epsilon_{1,2} + 2 \epsilon_{1,3}} \lambda^3 \eta_{2,3}^2 \xi_{1,2} \xi_{1,3} + e^{2 \epsilon_{1,3}} \lambda^4 \eta_{2,3} \eta_{2,3} \xi_{1,2} \xi_{1,3}\}$$

In[*]:= **Sort@sol1**

Out[*]=

$$\begin{aligned}
& \{f_{1,1}[\lambda] \rightarrow e^{\xi_{1,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,3} + e^{\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{1,2}[\lambda] \rightarrow e^{-\xi_{1,2,2} + \xi_{1,3,3} - \xi_{2,1,1}} \lambda^2 \eta_{2,2,3} \xi_{1,1,3} - e^{\xi_{1,3,3} - \xi_{2,1,1}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} + \\
& \quad \lambda^3 \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,2} + e^{\xi_{1,3,3}} \lambda^4 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2}, \\
& f_{1,3}[\lambda] \rightarrow \lambda^3 \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,3}, \\
& f_{2,2}[\lambda] \rightarrow -e^{\xi_{1,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3,3}} \lambda^2 \eta_{2,2,3} \xi_{1,2,3} - e^{\xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\
& f_{2,3}[\lambda] \rightarrow -e^{-\xi_{2,2,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,3} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,2,3}, \\
& f_{3,3}[\lambda] \rightarrow -e^{\xi_{1,3,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,3} - e^{\xi_{1,3,3}} \lambda^2 \eta_{2,2,3} \xi_{1,2,3}, \\
& g_{1,2}[\lambda] \rightarrow -e^{-\xi_{1,1,1} + 2\xi_{1,2,2}} \lambda^3 \eta_{2,1,2}^2 \xi_{1,1,2} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,3} - 2e^{-\xi_{1,1,1} + 2\xi_{1,2,2} + \xi_{1,3,3}} \lambda^4 \eta_{2,1,2} \\
& \quad \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - e^{-\xi_{1,1,1} + 2\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^5 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\
& g_{1,3}[\lambda] \rightarrow e^{-\xi_{1,1,1} + \xi_{1,2,2}} \lambda^2 \eta_{1,2,3} \eta_{2,1,2} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} - e^{-\xi_{1,1,1} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,1,3}^2 \xi_{1,1,3} + \\
& \quad e^{-\xi_{1,1,1} + \xi_{1,2,2} + \xi_{1,3,3}} \lambda^3 \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - e^{-\xi_{1,1,1} + \xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\
& g_{2,3}[\lambda] \rightarrow e^{\xi_{1,3,3}} \lambda^3 \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + \\
& \quad e^{2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - e^{-\xi_{1,2,2} + 2\xi_{1,3,3}} \lambda^3 \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\xi_{1,3,3}} \lambda^4 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3}\}
\end{aligned}$$

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In[*]:= cm[i_, j_ -> k_] := E_{(i,j)->(k)} [CF@Sum[If[SSQ[b],
    b /. x_{\alpha,\alpha} -> (\xi_{1,\alpha,\alpha} + \xi_{2,\alpha,\alpha}) x_{\alpha,\alpha}[k],
    b[k] (b /. {x_{\alpha\beta} -> f_{\alpha\beta}[\lambda], y_{\alpha\beta} -> g_{\alpha\beta}[\lambda]} /. sol10)
  ], {b, $Basis}],
  CF@Sum[b[k] (b /. {x_{\alpha\beta} -> f_{\alpha\beta}[\lambda], y_{\alpha\beta} -> g_{\alpha\beta}[\lambda]} /. sol11), {b, $Basis}]
] /. {\xi_{1,\alpha\beta} -> \xi_{\alpha\beta}[i], \eta_{1,\alpha\beta} -> \eta_{\alpha\beta}[i], \xi_{2,\alpha\beta} -> \xi_{\alpha\beta}[j], \eta_{2,\alpha\beta} -> \eta_{\alpha\beta}[j], \lambda -> 1};
cm[i, j -> k]

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Out[*]=

$$\begin{aligned}
& \mathbb{E}_{(i,j) \rightarrow (k)} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \right. \\
& e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + \\
& x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + \\
& e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + \\
& x_{2,2}[k] \xi_{2,2}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - \\
& x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,3}[k] \eta_{1,2}[j] \eta_{2,3}[i] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - \\
& e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] + e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] + \\
& x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + \\
& e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + \\
& e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\
& e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + \\
& x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[i] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - \\
& e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + \\
& e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& 2 e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i] + \\
& e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
& e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - \\
& e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + \\
& x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j] \left. \right]
\end{aligned}$$

In[*]:= Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]

Out[*]//Short=

$$\begin{aligned} & \mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\epsilon_{1,1}[1] + \epsilon_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\ & e^{-\epsilon_{1,1}[1] - \epsilon_{1,1}[2] + \epsilon_{2,2}[1] + \epsilon_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + y_{1,3}[1] \eta_{1,3}[1] + e^{-\epsilon_{1,1}[1] + \epsilon_{3,3}[1]} y_{1,3}[1] \eta_{1,3}[2] + \\ & e^{-\epsilon_{1,1}[1] - \epsilon_{1,1}[2] + \epsilon_{3,3}[1] + \epsilon_{3,3}[2]} y_{1,3}[1] \eta_{1,3}[3] + y_{2,3}[1] \eta_{2,3}[1] + e^{-\epsilon_{2,2}[1] + \epsilon_{3,3}[1]} y_{2,3}[1] \eta_{2,3}[2] + \\ & e^{-\epsilon_{2,2}[1] - \epsilon_{2,2}[2] + \epsilon_{3,3}[1] + \epsilon_{3,3}[2]} y_{2,3}[1] \eta_{2,3}[3] + x_{1,1}[1] \xi_{1,1}[1] + x_{1,1}[1] \xi_{1,1}[2] + \\ & x_{1,1}[1] \xi_{1,1}[3] + e^{-\epsilon_{1,1}[2] - \epsilon_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] - e^{\epsilon_{3,3}[1]} y_{2,3}[1] \eta_{1,3}[2] \xi_{1,2}[1] - \\ & e^{-\epsilon_{1,1}[2] + \epsilon_{3,3}[1] + \epsilon_{3,3}[2]} y_{2,3}[1] \eta_{1,3}[3] \xi_{1,2}[1] + e^{-\epsilon_{1,1}[3] - \epsilon_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] - \\ & e^{-\epsilon_{2,2}[1] + \epsilon_{3,3}[1] + \epsilon_{3,3}[2]} y_{2,3}[1] \eta_{1,3}[3] \xi_{1,2}[2] + e^{-\epsilon_{\ll 1 \gg}[1] - \ll 1 \gg} x_{\ll 1 \gg}[1] \xi_{1,2}[3] + \ll 1 \gg + \\ & e^{-\epsilon_{1,1}[3] - \epsilon_{3,3}[1]} x_{1,3}[1] \xi_{1,3}[2] + e^{-\epsilon_{3,3}[1] - \epsilon_{3,3}[2]} x_{1,3}[1] \xi_{1,3}[3] + x_{2,2}[1] \xi_{2,2}[1] + x_{2,2}[1] \xi_{2,2}[2] + \\ & x_{2,2}[1] \xi_{2,2}[3] + e^{-\epsilon_{2,2}[2] - \epsilon_{2,2}[3]} x_{2,3}[1] \xi_{2,3}[1] + e^{-\epsilon_{1,1}[1] + \epsilon_{2,2}[1] + \epsilon_{3,3}[1]} y_{1,2}[1] \eta_{1,3}[2] \xi_{2,3}[1] + \\ & e^{-\epsilon_{1,1}[1] - \epsilon_{1,1}[2] + \epsilon_{2,2}[1] + \epsilon_{3,3}[1] + \epsilon_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[1] - e^{-\epsilon_{1,1}[3]} x_{1,3}[1] \xi_{1,2}[2] \xi_{2,3}[1] - \\ & e^{-\epsilon_{2,2}[2]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[1] + e^{-\epsilon_{2,2}[3] - \epsilon_{3,3}[1]} x_{2,3}[1] \xi_{2,3}[2] + \\ & e^{-\epsilon_{1,1}[1] - \epsilon_{1,1}[2] + \epsilon_{2,2}[1] + \epsilon_{2,2}[2] + \epsilon_{3,3}[2]} y_{1,2}[1] \eta_{1,3}[3] \xi_{2,3}[2] - e^{-\epsilon_{3,3}[1]} x_{1,3}[1] \xi_{1,2}[3] \xi_{2,3}[2] + \\ & \left. e^{-\epsilon_{3,3}[1] - \epsilon_{3,3}[2]} x_{2,3}[1] \xi_{2,3}[3] + x_{3,3}[1] \xi_{3,3}[1] + x_{3,3}[1] \xi_{3,3}[2] + x_{3,3}[1] \xi_{3,3}[3], \ll 1 \gg \right] \end{aligned}$$

In[*]:= Simplify[(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])]

Out[*]=

True

$$\begin{aligned} \text{In[*]:= osol} &= \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \right. \\ & e^{-\epsilon_{1,1}[i] + \epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + \\ & x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\epsilon_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + \\ & e^{-\epsilon_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\epsilon_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\epsilon_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + \\ & x_{2,2}[k] \xi_{2,2}[j] + e^{-\epsilon_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - \\ & x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\epsilon_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], \\ & - e^{-\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\epsilon_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - \\ & e^{\epsilon_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\epsilon_{1,1}[i] + 2 \epsilon_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + \\ & e^{\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\ & e^{-\epsilon_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - \\ & e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\epsilon_{1,1}[i] + 2 \epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + \\ & e^{-\epsilon_{1,1}[j] - \epsilon_{2,2}[i] + \epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\epsilon_{2,2}[i] + 2 \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\ & e^{-\epsilon_{1,1}[j] + \epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2 \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + \\ & x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - \\ & e^{-\epsilon_{1,1}[i] + 2 \epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\epsilon_{2,2}[i] + 2 \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + \\ & e^{\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & 2 e^{-\epsilon_{1,1}[i] + 2 \epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\ & e^{2 \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{-\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - \\ & e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + 2 \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - \\ & e^{-\epsilon_{1,1}[i] + 2 \epsilon_{2,2}[i] + 2 \epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 + \\ & x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j], \\ & \left. - \frac{1}{2} e^{2 \epsilon_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \epsilon_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \right. \end{aligned}$$

$$\begin{aligned}
 & e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \epsilon_{1,2}[i]^2 + e^{\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - \\
 & e^{\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i] + \\
 & e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] - e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] + e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] + \\
 & 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] - e^{2\epsilon_{3,3}[i]} y_{1,3}[k] \\
 & \eta_{1,2}[i] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] + e^{-\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,2}[j] \epsilon_{1,3}[i] - \\
 & e^{\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i] \epsilon_{1,2}[j] \epsilon_{1,3}[i] - \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + \\
 & \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,3}[i]^2 + \\
 & e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \epsilon_{1,3}[i]^2 - e^{-\epsilon_{1,1}[j]-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i]^2 + \\
 & e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i]^2 + \\
 & e^{-\epsilon_{1,1}[j]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i]^2 - e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \epsilon_{1,2}[i] \epsilon_{1,3}[i]^2 + \\
 & e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \epsilon_{2,3}[i] + e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - e^{\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - \\
 & 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - \\
 & e^{2\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i] - e^{2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] + e^{2\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] + \\
 & 3 e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]+\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i] - \\
 & e^{2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
 & 2 e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & 2 e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
 & e^{-\epsilon_{1,1}[j]-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & 2 e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{1,3}[i] \epsilon_{2,3}[i] - e^{\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\
 & \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + e^{\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & 4 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & e^{-\epsilon_{1,1}[j]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
 & 2 e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] - \\
 & e^{2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{1,2}[j] \epsilon_{1,3}[i] \epsilon_{2,3}[i] + \\
 & e^{-\epsilon_{1,1}[i]+\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \epsilon_{1,3}[i]^2 \epsilon_{2,3}[i] - \\
 & \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + \frac{1}{2} e^{2\epsilon_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + \\
 & e^{-\epsilon_{1,1}[i]+3\epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{2,3}[i]^2 + e^{-\epsilon_{2,2}[i]+3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \epsilon_{2,3}[i]^2 - \\
 & 2 e^{-\epsilon_{1,1}[i]+2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i] \epsilon_{2,3}[i]^2 - \\
 & e^{3\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \epsilon_{1,2}[i] \epsilon_{2,3}[i]^2 - \frac{1}{2} e^{2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,1}[k] \\
 & \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \frac{1}{2} e^{2\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \\
 & 3 e^{-\epsilon_{1,1}[i]+3\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \epsilon_{1,2}[i]^2 \epsilon_{2,3}[i]^2 + \\
 & e^{-\epsilon_{2,2}[i]+2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \epsilon_{1,2}[j] \epsilon_{2,3}[i]^2 - e^{2\epsilon_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \epsilon_{1,2}[i]
 \end{aligned}$$

$$\xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 + e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3];$$

osol = (List@@osol).Table[e^k, {k, 0, Length[osol] - 1}]

Out[*]=

$$\begin{aligned} & y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + \\ & y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + \\ & e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + \\ & e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j] + \\ & e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + \\ & \epsilon^2 \left(-\frac{1}{2} e^{2\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \right. \\ & e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\ & e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] + \\ & e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\ & \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + \\ & 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - e^{2\xi_{3,3}[i]} y_{1,3}[k] \\ & \eta_{1,2}[i] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] - \\ & e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] - \frac{1}{2} e^{2\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\ & \frac{1}{2} e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\ & e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\ & e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\ & e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i]^2 - e^{3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i]^2 + \\ & e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\ & \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{2\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\ & \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + e^{2\xi_{2,2}[i]+3\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + \\ & 3 e^{-\xi_{1,1}[i]+3\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] - \\ & e^{2\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\ & 2 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\ & 2 e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + \\ & 2 e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\ & \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\ & 4 e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[j]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] - \\ & 2 e^{3\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\ & e^{-\xi_{2,2}[i]+2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] - e^{2\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \\ & \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{-\xi_{1,1}[i]+2\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 \xi_{2,3}[i] - \\ & \frac{1}{2} e^{2\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \\ & e^{-\xi_{1,1}[i]+3\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + e^{-\xi_{2,2}[i]+3\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \xi_{2,3}[i]^2 - \end{aligned}$$

$$\begin{aligned}
 & 2 e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 - \\
 & e^{3 \epsilon_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 - \frac{1}{2} e^{2 \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \\
 & \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \frac{1}{2} e^{2 \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \\
 & 3 e^{-\epsilon_{1,1}[\mathbf{i}] + 3 \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^2 + \\
 & e^{-\epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}]^2 - e^{2 \epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \\
 & \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}]^2 + 2 e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{2,2}[\mathbf{i}] + 3 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^3 \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 + \\
 & e^{-\epsilon_{1,1}[\mathbf{i}] + 3 \epsilon_{2,2}[\mathbf{i}] + 3 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^3 \xi_{1,2}[\mathbf{i}]^2 \xi_{2,3}[\mathbf{i}]^3 \Big) + e^{-\epsilon_{3,3}[\mathbf{i}]} x_{2,3}[\mathbf{k}] \xi_{2,3}[\mathbf{j}] + \\
 \in & \left(-e^{-\epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{i}] \eta_{2,3}[\mathbf{j}] + e^{\epsilon_{2,2}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] - \right. \\
 & e^{\epsilon_{2,2}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] - e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{2,2}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] + \\
 & e^{\epsilon_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{i}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] + x_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{2,2}[\mathbf{j}]} x_{2,3}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] + e^{\epsilon_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{\epsilon_{3,3}[\mathbf{i}]} x_{3,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{1,1}[\mathbf{i}] + \epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,3}[\mathbf{i}] + \\
 & e^{-\epsilon_{1,1}[\mathbf{j}] - \epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - e^{-\epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{1,1}[\mathbf{j}] + \epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] + e^{2 \epsilon_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{1,3}[\mathbf{i}] + \\
 & x_{1,3}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \xi_{1,2}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] + x_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] \xi_{1,3}[\mathbf{j}] + e^{\epsilon_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
 & e^{\epsilon_{3,3}[\mathbf{i}]} x_{3,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] + e^{-\epsilon_{1,1}[\mathbf{i}] + \epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - e^{-\epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}]^2 \xi_{2,3}[\mathbf{i}] + \\
 & e^{\epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} x_{1,1}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - e^{\epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} x_{2,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - \\
 & 2 e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,2}[\mathbf{j}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] + \\
 & e^{2 \epsilon_{3,3}[\mathbf{i}]} y_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{2,2}[\mathbf{i}] + \epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] + e^{\epsilon_{3,3}[\mathbf{i}]} x_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,2}[\mathbf{i}] \xi_{1,2}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{1,1}[\mathbf{i}] + \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{i}] + x_{1,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{1,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] - \\
 & e^{-\epsilon_{1,1}[\mathbf{i}] + 2 \epsilon_{2,2}[\mathbf{i}] + 2 \epsilon_{3,3}[\mathbf{i}]} y_{1,2}[\mathbf{k}] \eta_{1,3}[\mathbf{j}]^2 \xi_{1,2}[\mathbf{i}] \xi_{2,3}[\mathbf{i}]^2 + x_{2,3}[\mathbf{k}] \eta_{1,3}[\mathbf{j}] \xi_{1,3}[\mathbf{i}] \xi_{2,3}[\mathbf{j}] + \\
 & \left. x_{2,3}[\mathbf{k}] \eta_{2,3}[\mathbf{j}] \xi_{2,3}[\mathbf{i}] \xi_{2,3}[\mathbf{j}] \right) + x_{3,3}[\mathbf{k}] \xi_{3,3}[\mathbf{i}] + x_{3,3}[\mathbf{k}] \xi_{3,3}[\mathbf{j}]
 \end{aligned}$$

In[*]:= sol0

Out[*]=

$$\begin{aligned}
 & \{ \mathbf{f}_{1,2}[\lambda] \rightarrow e^{-\epsilon_{2,1,1}} \lambda \xi_{1,2} + e^{-\epsilon_{1,2,2}} \lambda \xi_{2,1,2}, \mathbf{g}_{2,3}[\lambda] \rightarrow \lambda (\eta_{1,2,3} + e^{\epsilon_{1,3,3}} (e^{-\epsilon_{1,2,2}} \eta_{2,2,3} - \lambda \eta_{2,1,3} \xi_{1,1,2})), \\
 & \mathbf{f}_{1,3}[\lambda] \rightarrow \lambda (e^{-\epsilon_{2,1,1}} \xi_{1,1,3} - \lambda \xi_{1,2,3} \xi_{2,1,2} + e^{-\epsilon_{1,3,3}} \xi_{2,1,3}), \mathbf{f}_{2,3}[\lambda] \rightarrow e^{-\epsilon_{2,2,2}} \lambda \xi_{1,2,3} + e^{-\epsilon_{1,3,3}} \lambda \xi_{2,2,3}, \\
 & \mathbf{g}_{1,2}[\lambda] \rightarrow \lambda (\eta_{1,1,2} + e^{-\epsilon_{1,1,1} + \epsilon_{1,2,2}} (\eta_{2,1,2} + e^{\epsilon_{1,3,3}} \lambda \eta_{2,1,3} \xi_{1,2,3})), \mathbf{g}_{1,3}[\lambda] \rightarrow \lambda (\eta_{1,1,3} + e^{-\epsilon_{1,1,1} + \epsilon_{1,3,3}} \eta_{2,1,3}) \}
 \end{aligned}$$

In[*]:= **CF[sol1 /. λ → 1]**

Out[*]=

$$\begin{aligned} & \{f_{1,1}[1] \rightarrow e^{\xi_{1,2}} \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3}} \eta_{2,1,3} \xi_{1,1,3} + e^{\xi_{1,2} + \xi_{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\ & f_{2,2}[1] \rightarrow -e^{\xi_{1,2}} \eta_{2,1,2} \xi_{1,1,2} + e^{\xi_{1,3}} \eta_{2,2,3} \xi_{1,2,3} - e^{\xi_{1,2} + \xi_{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3}, \\ & f_{3,3}[1] \rightarrow -e^{\xi_{1,3}} \eta_{2,1,3} \xi_{1,1,3} - e^{\xi_{1,3}} \eta_{2,2,3} \xi_{1,2,3}, \\ & f_{1,2}[1] \rightarrow e^{-\xi_{1,2} + \xi_{1,3} - \xi_{2,1}} \eta_{2,2,3} \xi_{1,1,3} - e^{\xi_{1,3} - \xi_{2,1}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} + \\ & \quad \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\xi_{1,2} + \xi_{1,3}} \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,2} + e^{\xi_{1,3}} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2}, \\ & f_{2,3}[1] \rightarrow -e^{-\xi_{2,2}} \eta_{2,1,2} \xi_{1,1,3} + \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + \eta_{2,2,3} \xi_{1,2,3} \xi_{2,2,3}, \\ & f_{1,3}[1] \rightarrow \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} + \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,3}, \\ & g_{1,3}[1] \rightarrow e^{-\xi_{1,1} + \xi_{1,2}} \eta_{1,2,3} \eta_{2,1,2} - e^{-\xi_{1,1} + \xi_{1,2} + \xi_{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} - e^{-\xi_{1,1} + 2\xi_{1,3}} \eta_{2,1,3}^2 \xi_{1,1,3} + \\ & \quad e^{-\xi_{1,1} + \xi_{1,2} + \xi_{1,3}} \eta_{1,2,3} \eta_{2,1,3} \xi_{1,2,3} - e^{-\xi_{1,1} + \xi_{1,2} + 2\xi_{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \quad g_{1,2}[1] \rightarrow \\ & \quad -e^{-\xi_{1,1} + 2\xi_{1,2}} \eta_{1,2}^2 \xi_{1,1,2} - e^{-\xi_{1,1} + \xi_{1,2} + \xi_{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,3} - 2e^{-\xi_{1,1} + 2\xi_{1,2} + \xi_{1,3}} \eta_{2,1,2} \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} - \\ & \quad e^{-\xi_{1,1} + \xi_{1,2} + 2\xi_{1,3}} \eta_{2,1,3}^2 \xi_{1,1,3} \xi_{1,2,3} - e^{-\xi_{1,1} + 2\xi_{1,2} + 2\xi_{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,2,3}, \\ & g_{2,3}[1] \rightarrow e^{\xi_{1,3}} \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\xi_{1,2} + 2\xi_{1,3}} \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} + e^{2\xi_{1,3}} \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - \\ & \quad e^{-\xi_{1,2} + 2\xi_{1,3}} \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\xi_{1,3}} \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} \} \end{aligned}$$

In[*]:= **Coefficient[osol, x_{1,1}[k] ε]**

Out[*]=

$$e^{\xi_{2,2}[i]} \eta_{1,2}[j] \xi_{1,2}[i] + e^{\xi_{3,3}[i]} \eta_{1,3}[j] \xi_{1,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i]$$

In[*]:= **CF[(Expand@sol[\$k].\$Basis /. {ξ_{1_{αβ}} → ξ_{αβ}[i], η_{1_{αβ}} → η_{αβ}[i], ξ_{2_{αβ}} → ξ_{αβ}[j], η_{2_{αβ}} → η_{αβ}[j], λ → 1}) - (osol /. v_[k] → v)]**

Out[*]=

$$\begin{aligned} & \text{sol}[1] \cdot \{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\} - y_{1,2} \eta_{1,2}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2} \eta_{1,2}[j] - y_{1,3} \eta_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3} \eta_{1,3}[j] - y_{2,3} \eta_{2,3}[i] - \\ & e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3} \eta_{2,3}[j] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} \in y_{1,3} \eta_{1,2}[i] \eta_{2,3}[j] - x_{1,1} \xi_{1,1}[i] - x_{1,1} \xi_{1,1}[j] - \\ & e^{-\xi_{1,1}[j]} x_{1,2} \xi_{1,2}[i] - e^{\xi_{2,2}[i]} \in x_{1,1} \eta_{1,2}[j] \xi_{1,2}[i] + e^{\xi_{2,2}[i]} \in x_{2,2} \eta_{1,2}[j] \xi_{1,2}[i] + \\ & e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i]} \in y_{1,2} \eta_{1,2}[j]^2 \xi_{1,2}[i] + e^{\xi_{3,3}[i]} y_{2,3} \eta_{1,3}[j] \xi_{1,2}[i] - \\ & e^{\xi_{3,3}[i]} \in y_{1,3} \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] - e^{-\xi_{2,2}[i]} x_{1,2} \xi_{1,2}[j] - \in x_{1,2} \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - \\ & e^{-\xi_{1,1}[j]} x_{1,3} \xi_{1,3}[i] + e^{-\xi_{2,2}[j]} \in x_{2,3} \eta_{1,2}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} \in x_{1,1} \eta_{1,3}[j] \xi_{1,3}[i] + \\ & e^{\xi_{3,3}[i]} \in x_{3,3} \eta_{1,3}[j] \xi_{1,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] + \\ & e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} \in y_{1,3} \eta_{1,3}[j]^2 \xi_{1,3}[i] - e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} \in x_{1,2} \eta_{2,3}[j] \xi_{1,3}[i] + \\ & e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] + e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} \in x_{1,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - \\ & e^{2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] - \in x_{1,3} \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] - e^{-\xi_{3,3}[i]} x_{1,3} \xi_{1,3}[j] - \\ & \in x_{1,3} \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] - x_{2,2} \xi_{2,2}[i] - x_{2,2} \xi_{2,2}[j] - e^{-\xi_{2,2}[j]} x_{2,3} \xi_{2,3}[i] - \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2} \eta_{1,3}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} \in x_{2,2} \eta_{2,3}[j] \xi_{2,3}[i] + \\ & e^{\xi_{3,3}[i]} \in x_{3,3} \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} \in y_{1,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} \in y_{2,3} \eta_{2,3}[j]^2 \xi_{2,3}[i] - \\ & e^{\xi_{2,2}[i] + \xi_{3,3}[i]} \in x_{1,1} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} \in x_{2,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + \\ & 2e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + \xi_{3,3}[i]} \in y_{1,2} \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\ & e^{2\xi_{3,3}[i]} \in y_{2,3} \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + x_{1,3} \xi_{1,2}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} \in x_{1,2} \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} \in x_{1,2} \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} \in y_{1,2} \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - \in x_{1,3} \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] + \\ & e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + 2\xi_{3,3}[i]} \in y_{1,2} \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 - e^{-\xi_{3,3}[i]} x_{2,3} \xi_{2,3}[j] - \\ & \in x_{2,3} \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] - \in x_{2,3} \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j] - x_{3,3} \xi_{3,3}[i] - x_{3,3} \xi_{3,3}[j] \} \end{aligned}$$

gl_{4,ε} Experiments

```

In[*]:= SetAlgebra[gl4,ε];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
{b1, $Basis}, {b2, $Basis}]
DeleteCases[Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]],
{a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]

Out[*]=
1

Out[*]=
{y1,2 → 1, y1,3 → 2, y2,3 → 3, y1,4 → 4, y2,4 → 5, y3,4 → 6, x1,1 → 7, x1,2 → 8,
x2,2 → 9, x1,3 → 10, x2,3 → 11, x3,3 → 12, x1,4 → 13, x2,4 → 14, x3,4 → 15, x4,4 → 16}

Out[*]//MatrixForm=
{
  {y1,2, y1,2} → 0,      {y1,2, y1,3} → 0,      {y1,2, y2,3} → -ε y1,3,      {y1,2, y1,4}
  {y1,3, y1,2} → 0,      {y1,3, y1,3} → 0,      {y1,3, y2,3} → 0,      {y1,3, y1,4}
  {y2,3, y1,2} → ε y1,3,  {y2,3, y1,3} → 0,      {y2,3, y2,3} → 0,      {y2,3, y1,4}
  {y1,4, y1,2} → 0,      {y1,4, y1,3} → 0,      {y1,4, y2,3} → 0,      {y1,4, y1,4}
  {y2,4, y1,2} → ε y1,4,  {y2,4, y1,3} → 0,      {y2,4, y2,3} → 0,      {y2,4, y1,4}
  {y3,4, y1,2} → 0,      {y3,4, y1,3} → ε y1,4,  {y3,4, y2,3} → ε y2,4,  {y3,4, y1,4}
  {x1,1, y1,2} → -y1,2,   {x1,1, y1,3} → -y1,3,   {x1,1, y2,3} → 0,      {x1,1, y1,4}
  {x1,2, y1,2} → ε x1,1 - ε x2,2, {x1,2, y1,3} → -y2,3,   {x1,2, y2,3} → 0,      {x1,2, y1,4}
  {x2,2, y1,2} → y1,2,    {x2,2, y1,3} → 0,      {x2,2, y2,3} → -y2,3,   {x2,2, y1,4}
  {x1,3, y1,2} → -ε x2,3,  {x1,3, y1,3} → ε x1,1 - ε x3,3, {x1,3, y2,3} → ε x1,2,   {x1,3, y1,4}
  {x2,3, y1,2} → 0,      {x2,3, y1,3} → y1,2,    {x2,3, y2,3} → ε x2,2 - ε x3,3, {x2,3, y1,4}
  {x3,3, y1,2} → 0,      {x3,3, y1,3} → y1,3,   {x3,3, y2,3} → y2,3,    {x3,3, y1,4}
  {x1,4, y1,2} → -ε x2,4,  {x1,4, y1,3} → -ε x3,4,   {x1,4, y2,3} → 0,      {x1,4, y1,4} → ε
  {x2,4, y1,2} → 0,      {x2,4, y1,3} → 0,      {x2,4, y2,3} → -ε x3,4,   {x2,4, y1,4}
  {x3,4, y1,2} → 0,      {x3,4, y1,3} → 0,      {x3,4, y2,3} → 0,      {x3,4, y1,4}
  {x4,4, y1,2} → 0,      {x4,4, y1,3} → 0,      {x4,4, y2,3} → 0,      {x4,4, y1,4}
}

```

Out[]//MatrixForm=

```
{
  {y1,2, y1,2} -> True {y1,2, y1,3} -> True {y1,2, y2,3} -> True {y1,2, y1,4} -> True {y1,2, y2,4} -> True {
  {y1,3, y1,2} -> True {y1,3, y1,3} -> True {y1,3, y2,3} -> True {y1,3, y1,4} -> True {y1,3, y2,4} -> True {
  {y2,3, y1,2} -> True {y2,3, y1,3} -> True {y2,3, y2,3} -> True {y2,3, y1,4} -> True {y2,3, y2,4} -> True {
  {y1,4, y1,2} -> True {y1,4, y1,3} -> True {y1,4, y2,3} -> True {y1,4, y1,4} -> True {y1,4, y2,4} -> True {
  {y2,4, y1,2} -> True {y2,4, y1,3} -> True {y2,4, y2,3} -> True {y2,4, y1,4} -> True {y2,4, y2,4} -> True {
  {y3,4, y1,2} -> True {y3,4, y1,3} -> True {y3,4, y2,3} -> True {y3,4, y1,4} -> True {y3,4, y2,4} -> True {
  {x1,1, y1,2} -> True {x1,1, y1,3} -> True {x1,1, y2,3} -> True {x1,1, y1,4} -> True {x1,1, y2,4} -> True {
  {x1,2, y1,2} -> True {x1,2, y1,3} -> True {x1,2, y2,3} -> True {x1,2, y1,4} -> True {x1,2, y2,4} -> True {
  {x2,2, y1,2} -> True {x2,2, y1,3} -> True {x2,2, y2,3} -> True {x2,2, y1,4} -> True {x2,2, y2,4} -> True {
  {x1,3, y1,2} -> True {x1,3, y1,3} -> True {x1,3, y2,3} -> True {x1,3, y1,4} -> True {x1,3, y2,4} -> True {
  {x2,3, y1,2} -> True {x2,3, y1,3} -> True {x2,3, y2,3} -> True {x2,3, y1,4} -> True {x2,3, y2,4} -> True {
  {x3,3, y1,2} -> True {x3,3, y1,3} -> True {x3,3, y2,3} -> True {x3,3, y1,4} -> True {x3,3, y2,4} -> True {
  {x1,4, y1,2} -> True {x1,4, y1,3} -> True {x1,4, y2,3} -> True {x1,4, y1,4} -> True {x1,4, y2,4} -> True {
  {x2,4, y1,2} -> True {x2,4, y1,3} -> True {x2,4, y2,3} -> True {x2,4, y1,4} -> True {x2,4, y2,4} -> True {
  {x3,4, y1,2} -> True {x3,4, y1,3} -> True {x3,4, y2,3} -> True {x3,4, y1,4} -> True {x3,4, y2,4} -> True {
  {x4,4, y1,2} -> True {x4,4, y1,3} -> True {x4,4, y2,3} -> True {x4,4, y1,4} -> True {x4,4, y2,4} -> True {
```

Out[]=

```
{ }
```

```

In[*]:= {sol0} = Block[{k = 0},
  lhs = λTangent[Join[
    Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
    Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
  ]];
  rhs = λTangent[
    Table[If[SSQ[b], b /. xα,α := (ξ1α,α + ξ2α,α + ε fα,α[λ]) xα,α,
      (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b], {b, $Basis}]
  ];
  Simplify@DSolve[
    And@@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      Select[$Basis, NilQ] /. {xαβ := fαβ[0] == 0, yαβ := gαβ[0] == 0}
    ],
    Select[$Basis, NilQ] /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
    λ
  ]
]

Out[*]= {
  {f1,2[λ] → e-ε2,1 λ ξ1,2 + e-ε1,2 λ ξ2,2,
  f2,3[λ] → e-ε2,2 λ ξ1,3 + e-ε1,3 λ ξ2,3, f1,3[λ] → λ (e-ε2,1 ξ1,3 - λ ξ1,3 ξ2,2 + e-ε1,3 ξ2,3),
  f3,4[λ] → e-ε2,3 λ ξ1,4 + e-ε1,4 λ ξ2,4, f2,4[λ] → λ (e-ε2,2 ξ1,4 - λ ξ1,4 ξ2,3 + e-ε1,4 ξ2,4),
  f1,4[λ] → λ (e-ε2,1 ξ1,4 - λ ξ1,4 ξ2,2 - λ ξ1,3 ξ2,3 + e-ε1,4 ξ2,4), g1,2[λ] →
  λ (η1,2 + e-ε1,1+ε1,2 (η2,2 + eε1,3 λ η2,3 ξ1,2 + eε1,4 λ η2,4 (ξ1,2 + eε1,3 λ ξ1,3 ξ1,4))),
  g1,3[λ] → λ (η1,3 + e-ε1,1+ε1,3 (η2,3 + eε1,4 λ η2,4 ξ1,3)), g2,3[λ] →
  λ (η1,2 + e-ε1,2+ε1,3 (η2,3 - eε1,2 λ η2,3 ξ1,2 + eε1,4 λ (η2,4 - eε1,2 λ η2,4 ξ1,2) ξ1,3)),
  g2,4[λ] → λ (η1,4 + eε1,4 (e-ε1,2 η2,4 - λ η2,4 ξ1,2)),
  g3,4[λ] → λ (η1,3 + e-ε1,3+ε1,4 η2,4 - eε1,4 λ (η2,4 ξ1,3 + η2,4 ξ1,2)),
  g1,4[λ] → λ (η1,4 + e-ε1,1+ε1,4 η2,4) }
}

```

```

In[*]:= $k = 1; SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
lhs = λTangent[Join[
  Table[If[SSQ[b1], (b1 /. x → ξ1) b1, λ (b1 /. {x → ξ1, y → η1}) b1], {b1, $Basis}],
  Table[If[SSQ[b2], (b2 /. x → ξ2) b2, λ (b2 /. {x → ξ2, y → η2}) b2], {b2, $Basis}]
]];
rhs = λTangent[
  Echo@Table[If[SSQ[b],
    b /. xα,α ⇒ (ξ1α,α + ξ2α,α + e fα,α[λ]) xα,α,
    (b /. {xαβ → fαβ[λ], yαβ → gαβ[λ]}) b /. (sol0 /. ((v- → s-) ⇒ (v → e v + s)))
  ], {b, $Basis}]
];
eqns = Transpose@{
  (Coefficient[CF[lhs - rhs], #] == 0) & /@SBasis,
  SBasis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
};
unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
(*{sol1}=DSolve[Flatten@eqns, λ]*)
sol1 = TriangularDSolve[eqns, unknowns, λ];

```

$$\begin{aligned}
& \gg \{y_{1,2} (\lambda (\eta_{1,2} + e^{-\xi_{1,1} + \xi_{1,2}} (\eta_{2,2} + e^{\xi_{1,3}} \lambda \eta_{2,3} \xi_{1,2,3} + e^{\xi_{1,4}} \lambda \eta_{2,4} (\xi_{1,2,4} + e^{\xi_{1,3}} \lambda \xi_{1,2,3} \xi_{1,3,4}))) + e g_{1,2}[\lambda]), \\
& y_{1,3} (\lambda (\eta_{1,3} + e^{-\xi_{1,1} + \xi_{1,3}} (\eta_{2,3} + e^{\xi_{1,4}} \lambda \eta_{2,4} \xi_{1,3,4})) + e g_{1,3}[\lambda]), \\
& y_{2,3} (\lambda (\eta_{2,3} + e^{-\xi_{1,2} + \xi_{1,3}} (\eta_{2,3} - e^{\xi_{1,2}} \lambda \eta_{2,3} \xi_{1,1,2} + e^{\xi_{1,4}} \lambda (\eta_{2,4} - e^{\xi_{1,2}} \lambda \eta_{2,4} \xi_{1,1,2}) \xi_{1,3,4})) + e g_{2,3}[\lambda]), \\
& y_{1,4} (\lambda (\eta_{1,4} + e^{-\xi_{1,1} + \xi_{1,4}} \eta_{2,4}) + e g_{1,4}[\lambda]), \\
& y_{2,4} (\lambda (\eta_{2,4} + e^{\xi_{1,4}} (e^{-\xi_{1,2}} \eta_{2,4} - \lambda \eta_{2,4} \xi_{1,1,2})) + e g_{2,4}[\lambda]), \\
& y_{3,4} (\lambda (\eta_{3,4} + e^{-\xi_{1,3} + \xi_{1,4}} \eta_{2,4} - e^{\xi_{1,4}} \lambda (\eta_{2,4} \xi_{1,1,3} + \eta_{2,4} \xi_{1,2,3})) + e g_{3,4}[\lambda]), \\
& x_{1,1} (\xi_{1,1} + \xi_{2,1} + e f_{1,1}[\lambda]), x_{1,2} (e^{-\xi_{2,1}} \lambda \xi_{1,2} + e^{-\xi_{1,2}} \lambda \xi_{2,2} + e f_{1,2}[\lambda]), \\
& x_{2,2} (\xi_{1,2} + \xi_{2,2} + e f_{2,2}[\lambda]), x_{1,3} (\lambda (e^{-\xi_{2,1}} \xi_{1,3} - \lambda \xi_{1,2,3} \xi_{2,2} + e^{-\xi_{1,3}} \xi_{2,3}) + e f_{1,3}[\lambda]), \\
& x_{2,3} (e^{-\xi_{2,2}} \lambda \xi_{1,2,3} + e^{-\xi_{1,3}} \lambda \xi_{2,3} + e f_{2,3}[\lambda]), x_{3,3} (\xi_{1,3} + \xi_{2,3} + e f_{3,3}[\lambda]), \\
& x_{1,4} (\lambda (e^{-\xi_{2,1}} \xi_{1,4} - \lambda \xi_{1,2,4} \xi_{2,2} - \lambda \xi_{1,3,4} \xi_{2,3} + e^{-\xi_{1,4}} \xi_{2,4}) + e f_{1,4}[\lambda]), \\
& x_{2,4} (\lambda (e^{-\xi_{2,2}} \xi_{1,2,4} - \lambda \xi_{1,3,4} \xi_{2,3} + e^{-\xi_{1,4}} \xi_{2,4}) + e f_{2,4}[\lambda]), \\
& x_{3,4} (e^{-\xi_{2,3}} \lambda \xi_{1,3,4} + e^{-\xi_{1,4}} \lambda \xi_{2,3,4} + e f_{3,4}[\lambda]), x_{4,4} (\xi_{1,4} + \xi_{2,4} + e f_{4,4}[\lambda])\} \\
& \gg \{2 e^{\xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} + 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} + 2 e^{\xi_{1,4}} \in \lambda \eta_{2,4} \xi_{1,4} + \\
& 3 e^{\xi_{1,2} + \xi_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{1,2,3} + 3 e^{\xi_{1,2} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,2} \xi_{1,2,4} + 3 e^{\xi_{1,3} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,3} \xi_{1,3,4} + \\
& 4 e^{\xi_{1,2} + \xi_{1,3} + \xi_{1,4}} \in \lambda^3 \eta_{2,4} \xi_{1,2} \xi_{1,2,3} \xi_{1,3,4} - e f_{1,1}'[\lambda] = 0, f_{1,1}[0] = 0\} \\
& \gg \{f_{1,1}[\lambda] \rightarrow e^{\xi_{1,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} + e^{\xi_{1,4}} \lambda^2 \eta_{2,4} \xi_{1,4} + e^{\xi_{1,2} + \xi_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,2,3} + \\
& e^{\xi_{1,2} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,2} \xi_{1,2,4} + e^{\xi_{1,3} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,3} \xi_{1,3,4} + e^{\xi_{1,2} + \xi_{1,3} + \xi_{1,4}} \lambda^4 \eta_{2,4} \xi_{1,2} \xi_{1,2,3} \xi_{1,3,4}\} \\
& \gg \{-2 e^{\xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,2} + 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} - 3 e^{\xi_{1,2} + \xi_{1,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2} \xi_{1,2,3} + \\
& 2 e^{\xi_{1,4}} \in \lambda \eta_{2,4} \xi_{1,2,4} - 3 e^{\xi_{1,2} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,2} \xi_{1,2,4} + 3 e^{\xi_{1,3} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,2,3} \xi_{1,3,4} - \\
& 4 e^{\xi_{1,2} + \xi_{1,3} + \xi_{1,4}} \in \lambda^3 \eta_{2,4} \xi_{1,2} \xi_{1,2,3} \xi_{1,3,4} - e f_{2,2}'[\lambda] = 0, f_{2,2}[0] = 0\} \\
& \gg \{f_{2,2}[\lambda] \rightarrow -e^{\xi_{1,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} - e^{\xi_{1,2} + \xi_{1,3}} \lambda^3 \eta_{2,3} \xi_{1,2} \xi_{1,2,3} + e^{\xi_{1,4}} \lambda^2 \eta_{2,4} \xi_{1,2,4} - \\
& e^{\xi_{1,2} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,2} \xi_{1,2,4} + e^{\xi_{1,3} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,2,3} \xi_{1,3,4} - e^{\xi_{1,2} + \xi_{1,3} + \xi_{1,4}} \lambda^4 \eta_{2,4} \xi_{1,2} \xi_{1,2,3} \xi_{1,3,4}\} \\
& \gg \{-2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,3} - 2 e^{\xi_{1,3}} \in \lambda \eta_{2,3} \xi_{1,2,3} + 2 e^{\xi_{1,4}} \in \lambda \eta_{2,4} \xi_{1,3,4} - \\
& 3 e^{\xi_{1,3} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,3} \xi_{1,3,4} - 3 e^{\xi_{1,3} + \xi_{1,4}} \in \lambda^2 \eta_{2,4} \xi_{1,2,3} \xi_{1,3,4} - e f_{3,3}'[\lambda] = 0, f_{3,3}[0] = 0\} \\
& \gg \{f_{3,3}[\lambda] \rightarrow -e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,3} - e^{\xi_{1,3}} \lambda^2 \eta_{2,3} \xi_{1,2,3} + \\
& e^{\xi_{1,4}} \lambda^2 \eta_{2,4} \xi_{1,3,4} - e^{\xi_{1,3} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,3} \xi_{1,3,4} - e^{\xi_{1,3} + \xi_{1,4}} \lambda^3 \eta_{2,4} \xi_{1,2,3} \xi_{1,3,4}\}
\end{aligned}$$

- » $\{-2 e^{\epsilon_{14,4}} \in \lambda \eta_{2,1,4} \xi_{1,1,4} - 2 e^{\epsilon_{14,4}} \in \lambda \eta_{2,2,4} \xi_{1,2,4} - 2 e^{\epsilon_{14,4}} \in \lambda \eta_{2,3,4} \xi_{1,3,4} - \in f_{4,4}'[\lambda] = 0, f_{4,4}[0] = 0\}$
- » $\{f_{4,4}[\lambda] \rightarrow -e^{\epsilon_{14,4}} \lambda^2 \eta_{2,1,4} \xi_{1,1,4} - e^{\epsilon_{14,4}} \lambda^2 \eta_{2,2,4} \xi_{1,2,4} - e^{\epsilon_{14,4}} \lambda^2 \eta_{2,3,4} \xi_{1,3,4}\}$
- » $\{2 e^{\epsilon_{13,3}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda \eta_{2,2,3} \xi_{1,1,3} - 3 e^{\epsilon_{12,2}+\epsilon_{13,3}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} +$
 $2 e^{\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda \eta_{2,2,4} \xi_{1,1,4} - 3 e^{\epsilon_{12,2}+\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,4} +$
 $3 e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4} - 4 e^{\epsilon_{12,2}+\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{22,2}} \in \lambda^3 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4} +$
 $3 e^{\epsilon_{12,2}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - 3 e^{\epsilon_{13,3}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,2} +$
 $4 e^{\epsilon_{12,2}+\epsilon_{13,3}+\epsilon_{22,2}} \in \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2} - 3 e^{\epsilon_{14,4}+\epsilon_{22,2}} \in \lambda^2 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,1,2} +$
 $4 e^{\epsilon_{12,2}+\epsilon_{14,4}+\epsilon_{22,2}} \in \lambda^3 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,2,4} \xi_{2,1,2} - 4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{22,2}} \in \lambda^3 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,2} +$
 $5 e^{\epsilon_{12,2}+\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{22,2}} \in \lambda^4 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,2} - e^{\epsilon_{12,2}+\epsilon_{22,2}} \in f_{1,2}'[\lambda] = 0, f_{1,2}[0] = 0\}$
- » $\{f_{1,2}[\lambda] \rightarrow e^{-\epsilon_{12,2}+\epsilon_{13,3}-\epsilon_{21,1}} \lambda^2 \eta_{2,2,3} \xi_{1,1,3} - e^{\epsilon_{13,3}-\epsilon_{21,1}} \lambda^3 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,1,3} +$
 $e^{-\epsilon_{12,2}+\epsilon_{14,4}-\epsilon_{21,1}} \lambda^2 \eta_{2,2,4} \xi_{1,1,4} - e^{\epsilon_{14,4}-\epsilon_{21,1}} \lambda^3 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,4} + e^{-\epsilon_{12,2}+\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}} \lambda^3 \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4} -$
 $e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}} \lambda^4 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4} + \lambda^3 \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} - e^{-\epsilon_{12,2}+\epsilon_{13,3}} \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,2} +$
 $e^{\epsilon_{13,3}} \lambda^4 \eta_{2,1,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{2,1,2} - e^{-\epsilon_{12,2}+\epsilon_{14,4}} \lambda^3 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,1,2} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,2,4} \xi_{2,1,2} -$
 $e^{-\epsilon_{12,2}+\epsilon_{13,3}+\epsilon_{14,4}} \lambda^4 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,2} + e^{\epsilon_{13,3}+\epsilon_{14,4}} \lambda^5 \eta_{2,1,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,2}\}$
- » $\{-2 e^{\epsilon_{13,3}-\epsilon_{22,2}+\epsilon_{23,3}} \in \lambda \eta_{2,1,2} \xi_{1,1,3} + 2 e^{\epsilon_{14,4}-\epsilon_{22,2}+\epsilon_{23,3}} \in \lambda \eta_{2,3,4} \xi_{1,2,4} -$
 $3 e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{22,2}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,2,4} - 3 e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{22,2}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,2,4} +$
 $3 e^{\epsilon_{13,3}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + 3 e^{\epsilon_{13,3}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,2,3} -$
 $3 e^{\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,2,3} + 4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,3,4} \xi_{2,2,3} +$
 $4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,2,3} - e^{\epsilon_{13,3}+\epsilon_{23,3}} \in f_{2,3}'[\lambda] = 0, f_{2,3}[0] = 0\}$
- » $\{f_{2,3}[\lambda] \rightarrow -e^{-\epsilon_{22,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,3} + e^{-\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{22,2}} \lambda^2 \eta_{2,3,4} \xi_{1,2,4} -$
 $e^{\epsilon_{14,4}-\epsilon_{22,2}} \lambda^3 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,2,4} - e^{\epsilon_{14,4}-\epsilon_{22,2}} \lambda^3 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,2,4} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,2,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,2,3} -$
 $e^{-\epsilon_{13,3}+\epsilon_{14,4}} \lambda^3 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,2,3} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,3,4} \xi_{2,2,3} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,2,3}\}$
- » $\{-2 e^{\epsilon_{14,4}-\epsilon_{23,3}+\epsilon_{24,4}} \in \lambda \eta_{2,1,3} \xi_{1,1,4} - 2 e^{\epsilon_{14,4}-\epsilon_{23,3}+\epsilon_{24,4}} \in \lambda \eta_{2,2,3} \xi_{1,2,4} + 3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,1,4} \xi_{1,1,4} \xi_{2,3,4} +$
 $3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,3,4} + 3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,3,4} - e^{\epsilon_{14,4}+\epsilon_{24,4}} \in f_{3,4}'[\lambda] = 0, f_{3,4}[0] = 0\}$
- » $\{f_{3,4}[\lambda] \rightarrow$
 $-e^{-\epsilon_{23,3}} \lambda^2 \eta_{2,1,3} \xi_{1,1,4} - e^{-\epsilon_{23,3}} \lambda^2 \eta_{2,2,3} \xi_{1,2,4} + \lambda^3 \eta_{2,1,4} \xi_{1,1,4} \xi_{2,3,4} + \lambda^3 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,3,4} + \lambda^3 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,3,4}\}$
- » $\{2 e^{\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{23,3}} \in \lambda \eta_{2,3,4} \xi_{1,1,4} -$
 $3 e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,1,4} - 3 e^{\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,2,4} \xi_{1,1,4} \xi_{1,2,3} +$
 $3 e^{\epsilon_{13,3}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} - 3 e^{\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,3,4} \xi_{1,2,4} \xi_{2,1,2} +$
 $4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,2,4} \xi_{2,1,2} + 4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,2,4} \xi_{2,1,2} +$
 $3 e^{\epsilon_{13,3}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + 3 e^{\epsilon_{13,3}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,3} \xi_{1,2,3} \xi_{2,1,3} -$
 $3 e^{\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^2 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,1,3} + 4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,3,4} \xi_{2,1,3} +$
 $4 e^{\epsilon_{13,3}+\epsilon_{14,4}+\epsilon_{23,3}} \in \lambda^3 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,3} - e^{\epsilon_{13,3}+\epsilon_{23,3}} \in f_{1,3}'[\lambda] = 0, f_{1,3}[0] = 0\}$
- » $\{f_{1,3}[\lambda] \rightarrow e^{-\epsilon_{13,3}+\epsilon_{14,4}-\epsilon_{21,1}} \lambda^2 \eta_{2,3,4} \xi_{1,1,4} - e^{\epsilon_{14,4}-\epsilon_{21,1}} \lambda^3 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,1,4} -$
 $e^{\epsilon_{14,4}-\epsilon_{21,1}} \lambda^3 \eta_{2,2,4} \xi_{1,1,4} \xi_{1,2,3} + \lambda^3 \eta_{2,1,2} \xi_{1,1,3} \xi_{2,1,2} - e^{-\epsilon_{13,3}+\epsilon_{14,4}} \lambda^3 \eta_{2,3,4} \xi_{1,2,4} \xi_{2,1,2} +$
 $e^{\epsilon_{14,4}} \lambda^4 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,2,4} \xi_{2,1,2} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,2,4} \xi_{2,1,2} + \lambda^3 \eta_{2,1,3} \xi_{1,1,3} \xi_{2,1,3} + \lambda^3 \eta_{2,2,3} \xi_{1,2,3} \xi_{2,1,3} -$
 $e^{-\epsilon_{13,3}+\epsilon_{14,4}} \lambda^3 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,1,3} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,1,4} \xi_{1,1,3} \xi_{1,3,4} \xi_{2,1,3} + e^{\epsilon_{14,4}} \lambda^4 \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} \xi_{2,1,3}\}$
- » $\{-2 e^{\epsilon_{14,4}-\epsilon_{22,2}+\epsilon_{24,4}} \in \lambda \eta_{2,1,2} \xi_{1,1,4} + 3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,1,3} \xi_{1,1,4} \xi_{2,2,3} +$
 $3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,2,3} \xi_{1,2,4} \xi_{2,2,3} + 3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,1,4} \xi_{1,1,4} \xi_{2,2,4} + 3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,2,4} +$
 $3 e^{\epsilon_{14,4}+\epsilon_{24,4}} \in \lambda^2 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,2,4} - e^{\epsilon_{14,4}+\epsilon_{24,4}} \in f_{2,4}'[\lambda] = 0, f_{2,4}[0] = 0\}$
- » $\{f_{2,4}[\lambda] \rightarrow -e^{-\epsilon_{22,2}} \lambda^2 \eta_{2,1,2} \xi_{1,1,4} + \lambda^3 \eta_{2,1,3} \xi_{1,1,4} \xi_{2,2,3} +$
 $\lambda^3 \eta_{2,2,3} \xi_{1,2,4} \xi_{2,2,3} + \lambda^3 \eta_{2,1,4} \xi_{1,1,4} \xi_{2,2,4} + \lambda^3 \eta_{2,2,4} \xi_{1,2,4} \xi_{2,2,4} + \lambda^3 \eta_{2,3,4} \xi_{1,3,4} \xi_{2,2,4}\}$

- $$\gg \{g_{2,3}[\lambda] \rightarrow e^{\epsilon_{1,3}} \lambda^3 \eta_{2,1,2} \eta_{2,2,3} \xi_{1,1,2} - e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3}} \lambda^3 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,3} +$$

$$e^{2\epsilon_{1,3,3}} \lambda^4 \eta_{2,1,3}^2 \xi_{1,1,2} \xi_{1,1,3} - e^{-\epsilon_{1,2,2} + \epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^3 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,4} + e^{\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,1,3} \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,4} -$$

$$e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3}} \lambda^3 \eta_{2,2,3}^2 \xi_{1,2,3} + e^{2\epsilon_{1,3,3}} \lambda^4 \eta_{2,1,3} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} - e^{-\epsilon_{1,2,2} + \epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^3 \eta_{2,2,3} \eta_{2,2,4} \xi_{1,2,4} +$$

$$e^{\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,4} + e^{\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,1,2} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,3,4} -$$

$$e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,3} \xi_{1,3,4} - e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4} +$$

$$2 e^{2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^5 \eta_{2,1,3} \eta_{2,1,4} \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4} - e^{-\epsilon_{1,2,2} + \epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,4} \xi_{1,3,4} +$$

$$e^{\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^5 \eta_{2,1,4}^2 \xi_{1,1,2} \xi_{1,1,4} \xi_{1,3,4} - 2 e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^4 \eta_{2,2,3} \eta_{2,2,4} \xi_{1,2,3} \xi_{1,3,4} +$$

$$e^{2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,3} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} + e^{2\epsilon_{1,3,3} + \epsilon_{1,4,4}} \lambda^5 \eta_{2,1,3} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4} -$$

$$e^{-\epsilon_{1,2,2} + \epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^4 \eta_{2,2,4}^2 \xi_{1,2,4} \xi_{1,3,4} + e^{\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,4} \xi_{1,3,4} -$$

$$e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^5 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,3,4}^2 + e^{2\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^6 \eta_{2,1,4}^2 \xi_{1,1,2} \xi_{1,1,3} \xi_{1,3,4}^2 -$$

$$e^{-\epsilon_{1,2,2} + 2\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^5 \eta_{2,2,4}^2 \xi_{1,2,3} \xi_{1,3,4}^2 + e^{2\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^6 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,2} \xi_{1,2,3} \xi_{1,3,4}^2 \}$$
- $$\gg \{3 e^{\epsilon_{1,3,3} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,1,2} \eta_{2,2,4} \xi_{1,1,3} +$$

$$3 e^{\epsilon_{1,3,3} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,1,3} \eta_{2,3,4} \xi_{1,1,3} - 3 e^{\epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,4} +$$

$$4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,1,4}^2 \xi_{1,1,3} \xi_{1,1,4} + 3 e^{\epsilon_{1,3,3} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,2,3} \eta_{2,3,4} \xi_{1,2,3} +$$

$$4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,4} \xi_{1,2,3} - 3 e^{\epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,4} +$$

$$4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,2,4} + 4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,2,4}^2 \xi_{1,2,3} \xi_{1,2,4} -$$

$$3 e^{\epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^2 \eta_{2,2,4}^2 \xi_{1,3,4} + 4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,3} \xi_{1,3,4} +$$

$$4 e^{\epsilon_{1,3,3} + \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in \lambda^3 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,3} \xi_{1,3,4} - e^{\epsilon_{1,3,3} - \epsilon_{1,4,4} + \epsilon_{2,3,3} - \epsilon_{2,4,4}} \in g_{3,4}'[\lambda] = 0, g_{3,4}[0] = 0 \}$$
- $$\gg \{g_{3,4}[\lambda] \rightarrow e^{\epsilon_{1,4,4}} \lambda^3 \eta_{2,1,2} \eta_{2,2,4} \xi_{1,1,3} + e^{\epsilon_{1,4,4}} \lambda^3 \eta_{2,1,3} \eta_{2,3,4} \xi_{1,1,3} - e^{-\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^3 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,4} +$$

$$e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4}^2 \xi_{1,1,3} \xi_{1,1,4} + e^{\epsilon_{1,4,4}} \lambda^3 \eta_{2,2,3} \eta_{2,3,4} \xi_{1,2,3} + e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,4} \xi_{1,2,3} -$$

$$e^{-\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^3 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,4} + e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,2,4} \xi_{1,1,3} \xi_{1,2,4} + e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,2,4}^2 \xi_{1,2,3} \xi_{1,2,4} -$$

$$e^{-\epsilon_{1,3,3} + 2\epsilon_{1,4,4}} \lambda^3 \eta_{2,3,4}^2 \xi_{1,3,4} + e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,1,4} \eta_{2,3,4} \xi_{1,1,3} \xi_{1,3,4} + e^{2\epsilon_{1,4,4}} \lambda^4 \eta_{2,2,4} \eta_{2,3,4} \xi_{1,2,3} \xi_{1,3,4} \}$$

```
In[*]:= cm[i_, j_ -> k_] := E[{i,j} -> {k}] [CF@Sum[If[SSQ[b],
  b /. x_{a, a} -> (\xi_{1,a,a} + \xi_{2,a,a}) x_{a,a}[k],
  b[k] (b /. {x_{a,b} -> f_{a,b}[\lambda], y_{a,b} -> g_{a,b}[\lambda]} /. sol0)
], {b, $Basis}],
  CF@Sum[b[k] (b /. {x_{a,b} -> f_{a,b}[\lambda], y_{a,b} -> g_{a,b}[\lambda]} /. sol1), {b, $Basis}]
] /. {\xi_{1,a,b} -> \xi_{a,b}[i], \eta_{1,a,b} -> \eta_{a,b}[i], \xi_{2,a,b} -> \xi_{a,b}[j], \eta_{2,a,b} -> \eta_{a,b}[j], \lambda -> 1};
Short[cm[i, j -> k], 10]
```

Out[*]//Short=

$$E_{\{i,j\} \rightarrow \{k\}} \left[y_{1,2}[k] \eta_{1,2}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + \right.$$

$$e^{-\epsilon_{1,1}[i] + \epsilon_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{1,4}[k] \eta_{1,4}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{4,4}[i]} y_{1,4}[k] \eta_{1,4}[j] + y_{2,3}[k] \eta_{2,3}[i] +$$

$$e^{-\epsilon_{2,2}[i] + \epsilon_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + y_{2,4}[k] \eta_{2,4}[i] + e^{-\epsilon_{2,2}[i] + \epsilon_{4,4}[i]} y_{2,4}[k] \eta_{2,4}[j] +$$

$$y_{3,4}[k] \eta_{3,4}[i] + e^{-\epsilon_{3,3}[i] + \epsilon_{4,4}[i]} y_{3,4}[k] \eta_{3,4}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] +$$

$$e^{-\epsilon_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\epsilon_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] - e^{\epsilon_{4,4}[i]} y_{2,4}[k] \eta_{1,4}[j] \xi_{1,2}[i] +$$

$$e^{-\epsilon_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\epsilon_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] - e^{\epsilon_{4,4}[i]} y_{3,4}[k] \eta_{1,4}[j] \xi_{1,3}[i] + \ll 11 \gg +$$

$$e^{-\epsilon_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + e^{-\epsilon_{2,2}[j]} x_{2,4}[k] \xi_{2,4}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + \epsilon_{4,4}[i]} y_{1,2}[k] \eta_{1,4}[j] \xi_{2,4}[i] -$$

$$x_{1,4}[k] \xi_{1,2}[j] \xi_{2,4}[i] + e^{-\epsilon_{4,4}[i]} x_{2,4}[k] \xi_{2,4}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j] +$$

$$e^{-\epsilon_{3,3}[j]} x_{3,4}[k] \xi_{3,4}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{3,3}[i] + \epsilon_{4,4}[i]} y_{1,3}[k] \eta_{1,4}[j] \xi_{3,4}[i] +$$

$$e^{-\epsilon_{2,2}[i] + \epsilon_{3,3}[i] + \epsilon_{4,4}[i]} y_{2,3}[k] \eta_{2,4}[j] \xi_{3,4}[i] - e^{\epsilon_{3,3}[i] + \epsilon_{4,4}[i]} y_{2,3}[k] \eta_{1,4}[j] \xi_{1,2}[i] \xi_{3,4}[i] -$$

$$x_{1,4}[k] \xi_{1,3}[j] \xi_{3,4}[i] + e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i] + \epsilon_{3,3}[i] + \epsilon_{4,4}[i]} y_{1,2}[k] \eta_{1,4}[j] \xi_{2,3}[i] \xi_{3,4}[i] -$$

$$x_{2,4}[k] \xi_{2,3}[j] \xi_{3,4}[i] + e^{-\epsilon_{4,4}[i]} x_{3,4}[k] \xi_{3,4}[j] + x_{4,4}[k] \xi_{4,4}[i] + x_{4,4}[k] \xi_{4,4}[j],$$

$$e^{-\epsilon_{1,1}[i] + \epsilon_{2,2}[i]} y_{1,3}[k] \eta_{1,2}[j] \eta_{2,3}[i] + \ll 264 \gg + x_{3,4}[k] \eta_{3,4}[j] \xi_{3,4}[i] \xi_{3,4}[j] \}$$

In[*]:= Short[cm[1, 2 → 2] // cm[2, 3 → 1], 10]

Out[*]//Short=

$$\begin{aligned}
& \mathbb{E}_{\{1,2,3\} \rightarrow \{1\}} \left[y_{1,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] + \right. \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] + y_{1,3}[1] \eta_{1,3}[1] + e^{-\xi_{1,1}[1] + \xi_{3,3}[1]} y_{1,3}[1] \eta_{1,3}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{1,3}[1] \eta_{1,3}[3] + y_{1,4}[1] \eta_{1,4}[1] + e^{-\xi_{1,1}[1] + \xi_{4,4}[1]} y_{1,4}[1] \eta_{1,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{4,4}[1] + \xi_{4,4}[2]} y_{1,4}[1] \eta_{1,4}[3] + y_{2,3}[1] \eta_{2,3}[1] + e^{-\xi_{2,2}[1] + \xi_{3,3}[1]} y_{2,3}[1] \eta_{2,3}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2]} y_{2,3}[1] \eta_{2,3}[3] + y_{2,4}[1] \eta_{2,4}[1] + e^{-\xi_{2,2}[1] + \xi_{4,4}[1]} y_{2,4}[1] \eta_{2,4}[2] + \\
& \ll 93 \gg + e^{-\xi_{3,3}[3] - \xi_{4,4}[1]} x_{3,4}[1] \xi_{3,4}[2] + e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{1,3}[1] \eta_{1,4}[3] \xi_{3,4}[2] + \\
& e^{-\xi_{2,2}[1] - \xi_{2,2}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{2,4}[3] \xi_{3,4}[2] - \\
& e^{-\xi_{1,1}[2] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{1,4}[3] \xi_{1,2}[1] \xi_{3,4}[2] - \\
& e^{-\xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{2,3}[1] \eta_{1,4}[3] \xi_{1,2}[2] \xi_{3,4}[2] - e^{-\xi_{4,4}[1]} x_{1,4}[1] \xi_{1,3}[3] \xi_{3,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{3,3}[1] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{1,2}[1] \eta_{1,4}[3] \xi_{2,3}[1] \xi_{3,4}[2] + \\
& e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1] + \xi_{2,2}[2] + \xi_{3,3}[2] + \xi_{4,4}[2]} y_{1,2}[1] \eta_{1,4}[3] \xi_{2,3}[2] \xi_{3,4}[2] - \\
& e^{-\xi_{4,4}[1]} x_{2,4}[1] \xi_{2,3}[3] \xi_{3,4}[2] + e^{-\xi_{4,4}[1] - \xi_{4,4}[2]} x_{3,4}[1] \xi_{3,4}[3] + \\
& x_{4,4}[1] \xi_{4,4}[1] + x_{4,4}[1] \xi_{4,4}[2] + x_{4,4}[1] \xi_{4,4}[3], \\
& \left. e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} y_{1,3}[1] \eta_{1,2}[2] \eta_{2,3}[1] + \ll 1981 \gg + e^{-\ll 1 \gg [1]} \ll 3 \gg \xi_{\ll 1 \gg [3]} \right]
\end{aligned}$$

In[*]:= (cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])

Out[*]=

True