

Pensieve header: The PBW multiplication tensor for  $\mathfrak{gl}_n(\epsilon)$  using the  $\lambda$ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[*]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.  
Read more at <http://katlas.org/wiki/KnotTheory>.

## Prolog

```
In[*]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends {\*\*, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU,  $\delta$ ,  $\epsilon$ , \$Basis, \$k, \$PBWRule}.

## Utilities

```
In[*]:=  $\delta_{i,j} := \text{If}[i == j, 1, 0];$ 
```

```
In[*]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[*]:= $k = 1;
```

```
In[*]:= CF[ $\mathcal{E}_-$ ] := Expand[ $\mathcal{E} / . e^L \rightarrow e^{\text{Expand}[L]}$ ] /. { $e^{k \cdot} /; k > $k \rightarrow 0$ };
```

## Implementing general universal enveloping algebras

```
In[*]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_*x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_*x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x^p_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2->i};
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i->j}[ε_] := ε /. U_i -> U_j;
```

```
In[*]:= m_{i,j->k}[c_. U_i[x__] U_j[]] := c U_k[x];
m_{i,j->k}[c_. U_i[] U_j[y__]] := c U_k[y];
m_{i,j->k}[c_. U_i[xx__, x_] U_j[y_, yy__]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j->i}) U_j[yy] // Expand // m_{i,j->k})
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is__}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[is, d-j]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[ε, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σi[z], {i, is}];
  z = Expand[y z]; Do[z = mσi→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

## Epilog

```
In[*]:= End[]; EndPackage[];
```

## Predefined Algebras

### sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: ⟨e,h,f⟩ / ([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Algebra = "sl2";
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

### $gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the ε-nilpotent algebra gln,ε."];
```

UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n,\epsilon}$ .

```
In[ ]:= SetAlgebra[gln,ε] := (
  $Algebra = gln,ε;
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[ε δj,k yi,l - ε δl,i yk,j];
  B[xi,j, yk,l] := CF[δj,k xi,l - δl,i xk,j / . xα,β → If[α ≤ β, ε xα,β, yβ,α]];
);
```

```
SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[ ]:= {y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}
```

```
Out[ ]//MatrixForm=
{
  {y1,2, y1,2} → 0, {y1,2, y1,3} → 0, {y1,2, y2,3} → 0, {y1,2, x1,1} → y1,2, {y1,2, x1,2} → 0,
  {y1,3, y1,2} → 0, {y1,3, y1,3} → 0, {y1,3, y2,3} → 0, {y1,3, x1,1} → y1,3, {y1,3, x1,2} → y2,3,
  {y2,3, y1,2} → 0, {y2,3, y1,3} → 0, {y2,3, y2,3} → 0, {y2,3, x1,1} → 0, {y2,3, x1,2} → 0,
  {x1,1, y1,2} → -y1,2, {x1,1, y1,3} → -y1,3, {x1,1, y2,3} → 0, {x1,1, x1,1} → 0, {x1,1, x1,2} → x1,2,
  {x1,2, y1,2} → 0, {x1,2, y1,3} → -y2,3, {x1,2, y2,3} → 0, {x1,2, x1,1} → -x1,2, {x1,2, x1,2} → 0,
  {x2,2, y1,2} → y1,2, {x2,2, y1,3} → 0, {x2,2, y2,3} → -y2,3, {x2,2, x1,1} → 0, {x2,2, x1,2} → -x1,2,
  {x1,3, y1,2} → 0, {x1,3, y1,3} → 0, {x1,3, y2,3} → 0, {x1,3, x1,1} → -x1,3, {x1,3, x1,2} → 0,
  {x2,3, y1,2} → 0, {x2,3, y1,3} → y1,2, {x2,3, y2,3} → 0, {x2,3, x1,1} → 0, {x2,3, x1,2} → -x1,3,
  {x3,3, y1,2} → 0, {x3,3, y1,3} → y1,3, {x3,3, y2,3} → y2,3, {x3,3, x1,1} → 0, {x3,3, x1,2} → 0
}
```

```
In[ ]:= adPower[x_, k_, Env_] [c * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_ , 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env] [y] = B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, {$Algebra, $k}] [y];
```

```
In[ ]:= x0 = ($Basis / . {x → ξ0, y → η0}) . $Basis
x1 = ($Basis / . {x → ξ1, y → η1}) . $Basis
```

```
Out[ ]:= y1,2 η0,2 + x1,1 ξ0,1 + x1,2 ξ0,2 + x2,2 ξ0,2
```

```
Out[ ]:= y1,2 η1,2 + x1,1 ξ1,1 + x1,2 ξ1,2 + x2,2 ξ1,2
```

```
In[*]:= adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ε, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]
]
```

```
In[*]:= SSQ[ξ0,1 x1,1]
```

```
Out[*]=
True
```

```
In[*]:= adExp[ξ0,1 x1,1][x1]
```

```
Out[*]=
eξ0,1 y1,2 η1,2 + x1,1 ξ1,1 + e-ξ0,1 x1,2 ξ1,2 + x2,2 ξ1,2
```

```
In[*]:= adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]
```

```
In[*]:= adExp[ξ0,2 x1,2][x1]
```

```
Out[*]=
y1,2 η1,2 - ε x1,1 η1,2 ξ0,2 + ε x2,2 η1,2 ξ0,2 - ε x1,2 η1,2 ξ0,22 +
x1,1 ξ1,1 + x1,2 ξ0,2 ξ1,1 + x1,2 ξ1,2 + x2,2 ξ1,2 - x1,2 ξ0,2 ξ1,2
```

```
In[*]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λx;
λTangent[xs_List] := λTangent@@xs
```

```
In[*]:= $Basis ($Basis /. {x → ξ, y → η})
```

```
Out[*]=
{y1,2 η1,2, x1,1 ξ1,1, x1,2 ξ1,2, x2,2 ξ2,2}
```

```
In[*]:= λTangent@(λ $Basis ($Basis /. {x → ξ, y → η}))
```

```
Out[*]=
eλ ξ1,1 - λ ξ2,2 y1,2 η1,2 + x1,1 ξ1,1 + eλ ξ2,2 x1,2 ξ1,2 - eλ ξ1,1 ε λ x1,1 η1,2 ξ1,2 +
eλ ξ1,1 ε λ x2,2 η1,2 ξ1,2 + eλ ξ2,2 λ x1,2 ξ1,1 ξ1,2 - eλ ξ1,1 + λ ξ2,2 ε λ2 x1,2 η1,2 ξ1,22 + x2,2 ξ2,2
```

```
In[*]:= $Basis
```

```
Out[*]=
{y1,2, x1,1, x1,2, x2,2}
```

In[\*]:= **Lhs =  $\lambda$ Tangent** [

**Join**[ $\lambda$  \$Basis (\$Basis /. {x  $\rightarrow$   $\xi_1$ , y  $\rightarrow$   $\eta_1$ }),  $\lambda$  \$Basis (\$Basis /. {x  $\rightarrow$   $\xi_2$ , y  $\rightarrow$   $\eta_2$ }) ]

Out[\*]=

$$\begin{aligned} & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{1,2} + \\ & e^{\lambda \xi_{1,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda x_{1,1} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda x_{2,2} \eta_{2,2} \xi_{1,2} + \\ & 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^2 y_{1,2} \eta_{1,2} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^2 y_{1,2} \eta_{2,2}^2 \xi_{1,2} + \\ & e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} - \\ & e^{\lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^3 y_{1,2} \eta_{2,2}^2 \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2}^2 + x_{2,2} \xi_{1,2} + \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{2,2} + \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{2,2} - e^{\lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{2,2} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{2,2} \xi_{2,2} + \\ & e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{2,2} - \\ & 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,2} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} + \\ & 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} - \\ & e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} - \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,2}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,2}^2 + \\ & e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{2,2}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2}^2 + x_{2,2} \xi_{2,2} \end{aligned}$$

In[\*]:= **rhs =  $\lambda$ Tangent@** (\$Basis (\$Basis /. {x <sub>$\alpha$</sub> ,  $\alpha$   $\rightarrow$  CF[f <sub>$\alpha$</sub> ,  $\alpha$ ][ $\lambda$ ], x <sub>$\alpha\beta$</sub>   $\rightarrow$  f <sub>$\alpha\beta$</sub> [ $\lambda$ ], y <sub>$\alpha\beta$</sub>   $\rightarrow$  g <sub>$\alpha\beta$</sub> [ $\lambda$ ]))

Out[\*]=

$$\begin{aligned} & x_{1,1} f_{1,1}'[\lambda] + e^{f_{2,2}[\lambda]} x_{1,2} f_{1,2}[\lambda] f_{1,1}'[\lambda] + e^{f_{2,2}[\lambda]} x_{1,2} f_{1,2}'[\lambda] + \\ & x_{2,2} f_{2,2}'[\lambda] + e^{f_{1,1}[\lambda] - f_{2,2}[\lambda]} y_{1,2} g_{1,2}'[\lambda] - e^{f_{1,1}[\lambda]} \in x_{1,1} f_{1,2}[\lambda] g_{1,2}'[\lambda] + \\ & e^{f_{1,1}[\lambda]} \in x_{2,2} f_{1,2}[\lambda] g_{1,2}'[\lambda] - e^{f_{1,1}[\lambda] + f_{2,2}[\lambda]} \in x_{1,2} f_{1,2}[\lambda]^2 g_{1,2}'[\lambda] \end{aligned}$$

In[\*]:= **SSBasis = Select**[\$Basis, SSQ]

Out[\*]=

{x<sub>1,1</sub>, x<sub>2,2</sub>}

In[\*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ \$Basis]

Out[\*]=

$$\begin{aligned} & \left\{ e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1} - \xi_{2,2})} \eta_{1,2} \left( 1 + 2 e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \right) = \right. \\ & \quad e^{\lambda (\xi_{1,2} + \xi_{2,1} - \xi_{2,2})} \in \lambda^2 \eta_{2,2}^2 \left( 1 + \lambda \xi_{1,1} \right) \xi_{1,2} + \\ & \quad e^{\lambda (\xi_{2,1} - \xi_{2,2})} \eta_{2,2} \left( -1 + \lambda \xi_{1,1} - \lambda \xi_{1,2} \right) + e^{f_{1,1}[\lambda] - f_{2,2}[\lambda]} g_{1,2}'[\lambda], \\ & \quad \xi_{2,1} + \xi_{1,1} \left( 1 + e^{\lambda^2 \eta_{2,2}} \left( e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2} \right) \right) + e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda] g_{1,2}'[\lambda] = \\ & \quad e^{\lambda (\xi_{1,1} - \xi_{1,2})} \in \lambda \eta_{1,2} \left( e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2} \right) + \\ & \quad \in \lambda \eta_{2,2} \left( -e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \left( 1 + \lambda \xi_{1,2} \right) \xi_{2,2} \right) + f_{1,1}'[\lambda], \\ & \quad e^{-\lambda (\xi_{1,2} + \xi_{2,1})} \left( -e^{\lambda (\xi_{1,1} + 2 \xi_{1,2} + \xi_{2,2})} \in \lambda^2 \eta_{1,2} \xi_{1,2}^2 + e^{\lambda (\xi_{1,2} + \xi_{2,2})} \xi_{1,2} \left( -2 e^{\lambda (\xi_{1,1} + \xi_{2,1})} \in \lambda^2 \eta_{1,2} \right. \right. \\ & \quad \left. \left. \xi_{2,2} + e^{\lambda \xi_{1,2}} \left( 1 + 2 e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{2,2} \right) + e^{\lambda \xi_{1,2}} \lambda \xi_{1,1} \left( 1 + 2 e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{2,2} \right) \right) - \right. \\ & \quad \left. e^{\lambda \xi_{2,1}} \left( e^{\lambda (\xi_{1,2} + \xi_{2,2})} \left( -1 - \lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} \right) \xi_{2,2} + \right. \right. \\ & \quad \left. \left. e^{\lambda (\xi_{2,1} + \xi_{2,2})} \in \lambda^2 \left( e^{\lambda \xi_{1,1}} \eta_{1,2} + e^{\lambda \xi_{1,2}} \eta_{2,2} \left( 1 - \lambda \xi_{1,1} + \lambda \xi_{1,2} \right) \right) \xi_{2,2}^2 + \right. \right. \\ & \quad \left. \left. e^{\lambda \xi_{1,2} + f_{2,2}[\lambda]} \left( f_{1,2}[\lambda] f_{1,1}'[\lambda] + f_{1,2}'[\lambda] - e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda]^2 g_{1,2}'[\lambda] \right) \right) \right) = 0, \\ & \quad \xi_{1,2} + e^{\lambda \eta_{1,2}} \left( e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,2} \right) + \xi_{2,2} = \\ & \quad \in \lambda \eta_{2,2} \left( e^{\lambda \xi_{1,2}} \left( 1 + \lambda \xi_{1,1} \right) \xi_{1,2} + e^{\lambda \xi_{2,1}} \left( -1 + \lambda \xi_{1,1} - \lambda \xi_{1,2} \right) \xi_{2,2} \right) + \\ & \quad \left. f_{2,2}'[\lambda] + e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda] g_{1,2}'[\lambda] \right\} \end{aligned}$$

In[\*]:= Length@eqns

Out[\*]=

4

In[\*]:= unknowns = \$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]}

Out[\*]=

{g<sub>1,2</sub>[λ], f<sub>1,1</sub>[λ], f<sub>1,2</sub>[λ], f<sub>2,2</sub>[λ]}

In[\*]:= \$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[0], y<sub>αβ</sub> → g<sub>αβ</sub>[0]}

Out[\*]=

{g<sub>1,2</sub>[0] == 0, f<sub>1,1</sub>[0] == 0, f<sub>1,2</sub>[0] == 0, f<sub>2,2</sub>[0] == 0}

In[\*]:= {sol0} = Block[{\$k = 0}, Simplify@DSolve[

lhs = λTangent[

Join[λ \$Basis (\$Basis /. {x → ξ<sub>1</sub>, y → η<sub>1</sub>}), λ \$Basis (\$Basis /. {x → ξ<sub>2</sub>, y → η<sub>2</sub>})]

];

rhs = λTangent@(\$Basis (\$Basis /. {x<sub>α</sub>, y → CF[f<sub>α,α</sub>[λ]], x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]}));

Join[

(Coefficient[CF[lhs - rhs], #] == 0) & /@ \$Basis,

\$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[0], y<sub>αβ</sub> → g<sub>αβ</sub>[0]}]

],

\$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]},

λ

]]

Out[\*]=

{ {f<sub>1,1</sub>[λ] → λ (ξ<sub>1,1</sub> + ξ<sub>2,1</sub>), f<sub>2,2</sub>[λ] → λ (ξ<sub>1,2</sub> + ξ<sub>2,2</sub>),  
f<sub>1,2</sub>[λ] → e<sup>-λ ξ<sub>2,1</sub></sup> λ ξ<sub>1,2</sub> + e<sup>-λ ξ<sub>1,2</sub></sup> λ ξ<sub>2,2</sub>, g<sub>1,2</sub>[λ] → λ (η<sub>1,2</sub> + e<sup>λ (-ξ<sub>1,1</sub> + ξ<sub>1,2</sub>)</sup> η<sub>2,2</sub>) } }

```
In[*]:= {sol1} = Block[{k = 1}, Simplify@DSolve[
  lhs = λTangent[
    Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
  ];
  rhs = λTangent[
    $Basis ($Basis /. {xα,α → CF[fα,α[λ]], xαβ → fαβ[λ], yαβ → gαβ[λ]}) /.
    (sol0 /. ((a → b) ⇒ (a → ε a + b)))
  ];
  Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ ⇒ fαβ[0] == 0, yαβ ⇒ gαβ[0] == 0}
  ],
  $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
  λ
]]
```

```
Out[*]= { {f1,1[λ] → eλ ξ1,2 λ2 η2,2 ξ1,2, f2,2[λ] → -eλ ξ1,2 λ2 η2,2 ξ1,2,
  f1,2[λ] → λ3 η2,2 ξ1,2 ξ2,2, g1,2[λ] → -e-λ ξ1,1 + 2 λ ξ1,2 λ3 η2,22 ξ1,2 }
```

```
In[*]:= SetAlgebra[gl3,ε];
k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]= 1
```

```
Out[*]= {y1,2 → 1, y1,3 → 2, y2,3 → 3, x1,1 → 4, x1,2 → 5, x2,2 → 6, x1,3 → 7, x2,3 → 8, x3,3 → 9}
```

```
Out[*]//MatrixForm=
{
  {y1,2, y1,2} → 0, {y1,2, y1,3} → 0, {y1,2, y2,3} → ε y1,3, {y1,2, x1,1} → y1,2,
  {y1,3, y1,2} → 0, {y1,3, y1,3} → 0, {y1,3, y2,3} → 0, {y1,3, x1,1} → y1,3,
  {y2,3, y1,2} → -ε y1,3, {y2,3, y1,3} → 0, {y2,3, y2,3} → 0, {y2,3, x1,1} → y2,3,
  {x1,1, y1,2} → -y1,2, {x1,1, y1,3} → -y1,3, {x1,1, y2,3} → 0, {x1,1, x1,1} → x1,1,
  {x1,2, y1,2} → ε x1,1 - ε x2,2, {x1,2, y1,3} → -y2,3, {x1,2, y2,3} → 0, {x1,2, x1,1} → -x1,2,
  {x2,2, y1,2} → y1,2, {x2,2, y1,3} → 0, {x2,2, y2,3} → -y2,3, {x2,2, x1,1} → -x2,2,
  {x1,3, y1,2} → -ε x2,3, {x1,3, y1,3} → ε x1,1 - ε x3,3, {x1,3, y2,3} → ε x1,2, {x1,3, x1,1} → -x1,3,
  {x2,3, y1,2} → 0, {x2,3, y1,3} → y1,2, {x2,3, y2,3} → ε x2,2 - ε x3,3, {x2,3, x1,1} → -x2,3,
  {x3,3, y1,2} → 0, {x3,3, y1,3} → y1,3, {x3,3, y2,3} → y2,3, {x3,3, x1,1} → -x3,3
}
```

```

In[*]:= sol[-1] = 0 $Basis;
Do[
  Block[{$k = k},
    s0 = Thread[$Basis → sol[k - 1]];
    lhs = λTangent[
      Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
    ];
    rhs = λTangent[
      $Basis ($Basis /. {xαβ → (xαβ /. s0) + εk fαβ[λ], yαβ → (yαβ /. s0) + εk gαβ[λ]})
    ];
    Echo@"computed lhs and rhs";
    eqns = EchoLabel["eqns"]@Join[
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
      $Basis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
    ];
    unknowns = $Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
    {s} = DSolve[eqns, unknowns, λ];
    sol[k] = CF[sol[k - 1] + εk ($Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]} /. s)]
  ],
  {k, 0, $k}
];
sol[$k].$Basis
» computed lhs and rhs

```











- ☹ Set: Part 8 of Asymptotics`AsymptoticRSolveValueDump`c\$15938[[Asymptotics`AsymptoticRSolveValueDump`i\$15938]] does not exist.
- ☹ Set: Part 9 of Asymptotics`AsymptoticRSolveValueDump`c\$15938[[Asymptotics`AsymptoticRSolveValueDump`i\$15938]] does not exist.
- ☹ General: Further output of Set::partw will be suppressed during this calculation.
- ☹ Part: Part 7 of {0, 0, 0, 0, 0, 0} does not exist.
- ☹ Part: Part 8 of {0, 0, 0, 0, 0, 0} does not exist.
- ☹ Part: Part 9 of {0, 0, 0, 0, 0, 0} does not exist.
- ☹ General: Further output of Part::partw will be suppressed during this calculation.

Out[\*]=

\$Aborted

☹ Dot: Tensors

$\{\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} - e^{-\lambda \xi_{1,1} + 2\lambda \xi_{1,2}} \in \lambda^3 \eta_{2,2}^2 \xi_{1,2}, \lambda \xi_{1,1} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{2,1}, e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,1} + \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}, -e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{1,2} + \lambda \xi_{2,2}\}$  and  $\{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\}$  have incompatible shapes.

Out[\*]=

$\{\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} - e^{-\lambda \xi_{1,1} + 2\lambda \xi_{1,2}} \in \lambda^3 \eta_{2,2}^2 \xi_{1,2}, \lambda \xi_{1,1} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{2,1}, e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,1} + \in \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}, -e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{1,2} + \lambda \xi_{2,2}\}.$   
 $\{y_{1,2}, y_{1,3}, y_{2,3}, x_{1,1}, x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}, x_{3,3}\}$

$osol = \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[ y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + y_{1,3}[k] \eta_{1,3}[i] + e^{-\xi_{1,1}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] + y_{2,3}[k] \eta_{2,3}[i] + e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] - e^{\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + e^{-\xi_{1,1}[j]} x_{1,3}[k] \xi_{1,3}[i] + e^{-\xi_{3,3}[i]} x_{1,3}[k] \xi_{1,3}[j] + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j] + e^{-\xi_{2,2}[j]} x_{2,3}[k] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j] \xi_{2,3}[i] - x_{1,3}[k] \xi_{1,2}[j] \xi_{2,3}[i] + e^{-\xi_{3,3}[i]} x_{2,3}[k] \xi_{2,3}[j] + x_{3,3}[k] \xi_{3,3}[i] + x_{3,3}[k] \xi_{3,3}[j], -e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j] + e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + e^{\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] - e^{-\xi_{2,2}[j]} x_{2,3}[k] \eta_{1,2}[j] \xi_{1,3}[i] + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] + e^{-\xi_{1,1}[j] - \xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[j] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,2}[j] \xi_{1,2}[j] \xi_{1,3}[i] + x_{1,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{1,3}[j] + e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j] \xi_{2,3}[i] - e^{\xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i] + \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - 2e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] + e^{2\xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{-\xi_{2,2}[i] + \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{2,3}[i] + e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + x_{1,3}[k] \eta_{2,3}[j] \xi_{1,3}[j] \xi_{2,3}[i] - e^{-\xi_{1,1}[i] + 2\xi_{2,2}[i] + 2\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 +$

$$\begin{aligned}
 & x_{2,3}[k] \eta_{1,3}[j] \xi_{1,3}[i] \xi_{2,3}[j] + x_{2,3}[k] \eta_{2,3}[j] \xi_{2,3}[i] \xi_{2,3}[j], \\
 & -\frac{1}{2} e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \\
 & e^{-\xi_{1,1}[i]+3 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 + e^{\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - \\
 & e^{\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] + \\
 & e^{-\xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] + \\
 & 2 e^{-\xi_{1,1}[i]+2 \xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] - e^{2 \xi_{3,3}[i]} y_{1,3}[k] \\
 & \eta_{1,2}[i] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] - \\
 & e^{\xi_{3,3}[i]} x_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] - \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
 & \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,3}[i]^2 + \\
 & e^{-\xi_{1,1}[i]+3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 - e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
 & e^{-\xi_{2,2}[i]+3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i]^2 + \\
 & e^{-\xi_{1,1}[j]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i]^2 - e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i]^2 + \\
 & e^{-\xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{2,3}[j]^2 \xi_{2,3}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
 & 2 e^{-\xi_{1,1}[i]+2 \xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - \\
 & e^{2 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,2}[i] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i] - e^{2 \xi_{2,2}[i]+\xi_{3,3}[i]} x_{1,1}[k] \eta_{1,2}[j] \\
 & \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + e^{2 \xi_{2,2}[i]+\xi_{3,3}[i]} x_{2,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] + \\
 & 3 e^{-\xi_{1,1}[i]+3 \xi_{2,2}[i]+\xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \eta_{1,3}[j] \xi_{1,2}[i]^2 \xi_{2,3}[i] - \\
 & e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
 & 2 e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & 2 e^{-\xi_{1,1}[i]+3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
 & e^{-\xi_{1,1}[j]-\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & 2 e^{-\xi_{2,2}[i]+3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,3}[i] \xi_{2,3}[i] - e^{\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,1}[k] \eta_{1,3}[j]^2 \\
 & \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + e^{\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & 4 e^{-\xi_{1,1}[i]+2 \xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & e^{-\xi_{1,1}[j]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] - \\
 & 2 e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j]^2 \eta_{2,3}[j] \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & e^{-\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] - \\
 & e^{2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i] \xi_{1,2}[j] \xi_{1,3}[i] \xi_{2,3}[i] + \\
 & e^{-\xi_{1,1}[i]+\xi_{2,2}[i]+3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,3}[i]^2 \xi_{2,3}[i] - \\
 & \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2 \xi_{3,3}[i]} x_{3,3}[k] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + \\
 & e^{-\xi_{1,1}[i]+3 \xi_{3,3}[i]} y_{1,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{2,3}[i]^2 + e^{-\xi_{2,2}[i]+3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{2,3}[j]^3 \xi_{2,3}[i]^2 - \\
 & 2 e^{-\xi_{1,1}[i]+2 \xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i] \xi_{2,3}[i]^2 - \\
 & e^{3 \xi_{3,3}[i]} y_{2,3}[k] \eta_{1,3}[j] \eta_{2,3}[j]^2 \xi_{1,2}[i] \xi_{2,3}[i]^2 - \frac{1}{2} e^{2 \xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,1}[k] \\
 & \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{2,2}[k] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
 & 3 e^{-\xi_{1,1}[i]+3 \xi_{2,2}[i]+2 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,2}[j] \eta_{1,3}[j]^2 \xi_{1,2}[i]^2 \xi_{2,3}[i]^2 + \\
 & e^{-\xi_{2,2}[i]+2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{2,3}[j]^2 \xi_{1,2}[j] \xi_{2,3}[i]^2 - e^{2 \xi_{3,3}[i]} x_{1,2}[k] \eta_{1,3}[j] \eta_{2,3}[j] \xi_{1,2}[i]
 \end{aligned}$$

$$\xi_{1,2}[j] \xi_{2,3}[i]^2 + 2 e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i] \xi_{1,3}[i] \xi_{2,3}[i]^2 + e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i] + 3 \xi_{3,3}[i]} y_{1,2}[k] \eta_{1,3}[j]^3 \xi_{1,2}[i]^2 \xi_{2,3}[i]^3];$$

```
osol = (List@@osol).Table[e^k, {k, 0, Length[osol] - 1}]
```

Out[\*]=

$$y_{1,2}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] + x_{1,1}[k] \xi_{1,1}[i] + x_{1,1}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] + \epsilon^2 \left( -\frac{1}{2} e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + \frac{1}{2} e^{2 \xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i]^2 + e^{-\xi_{1,1}[i] + 3 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^3 \xi_{1,2}[i]^2 \right) + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + \epsilon \left( e^{\xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j] \right) + x_{2,2}[k] \xi_{2,2}[i] + x_{2,2}[k] \xi_{2,2}[j]$$

```
In[*]:= CF[(Expand@sol[$k].$Basis /. {xi1_αβ_ -> xi_αβ[i], eta1_αβ_ -> eta_αβ[i], xi2_αβ_ -> xi_αβ[j], eta2_αβ_ -> eta_αβ[j], λ -> 1}) - (osol /. v_ [k] -> v)]
```

Out[\*]=

0