

Pensieve header: The PBW multiplication tensor for  $\mathfrak{gl}_n(\epsilon)$  using the  $\lambda$ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[*]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.  
Read more at <http://katlas.org/wiki/KnotTheory>.

## Prolog

```
In[*]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**",  $\epsilon$ , $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ , adPower, adExp},
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends {\*\*, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU,  $\delta$ ,  $\epsilon$ , \$Basis, \$k, \$PBWRule}.

## Utilities

```
In[*]:=  $\delta_{i,j} := \text{If}[i == j, 1, 0];$ 
```

```
In[*]:= SSQ[c_.*x_] /; MemberQ[$Basis, x] :=  $\neg$  NilQ[x]; (* Semi-Simple Q *)
NilQ[c_.*x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
In[*]:= $k = 1;
```

```
In[*]:= CF[ $\mathcal{E}_-$ ] := Expand[ $\mathcal{E} / . e^L \rightarrow e^{\text{Expand}[L]}$ ] / . { $e^{k \cdot} / ; k > $k \rightarrow 0$ };
```

## Implementing general universal enveloping algebras

```
In[*]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_*x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_*x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x^p_] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2->i};
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i->j}[ε_] := ε /. U_i → U_j;
```

```
In[*]:= m_{i,j->k}[c_. U_i[x__] U_j[]] := c U_k[x];
m_{i,j->k}[c_. U_i[] U_j[y__]] := c U_k[y];
m_{i,j->k}[c_. U_i[xx__, x_] U_j[y_, yy__]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j->i}) U_j[yy] // Expand // m_{i,j->k})
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is__}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[is, d-j]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σi[z], {i, is}];
  z = Expand[y z]; Do[z = mσi→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

## Epilog

```
In[*]:= End[]; EndPackage[];
```

## Predefined Algebras

### sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: ⟨e,h,f⟩ / ([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
  NilQ[e] = NilQ[f] = True; NilQ[h] = False;
);
```

### $gl_{n,\epsilon}$

```
In[*]:= Print["UEA`SetAlgebra knows the ε-nilpotent algebra gln,ε."];
```

UEA`SetAlgebra knows the  $\epsilon$ -nilpotent algebra  $gl_{n,\epsilon}$ .

```
In[*]:= SetAlgebra[gln,ε] := (
  $Basis = Flatten@{
    Table[yα,β, {β, 2, n}, {α, 1, β - 1}],
    Table[xα,β, {β, 1, n}, {α, 1, β}]
  };
  NilQ[y_] = True; NilQ[xα,β] := (α != β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
  B[yi,j, yk,l] := CF[ε δj,k yi,l - ε δl,i yk,j];
  B[xi,j, yl,k] := CF[δj,k xi,l - δl,i xk,j /. xα,β → If[α ≤ β, ε xα,β, yβ,α]];
);
```

```
In[*]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]= {y1,2 → 1, x1,1 → 2, x1,2 → 3, x2,2 → 4}
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -\epsilon x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0 \end{pmatrix}$$

```

```
In[*]:= Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_*y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_ , 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];
```

```
In[*]:= x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis
```

```
Out[*]= y1,2 η0,2 + x1,1 ξ0,1 + x1,2 ξ0,2 + x2,2 ξ0,2
```

```
Out[*]= y1,2 η1,2 + x1,1 ξ1,1 + x1,2 ξ1,2 + x2,2 ξ1,2
```

```
In[*]:= adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
  A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
  CF[$Basis.Normal[Series[MatrixExp[A], {ε, 0, $k}]]].
  Table[Coefficient[y, b], {b, $Basis}]
]
```

In[\*]:= **SSQ**[ $\xi_{0,1}$   $x_{1,1}$ ]

Out[\*]=  
True

In[\*]:= **adExp**[ $\xi_{0,1}$   $x_{1,1}$ ] [**x1**]

Out[\*]=  
 $e^{\xi_{0,1}}$   $y_{1,2}$   $\eta_{1,2}$  +  $x_{1,1}$   $\xi_{1,1}$  +  $e^{-\xi_{0,1}}$   $x_{1,2}$   $\xi_{1,2}$  +  $x_{2,2}$   $\xi_{1,2,2}$

```
In[*]:= adExp[x_] [y_] /; NilQ[x] := Expand@Module [{k = 0, s = 0},
While [adPower[x, k] [y] != 0,
s += adPower[x, k] [y] / k!;
++k
];
s
]
```

In[\*]:= **adExp**[ $\xi_{0,2}$   $x_{1,2}$ ] [**x1**]

Out[\*]=  
 $y_{1,2}$   $\eta_{1,2}$  -  $\epsilon$   $x_{1,1}$   $\eta_{1,2}$   $\xi_{0,2}$  +  $\epsilon$   $x_{2,2}$   $\eta_{1,2}$   $\xi_{0,2}$  -  $\epsilon$   $x_{1,2}$   $\eta_{1,2}$   $\xi_{0,2}^2$  +  
 $x_{1,1}$   $\xi_{1,1}$  +  $x_{1,2}$   $\xi_{0,2}$   $\xi_{1,1}$  +  $x_{1,2}$   $\xi_{1,2}$  +  $x_{2,2}$   $\xi_{1,2,2}$  -  $x_{1,2}$   $\xi_{0,2}$   $\xi_{1,2,2}$

```
In[*]:= λTangent [] = 0;
λTangent [xs____, x_] := adExp[x] [λTangent[xs]] +  $\partial_{\lambda}$  x;
λTangent [xs_List] := λTangent @@ xs
```

In[\*]:= **\$Basis** (**\$Basis** /. {**x** →  $\xi$ , **y** →  $\eta$ })

Out[\*]=  
{ $y_{1,2}$   $\eta_{1,2}$ ,  $x_{1,1}$   $\xi_{1,1}$ ,  $x_{1,2}$   $\xi_{1,2}$ ,  $x_{2,2}$   $\xi_{2,2}$ }

In[\*]:= **λTangent**@( $\lambda$  **\$Basis** (**\$Basis** /. {**x** →  $\xi$ , **y** →  $\eta$ }))

Out[\*]=  
 $e^{\lambda \xi_{1,1} - \lambda \xi_{2,2}}$   $y_{1,2}$   $\eta_{1,2}$  +  $x_{1,1}$   $\xi_{1,1}$  +  $e^{\lambda \xi_{2,2}}$   $x_{1,2}$   $\xi_{1,2}$  -  $e^{\lambda \xi_{1,1}}$   $\epsilon$   $\lambda$   $x_{1,1}$   $\eta_{1,2}$   $\xi_{1,2}$  +  
 $e^{\lambda \xi_{1,1}}$   $\epsilon$   $\lambda$   $x_{2,2}$   $\eta_{1,2}$   $\xi_{1,2}$  +  $e^{\lambda \xi_{2,2}}$   $\lambda$   $x_{1,2}$   $\xi_{1,1}$   $\xi_{1,2}$  -  $e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}}$   $\epsilon$   $\lambda^2$   $x_{1,2}$   $\eta_{1,2}$   $\xi_{1,2}^2$  +  $x_{2,2}$   $\xi_{2,2}$

In[\*]:= **\$Basis**

Out[\*]=  
{ $y_{1,2}$ ,  $x_{1,1}$ ,  $x_{1,2}$ ,  $x_{2,2}$ }

In[\*]:= **lhs =  $\lambda$ Tangent [**

**Join [ $\lambda$  \$Basis (\$Basis /. {x  $\rightarrow$   $\xi_1$ , y  $\rightarrow$   $\eta_1$ }),  $\lambda$  \$Basis (\$Basis /. {x  $\rightarrow$   $\xi_2$ , y  $\rightarrow$   $\eta_2$ })]**

Out[\*]=

$$\begin{aligned}
& e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - \\
& e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{1,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{1,2} + \\
& e^{\lambda \xi_{1,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda x_{1,1} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda x_{2,2} \eta_{2,2} \xi_{1,2} + \\
& 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^2 y_{1,2} \eta_{1,2} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^2 y_{1,2} \eta_{2,2}^2 \xi_{1,2} + \\
& e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} - \\
& e^{\lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda^3 y_{1,2} \eta_{2,2}^2 \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2}^2 + x_{2,2} \xi_{1,2} + \\
& e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{2,2} + \\
& e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{2,2} - e^{\lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{2,2} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{2,2} \xi_{2,2} + \\
& e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{2,2} - \\
& 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,2} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} + \\
& 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} - \\
& e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} - \\
& e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,2}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,2}^2 + \\
& e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{2,2}^2 - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,2}^2 + x_{2,2} \xi_{2,2}
\end{aligned}$$

In[\*]:= **rhs =  $\lambda$ Tangent@ (\$Basis (\$Basis /. {x <sub>$\alpha$</sub> ,  $\alpha$   $\rightarrow$  CF[f <sub>$\alpha$</sub> ,  $\alpha$ ][ $\lambda$ ], x <sub>$\alpha\beta$</sub>   $\rightarrow$  f <sub>$\alpha\beta$</sub> [ $\lambda$ ], y <sub>$\alpha\beta$</sub>   $\rightarrow$  g <sub>$\alpha\beta$</sub> [ $\lambda$ ])}**

Out[\*]=

$$\begin{aligned}
& x_{1,1} f_{1,1}'[\lambda] + e^{f_{2,2}[\lambda]} x_{1,2} f_{1,2}[\lambda] f_{1,1}'[\lambda] + e^{f_{2,2}[\lambda]} x_{1,2} f_{1,2}'[\lambda] + \\
& x_{2,2} f_{2,2}'[\lambda] + e^{f_{1,1}[\lambda] - f_{2,2}[\lambda]} y_{1,2} g_{1,2}'[\lambda] - e^{f_{1,1}[\lambda]} \in x_{1,1} f_{1,2}[\lambda] g_{1,2}'[\lambda] + \\
& e^{f_{1,1}[\lambda]} \in x_{2,2} f_{1,2}[\lambda] g_{1,2}'[\lambda] - e^{f_{1,1}[\lambda] + f_{2,2}[\lambda]} \in x_{1,2} f_{1,2}[\lambda]^2 g_{1,2}'[\lambda]
\end{aligned}$$

In[\*]:= **SSBasis = Select[\$Basis, SSQ]**

Out[\*]=

{x<sub>1,1</sub>, x<sub>2,2</sub>}

In[\*]:= eqns = Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ \$Basis]

Out[\*]=

$$\begin{aligned} & \left\{ e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1} - \xi_{2,2})} \eta_{1,2} \left( 1 + 2 e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,2} \right) = \right. \\ & \quad e^{\lambda (\xi_{1,2} + \xi_{2,1} - \xi_{2,2})} \in \lambda^2 \eta_{2,2}^2 \left( 1 + \lambda \xi_{1,1} \right) \xi_{1,2} + \\ & \quad e^{\lambda (\xi_{2,1} - \xi_{2,2})} \eta_{2,2} \left( -1 + \lambda \xi_{1,1} - \lambda \xi_{1,2} \right) + e^{f_{1,1}[\lambda] - f_{2,2}[\lambda]} g_{1,2}'[\lambda], \\ & \quad \xi_{2,1} + \xi_{1,1} \left( 1 + e^{\lambda^2 \eta_{2,2}} \left( e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2} \right) \right) + e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda] g_{1,2}'[\lambda] = \\ & \quad e^{\lambda (\xi_{1,1} - \xi_{1,2})} \in \lambda \eta_{1,2} \left( e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \xi_{2,2} \right) + \\ & \quad \in \lambda \eta_{2,2} \left( -e^{\lambda \xi_{1,2}} \xi_{1,2} + e^{\lambda \xi_{2,1}} \left( 1 + \lambda \xi_{1,2} \right) \xi_{2,2} \right) + f_{1,1}'[\lambda], \\ & \quad e^{-\lambda (\xi_{1,2} + \xi_{2,1})} \left( -e^{\lambda (\xi_{1,1} + 2 \xi_{1,2} + \xi_{2,2})} \in \lambda^2 \eta_{1,2} \xi_{1,2}^2 + e^{\lambda (\xi_{1,2} + \xi_{2,2})} \xi_{1,2} \left( -2 e^{\lambda (\xi_{1,1} + \xi_{2,1})} \in \lambda^2 \eta_{1,2} \right. \right. \\ & \quad \left. \left. \xi_{2,2} + e^{\lambda \xi_{1,2}} \left( 1 + 2 e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{2,2} \right) + e^{\lambda \xi_{1,2}} \lambda \xi_{1,1} \left( 1 + 2 e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{2,2} \right) \right) - \right. \\ & \quad \left. e^{\lambda \xi_{2,1}} \left( e^{\lambda (\xi_{1,2} + \xi_{2,2})} \left( -1 - \lambda \xi_{1,1} + \lambda \xi_{1,2} - \lambda \xi_{2,1} \right) \xi_{2,2} + \right. \right. \\ & \quad \left. \left. e^{\lambda (\xi_{2,1} + \xi_{2,2})} \in \lambda^2 \left( e^{\lambda \xi_{1,1}} \eta_{1,2} + e^{\lambda \xi_{1,2}} \eta_{2,2} \left( 1 - \lambda \xi_{1,1} + \lambda \xi_{1,2} \right) \right) \xi_{2,2}^2 + \right. \right. \\ & \quad \left. \left. e^{\lambda \xi_{1,2} + f_{2,2}[\lambda]} \left( f_{1,2}[\lambda] f_{1,1}'[\lambda] + f_{1,2}'[\lambda] - e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda]^2 g_{1,2}'[\lambda] \right) \right) \right) = 0, \\ & \quad \xi_{1,2} + e^{\lambda \eta_{1,2}} \left( e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,2} \right) + \xi_{2,2} = \\ & \quad \in \lambda \eta_{2,2} \left( e^{\lambda \xi_{1,2}} \left( 1 + \lambda \xi_{1,1} \right) \xi_{1,2} + e^{\lambda \xi_{2,1}} \left( -1 + \lambda \xi_{1,1} - \lambda \xi_{1,2} \right) \xi_{2,2} \right) + \\ & \quad \left. f_{2,2}'[\lambda] + e^{f_{1,1}[\lambda]} \in f_{1,2}[\lambda] g_{1,2}'[\lambda] \right\} \end{aligned}$$

In[\*]:= Length@eqns

Out[\*]=

4

In[\*]:= unknowns = \$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]}

Out[\*]=

{g<sub>1,2</sub>[λ], f<sub>1,1</sub>[λ], f<sub>1,2</sub>[λ], f<sub>2,2</sub>[λ]}

In[\*]:= \$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[0], y<sub>αβ</sub> → g<sub>αβ</sub>[0]}

Out[\*]=

{g<sub>1,2</sub>[0] == 0, f<sub>1,1</sub>[0] == 0, f<sub>1,2</sub>[0] == 0, f<sub>2,2</sub>[0] == 0}

In[\*]:= {sol0} = Block[{\$k = 0}, Simplify@DSolve[

lhs = λTangent[

Join[λ \$Basis (\$Basis /. {x → ξ<sub>1</sub>, y → η<sub>1</sub>}), λ \$Basis (\$Basis /. {x → ξ<sub>2</sub>, y → η<sub>2</sub>})]

];

rhs = λTangent@(\$Basis (\$Basis /. {x<sub>α</sub>, y → CF[f<sub>α,α</sub>[λ]], x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]}));

Join[

(Coefficient[CF[lhs - rhs], #] == 0) & /@ \$Basis,

\$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[0], y<sub>αβ</sub> → g<sub>αβ</sub>[0]}]

],

\$Basis /. {x<sub>αβ</sub> → f<sub>αβ</sub>[λ], y<sub>αβ</sub> → g<sub>αβ</sub>[λ]},

λ

]]

Out[\*]=

{ {f<sub>1,1</sub>[λ] → λ (ξ<sub>1,1</sub> + ξ<sub>2,1</sub>), f<sub>2,2</sub>[λ] → λ (ξ<sub>1,2</sub> + ξ<sub>2,2</sub>),  
f<sub>1,2</sub>[λ] → e<sup>-λ ξ<sub>2,1</sub></sup> λ ξ<sub>1,2</sub> + e<sup>-λ ξ<sub>1,2</sub></sup> λ ξ<sub>2,2</sub>, g<sub>1,2</sub>[λ] → λ (η<sub>1,2</sub> + e<sup>λ (-ξ<sub>1,1</sub> + ξ<sub>1,2</sub>)</sup> η<sub>2,2</sub>) } }

```
In[*]:= (sol0 /. (a_ -> b_) := (a -> e a + b))
```

```
Out[*]=
```

```
{f1,1[λ] -> λ (ξ1,1 + ξ2,1) + e f1,1[λ], f2,2[λ] -> λ (ξ1,2 + ξ2,2) + e f2,2[λ],
 f1,2[λ] -> e-λ ξ2,1 λ ξ1,2 + e-λ ξ1,2 λ ξ2,2 + e f1,2[λ],
 g1,2[λ] -> λ (η1,2 + eλ (-ξ1,1+ξ1,2) η2,2) + e g1,2[λ]}
```

```
In[*]:= {sol1} = Block[{k = 1}, Simplify@DSolve[
  lhs = λTangent[
    Join[λ $Basis ($Basis /. {x -> ξ1, y -> η1}), λ $Basis ($Basis /. {x -> ξ2, y -> η2})]
  ];
  rhs = EchoLabel["rhs"]@λTangent@Echo[
    $Basis ($Basis /. {xα -> CF[fα,α[λ]], xαβ -> fαβ[λ], yαβ -> gαβ[λ]}) /.
    (sol0 /. ((a_ -> b_) := (a -> e a + b)))
  ];
  EchoLabel["eqns"]@Join[
    (Coefficient[CF[lhs - rhs], #] == 0) & /@ $Basis,
    $Basis /. {xαβ -> fαβ[0] == 0, yαβ -> gαβ[0] == 0}
  ],
  $Basis /. {xαβ -> fαβ[λ], yαβ -> gαβ[λ]},
  λ
]]
```

```
» {y1,2 (λ (η1,2 + eλ (-ξ1,1+ξ1,2) η2,2) + e g1,2[λ]), x1,1 (λ (ξ1,1 + ξ2,1) + e f1,1[λ]),
 x1,2 (e-λ ξ2,1 λ ξ1,2 + e-λ ξ1,2 λ ξ2,2 + e f1,2[λ]), x2,2 (λ (ξ1,2 + ξ2,2) + e f2,2[λ])}
```

» rhs 
$$\begin{aligned} & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{1,2} + \\ & e^{\lambda \xi_{1,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda x_{1,1} \eta_{2,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda x_{2,2} \eta_{2,2} \xi_{1,2} + \\ & e^{\lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{1,2} - \\ & e^{\lambda \xi_{1,2}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} - \\ & e^{2 \lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} + e^{2 \lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} + \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2} \xi_{1,2} - \\ & e^{2 \lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,1} - \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{1,2} \xi_{2,1,2} + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{1,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1}} \in \lambda x_{1,1} \eta_{2,2} \xi_{2,1,2} + \\ & e^{\lambda \xi_{2,1}} \in \lambda x_{2,2} \eta_{2,2} \xi_{2,1,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,1,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,1} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,1} \xi_{2,1,2} - 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{1,2} \xi_{2,1,2} - \\ & 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,1,2} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{1,2} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1}} \in \lambda^2 x_{1,1} \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 x_{2,2} \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} - \\ & 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2} \xi_{1,2,2} \xi_{2,1,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1,1} \xi_{2,1,2} - \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{1,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 x_{1,2} \eta_{2,2} \xi_{2,1,2} + \\ & e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,1} \xi_{2,1,2} - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 x_{1,2} \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} + \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in y_{1,2} \eta_{1,2} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in y_{1,2} \eta_{2,2} f_{1,1}[\lambda] - \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda y_{1,2} \eta_{2,2} \xi_{1,1} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda y_{1,2} \eta_{2,2} \xi_{1,2} f_{1,1}[\lambda] + \\ & e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in x_{1,2} \xi_{1,1} f_{1,2}[\lambda] + e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in x_{1,2} \xi_{2,1} f_{1,2}[\lambda] - \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in y_{1,2} \eta_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in y_{1,2} \eta_{2,2} f_{2,2}[\lambda] + \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda y_{1,2} \eta_{2,2} \xi_{1,1} f_{2,2}[\lambda] + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in x_{1,2} \xi_{1,2} f_{2,2}[\lambda] + \\ & e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{1,1} \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda y_{1,2} \eta_{2,2} \xi_{1,2} f_{2,2}[\lambda] + \\ & e^{\lambda \xi_{2,2}} \in x_{1,2} \xi_{2,1,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{1,1} \xi_{2,1,2} f_{2,2}[\lambda] - \\ & e^{\lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{1,2,2} \xi_{2,1,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{2,1,1} \xi_{2,1,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda x_{1,1} f_{1,1}'[\lambda] + \\ & e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{1,1} f_{1,1}'[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda x_{1,2} \xi_{2,1,2} f_{1,1}'[\lambda] + \\ & e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in x_{1,2} f_{1,2}'[\lambda] + x_{2,2} (\xi_{1,2,2} + \xi_{2,2,2} + e^{\lambda \xi_{2,2}} f_{2,2}'[\lambda]) + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in y_{1,2} g_{1,2}'[\lambda] \end{aligned}$$

» eqns

$$\begin{aligned} & \left\{ - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{1,2} f_{1,1}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{2,2} f_{1,1}[\lambda] + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,1} f_{1,1}[\lambda] - \right. \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,2} f_{1,1}[\lambda] + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{1,2} f_{2,2}[\lambda] + \\ & e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \eta_{2,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,1} f_{2,2}[\lambda] + \\ & \left. e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \in \lambda \eta_{2,2} \xi_{1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} \in g_{1,2}'[\lambda] = 0, \right. \\ & e^{\lambda \xi_{1,1}} \in \lambda \eta_{1,2} \xi_{1,1,2} + e^{\lambda \xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{1,1,2} + \\ & e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,2} \xi_{1,1,2} + e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda \eta_{1,2} \xi_{2,1,2} + e^{\lambda \xi_{2,1}} \in \lambda \eta_{2,2} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{2,1,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{2,1,2} = 0, \\ & e^{\lambda \xi_{1,1} + \lambda \xi_{2,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{1,1,2} + e^{2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,2} - \\ & e^{2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1,1} \xi_{1,1,2} + e^{2 \lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1,2} \xi_{1,1,2} + \\ & 2 e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{1,1,2} \xi_{2,1,2} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,2} \xi_{2,1,2} - \\ & 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1,1} \xi_{1,1,2} \xi_{2,1,2} + 2 e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1,2} \xi_{1,1,2} \xi_{2,1,2} + \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{1,2} \xi_{2,1,2} + e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^2 \eta_{2,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,1,1} \xi_{2,1,2} + \\ & e^{\lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda^3 \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \xi_{1,1,1} f_{1,2}[\lambda] - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in \xi_{2,1,1} f_{1,2}[\lambda] - \\ & e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \xi_{1,1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda \xi_{1,1,1} \xi_{1,1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,2}} \in \xi_{2,1,2} f_{2,2}[\lambda] - \\ & e^{\lambda \xi_{2,2}} \in \lambda \xi_{1,1,1} \xi_{2,1,2} f_{2,2}[\lambda] + e^{\lambda \xi_{2,2}} \in \lambda \xi_{1,2,2} \xi_{2,1,2} f_{2,2}[\lambda] - e^{\lambda \xi_{2,2}} \in \lambda \xi_{2,1,1} \xi_{2,1,2} f_{2,2}[\lambda] - \\ & e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \in \lambda \xi_{1,1,2} f_{1,1}'[\lambda] - e^{\lambda \xi_{2,2}} \in \lambda \xi_{2,1,2} f_{1,1}'[\lambda] - e^{\lambda \xi_{1,2} + \lambda \xi_{2,2}} \in f_{1,2}'[\lambda] = 0, \\ & - e^{\lambda \xi_{1,1}} \in \lambda \eta_{1,2} \xi_{1,1,2} - e^{\lambda \xi_{1,2}} \in \lambda \eta_{2,2} \xi_{1,1,2} + e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{1,1,2} - e^{\lambda \xi_{1,2}} \in \lambda^2 \eta_{2,2} \xi_{1,1,2} \xi_{1,1,2} - \\ & e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1}} \in \lambda \eta_{1,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1}} \in \lambda \eta_{2,2} \xi_{2,1,2} + e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{2,1,2} - \\ & e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,2,2} \xi_{2,1,2} - e^{\lambda \xi_{2,1}} \in \lambda^2 \eta_{2,2} \xi_{1,1,1} \xi_{2,1,2} = 0, g_{1,2}[\theta] = 0, f_{1,1}[\theta] = 0, f_{1,2}[\theta] = 0, f_{2,2}[\theta] = 0 \} \end{aligned}$$

Out[0]=

$$\left\{ \left\{ f_{1,1}[\lambda] \rightarrow \frac{1}{\xi_{1,1,1}^2 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2} \right. \right.$$

$$\begin{aligned}
 & \eta_{1,2} \left( - \left( (-1 + e^{\lambda \xi_{1,1}}) \xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 \right) + \xi_{1,1} \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) \left( -2 + 2 e^{\lambda \xi_{1,1}} + \right. \right. \\
 & \quad \left. \left. e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - e^{\lambda \xi_{1,1}} \lambda \xi_{2,1} \right) + \lambda \xi_{1,1}^3 \left( e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,2} \right) + \right. \\
 & \quad \left. \xi_{1,1}^2 \left( -\xi_{1,2} \left( -1 + e^{\lambda \xi_{1,1}} + 2 e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - 2 e^{\lambda \xi_{1,1}} \lambda \xi_{2,1} \right) - \right. \right. \\
 & \quad \left. \left. \left( -1 + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{1,2} - e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{2,1} \right) \xi_{2,2} \right) \right) + \\
 & \frac{1}{\xi_{1,2}^3 \xi_{2,1}^3} \eta_{2,2} \left( \xi_{1,2} \xi_{1,2} \left( -1 + e^{\lambda \xi_{1,2}} - e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2 \right) \xi_{2,1}^3 + \xi_{1,2}^3 \right. \\
 & \quad \left( \xi_{2,1} \left( 1 - e^{\lambda \xi_{2,1}} + e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} \right) + \xi_{1,2} \left( -2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2 \right) \right) \\
 & \quad \xi_{2,2} - \xi_{1,1} \left( \xi_{1,2} \left( -2 + 2 e^{\lambda \xi_{1,2}} - 2 e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2 \right) \xi_{2,1}^3 + \right. \\
 & \quad \left. \xi_{1,2}^3 \left( -2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2 \right) \xi_{2,2} \right) \right), \\
 f_{2,2}[\lambda] \rightarrow & - \frac{1}{\xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2} \eta_{1,2} \left( - \left( (-1 + e^{\lambda \xi_{1,1}}) \xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 \right) + \right. \\
 & \quad \xi_{1,1} \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) \left( -2 + 2 e^{\lambda \xi_{1,1}} + e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - e^{\lambda \xi_{1,1}} \lambda \xi_{2,1} \right) + \\
 & \quad \lambda \xi_{1,1}^3 \left( e^{\lambda \xi_{1,1}} \xi_{1,2} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \xi_{2,2} \right) + \\
 & \quad \left. \xi_{1,1}^2 \left( -\xi_{1,2} \left( -1 + e^{\lambda \xi_{1,1}} + 2 e^{\lambda \xi_{1,1}} \lambda \xi_{1,2} - 2 e^{\lambda \xi_{1,1}} \lambda \xi_{2,1} \right) - \right. \right. \\
 & \quad \left. \left. \left( -1 + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} + e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{1,2} - e^{\lambda (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})} \lambda \xi_{2,1} \right) \xi_{2,2} \right) \right) + \\
 & \frac{1}{\xi_{1,2}^3 \xi_{2,1}^3} \eta_{2,2} \left( \xi_{1,1} \left( \xi_{1,2} \left( -2 + 2 e^{\lambda \xi_{1,2}} - 2 e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2 \right) \xi_{2,1}^3 + \right. \right. \\
 & \quad \left. \left. \xi_{1,2}^3 \left( -2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2 \right) \xi_{2,2} \right) - \right. \\
 & \quad \left. \xi_{1,2} \left( \xi_{1,2} \left( -1 + e^{\lambda \xi_{1,2}} - e^{\lambda \xi_{1,2}} \lambda \xi_{1,2} + e^{\lambda \xi_{1,2}} \lambda^2 \xi_{1,2}^2 \right) \xi_{2,1}^3 + \xi_{1,2}^2 \left( \xi_{2,1} \left( 1 - e^{\lambda \xi_{2,1}} + \right. \right. \right. \right. \\
 & \quad \left. \left. \left. e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} \right) + \xi_{1,2} \left( -2 + 2 e^{\lambda \xi_{2,1}} - 2 e^{\lambda \xi_{2,1}} \lambda \xi_{2,1} + e^{\lambda \xi_{2,1}} \lambda^2 \xi_{2,1}^2 \right) \right) \xi_{2,2} \right) \right), \\
 f_{1,2}[\lambda] \rightarrow & \frac{1}{4 \xi_{1,1}^3 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3} e^{-\lambda (\xi_{1,1} + \xi_{2,1})} \left( \frac{1}{(2 \xi_{1,1} - \xi_{1,2} + \xi_{2,1})^3} \right. \\
 & \left( \eta_{1,2} \left( 33 \xi_{1,1}^2 \xi_{1,2}^2 (\xi_{1,2} - \xi_{2,1})^4 - 9 \xi_{1,1} \xi_{1,2}^2 (\xi_{1,2} - \xi_{2,1})^5 + \xi_{1,2}^2 (\xi_{1,2} - \xi_{2,1})^6 + \right. \right. \\
 & \quad 8 \xi_{1,1}^6 (\xi_{1,2} + \xi_{2,2})^2 - 12 \xi_{1,1}^5 (\xi_{1,2} - \xi_{2,1}) \left( 3 \xi_{1,2}^2 + 4 \xi_{1,2} \xi_{2,2} + \xi_{2,2}^2 \right) + \\
 & \quad 6 \xi_{1,1}^4 (\xi_{1,2} - \xi_{2,1})^2 \left( 11 \xi_{1,2}^2 + 8 \xi_{1,2} \xi_{2,2} + \xi_{2,2}^2 \right) - \\
 & \quad \left. \left. \xi_{1,1}^3 (\xi_{1,2} - \xi_{2,1})^3 \left( 63 \xi_{1,2}^2 + 16 \xi_{1,2} \xi_{2,2} + \xi_{2,2}^2 \right) \right) + \right. \\
 & \left( 16 \eta_{2,2} \xi_{1,1}^3 \left( 2 \xi_{1,1}^2 - 3 \xi_{1,1} (\xi_{1,2} - \xi_{2,1}) + (\xi_{1,2} - \xi_{2,1})^2 \right)^3 \right. \\
 & \quad \left( -\xi_{1,2}^2 \xi_{1,2} (\xi_{1,2} - 2 \xi_{2,1})^4 \xi_{2,1}^4 - \xi_{1,2} \xi_{1,2}^2 (\xi_{1,2} - 2 \xi_{2,1})^4 \left( 3 \xi_{1,2} - \xi_{2,1} \right) \right. \\
 & \quad \left. \xi_{2,2} + \xi_{1,2}^2 \xi_{2,1}^4 \left( -2 \xi_{1,2} + \xi_{2,1} \right) \xi_{2,2}^2 + 2 \xi_{1,1}^9 (\xi_{1,2} + \xi_{2,2})^2 - 3 \xi_{1,1}^8 \right. \\
 & \quad \left. (\xi_{1,2} + \xi_{2,2}) (\xi_{1,2} \left( 3 \xi_{1,2} - 8 \xi_{2,1} \right) - (2 \xi_{1,2} + 3 \xi_{2,1}) \xi_{2,2}) - 2 \right. \\
 & \quad \left. \xi_{1,1} \left( \xi_{1,2}^2 (\xi_{1,2} - 2 \xi_{2,1})^3 \xi_{2,1}^3 \left( 2 \xi_{1,2}^2 - 7 \xi_{1,2} \xi_{2,1} + 2 \xi_{2,1}^2 \right) - \right. \right. \\
 & \quad \left. \left. 2 \xi_{1,2} \xi_{1,2}^3 (\xi_{1,2} - 2 \xi_{2,1})^3 \left( \xi_{1,2}^2 + 4 \xi_{1,2} \xi_{2,1} - 2 \xi_{2,1}^2 \right) \xi_{2,2} + \right. \right. \\
 & \quad \left. \left. \xi_{1,2}^3 \xi_{2,1}^3 \left( 4 \xi_{1,2}^2 + \xi_{1,2} \xi_{2,1} - 2 \xi_{2,1}^2 \right) \xi_{2,2}^2 \right) + 4 \right. \\
 & \quad \left. \xi_{1,1}^7 \left( \xi_{1,2}^2 \left( 4 \xi_{1,2}^2 - 23 \xi_{1,2} \xi_{2,1} + 31 \xi_{2,1}^2 \right) + 2 \xi_{1,2} \right. \right. \\
 & \quad \left. \left. \left( -2 \xi_{1,2}^2 + \xi_{1,2} \xi_{2,1} + 13 \xi_{2,1}^2 \right) \xi_{2,2} + \left( \xi_{1,2}^2 + 7 \xi_{1,2} \xi_{2,1} + 4 \xi_{2,1}^2 \right) \xi_{2,2}^2 \right) - \right. \\
 & \quad \left. 2 \xi_{1,1}^6 \left( \xi_{1,2}^2 \left( 7 \xi_{1,2}^2 - 68 \xi_{1,2} \xi_{2,1} + 199 \xi_{1,2} \xi_{2,1}^2 - 180 \xi_{2,1}^3 \right) - \right. \right. \\
 & \quad \left. \left. 2 \xi_{1,2} \left( 3 \xi_{1,2}^3 - 31 \xi_{1,2}^2 \xi_{2,1} + 32 \xi_{1,2} \xi_{2,1}^2 + 38 \xi_{2,1}^3 \right) \xi_{2,2} + \right. \right. \\
 & \quad \left. \left. \left( 2 \xi_{1,2}^3 - 11 \xi_{1,2}^2 \xi_{2,1} - 26 \xi_{1,2} \xi_{2,1}^2 - 7 \xi_{2,1}^3 \right) \xi_{2,2}^2 \right) - 6 \xi_{1,1}^5 \right. \\
 & \quad \left. \left( \xi_{1,2}^2 (\xi_{1,2} - 2 \xi_{2,1})^2 \xi_{2,1}^2 \left( \xi_{1,2}^3 - 8 \xi_{1,2}^2 \xi_{2,1} + 17 \xi_{1,2} \xi_{2,1}^2 - 8 \xi_{2,1}^3 \right) - \right. \right. \\
 & \quad \left. \left. 2 \xi_{1,2} \xi_{1,2}^2 (\xi_{1,2} - 2 \xi_{2,1})^2 \left( \xi_{1,2}^3 - 5 \xi_{1,2}^2 \xi_{2,1} + 2 \xi_{2,1}^3 \right) \xi_{2,2} + \xi_{1,2}^2 \xi_{2,1}^2 \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 & (2 \xi_{1,2}^3 + 3 \xi_{1,2}^2 \xi_{2,1} - 2 \xi_{1,2} \xi_{2,1}^2 - \xi_{2,1}^3) \xi_{2,1,2}^2 - 4 \xi_{1,1}^3 (\xi_{1,2,2} - \xi_{2,1,1}) \\
 & (\xi_{1,2}^2 \xi_{2,1,1} (\xi_{1,2,2}^4 - 16 \xi_{1,2,2}^3 \xi_{2,1,1} + 82 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 170 \xi_{1,2,2} \xi_{2,1,1}^3 + 124 \xi_{2,1,1}^4) + \\
 & 2 \xi_{1,1,2} \xi_{1,2,2} (2 \xi_{1,2,2}^4 - \xi_{1,2,2}^3 \xi_{2,1,1} - 28 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 40 \xi_{1,2,2} \xi_{2,1,1}^3 + 8 \xi_{2,1,1}^4) \xi_{2,1,2} + \\
 & \xi_{1,2,2} \xi_{2,1,1} (2 \xi_{1,2,2}^3 + 10 \xi_{1,2,2}^2 \xi_{2,1,1} + 8 \xi_{1,2,2} \xi_{2,1,1}^2 + \xi_{2,1,1}^3) \xi_{2,1,2}^2) + 6 \xi_{1,1,1}^5 \\
 & (\xi_{1,2}^2 (\xi_{1,2,2}^4 - 16 \xi_{1,2,2}^3 \xi_{2,1,1} + 80 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 158 \xi_{1,2,2} \xi_{2,1,1}^3 + 107 \xi_{2,1,1}^4) + \\
 & 4 \xi_{1,1,2} (\xi_{1,2,2}^4 - \xi_{1,2,2}^3 \xi_{2,1,1} - 11 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 14 \xi_{1,2,2} \xi_{2,1,1}^3 + 4 \xi_{2,1,1}^4) \xi_{2,1,2} + \\
 & (-\xi_{1,2,2}^4 - 2 \xi_{1,2,2}^3 \xi_{2,1,1} + 8 \xi_{1,2,2}^2 \xi_{2,1,1}^2 + 8 \xi_{1,2,2} \xi_{2,1,1}^3 + \xi_{2,1,1}^4) \xi_{2,1,2}^2) + \\
 & \xi_{1,1,1}^4 (-\xi_{1,2,2}^2 (\xi_{1,2,2}^5 - 32 \xi_{1,2,2}^4 \xi_{2,1,1} + 268 \xi_{1,2,2}^3 \xi_{2,1,1}^2 - 912 \xi_{1,2,2}^2 \xi_{2,1,1}^3 + \\
 & 1353 \xi_{1,2,2} \xi_{2,1,1}^4 - 720 \xi_{2,1,1}^5) - 2 \xi_{1,1,2} (9 \xi_{1,2,2}^5 - 87 \xi_{1,2,2}^4 \xi_{2,1,1} + \\
 & 168 \xi_{1,2,2}^3 \xi_{2,1,1}^2 + 28 \xi_{1,2,2}^2 \xi_{2,1,1}^3 - 152 \xi_{1,2,2} \xi_{2,1,1}^4 - 8 \xi_{2,1,1}^5) \xi_{2,1,2} + (-2 \xi_{1,2,2}^5 - \\
 & 23 \xi_{1,2,2}^4 \xi_{2,1,1} - 8 \xi_{1,2,2}^3 \xi_{2,1,1}^2 + 52 \xi_{1,2,2}^2 \xi_{2,1,1}^3 + 22 \xi_{1,2,2} \xi_{2,1,1}^4 + \xi_{2,1,1}^5) \xi_{2,1,2}^2)) / \\
 & ((\xi_{1,1,1} + \xi_{1,2,2})^4 (\xi_{1,1,1} + \xi_{2,1,1})^4 (\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})^4) - \\
 & \frac{1}{\xi_{1,2,2}^3 \xi_{2,1,1}^3} \xi_{1,1,1} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1}) \left( e^{\lambda (\xi_{1,1,1} + \xi_{1,2,2})} \xi_{1,1,1} \xi_{1,2,2}^2 \xi_{2,1,1}^3 \right. \\
 & \left( \frac{16 \eta_{2,1,2} \xi_{1,1,1} (2 \xi_{1,1,1} - \xi_{1,2,2}) \xi_{1,2,2}^3 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2}{(\xi_{1,1,1} + \xi_{1,2,2})^4} + \right. \\
 & \left. \frac{4 \lambda^3 \eta_{2,1,2} \xi_{1,1,1}^2 (\xi_{1,1,1} - \xi_{1,2,2}) \xi_{1,2,2}^2 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2}{\xi_{1,1,1} + \xi_{1,2,2}} - \right. \\
 & \left. \frac{1}{(\xi_{1,1,1} + \xi_{1,2,2})^2} 4 \lambda^2 \eta_{2,1,2} \xi_{1,1,1} \xi_{1,2,2} (2 \xi_{1,1,1}^3 + 3 \xi_{1,1,1}^2 \xi_{1,2,2} - 4 \xi_{1,1,1} \xi_{1,2,2}^2 + \xi_{1,2,2}^3) \right. \\
 & \left. (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 + \frac{1}{(\xi_{1,1,1} + \xi_{1,2,2})^3} 4 \lambda \eta_{2,1,2} \xi_{1,1,1} (2 \xi_{1,1,1}^4 + \right. \\
 & \left. 5 \xi_{1,1,1}^3 \xi_{1,2,2} + 3 \xi_{1,1,1}^2 \xi_{1,2,2}^2 - 9 \xi_{1,1,1} \xi_{1,2,2}^3 + 3 \xi_{1,2,2}^4) (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^2 + \right. \\
 & \left. e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2})} \eta_{1,1,2} \xi_{1,2,2}^3 (1 + 2 \lambda \xi_{1,1,1} (1 + 2 \lambda \xi_{1,2,2} - 2 \lambda \xi_{2,1,1})) \right) + \\
 & \frac{1}{\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1}} e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})} \xi_{1,1,1}^2 \xi_{1,2,2}^3 \\
 & \left( \frac{16 \eta_{2,1,2} \xi_{2,1,1}^3 (2 \xi_{1,1,1} - 2 \xi_{1,2,2} + \xi_{2,1,1}) (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3}{(\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})^4} + \right. \\
 & \left. \frac{4 \lambda^3 \eta_{2,1,2} (\xi_{1,1,1} - \xi_{1,2,2}) \xi_{2,1,1}^2 (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^4}{\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1}} + e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2})} \eta_{1,1,2} \right. \\
 & \left. \xi_{2,1,1}^3 (1 - 2 \lambda^2 \xi_{1,2,2}^2 + 2 \lambda \xi_{2,1,1} - 2 \lambda^2 \xi_{2,1,1}^2 + 2 \lambda \xi_{1,1,1} (1 + \lambda \xi_{1,2,2} - \lambda \xi_{2,1,1}) + \right. \\
 & \left. 2 \lambda \xi_{1,2,2} (-1 + 2 \lambda \xi_{2,1,1})) - (4 \lambda^2 \eta_{2,1,2} \xi_{2,1,1} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3 (2 \xi_{1,1,1}^3 - \right. \\
 & \left. 2 \xi_{1,2,2}^3 + 9 \xi_{1,2,2}^2 \xi_{2,1,1} - 8 \xi_{1,2,2} \xi_{2,1,1}^2 + 2 \xi_{2,1,1}^3 + \xi_{1,1,1}^2 (-6 \xi_{1,2,2} + 9 \xi_{2,1,1}) + \right. \\
 & \left. 2 \xi_{1,1,1} (3 \xi_{1,2,2}^2 - 9 \xi_{1,2,2} \xi_{2,1,1} + 4 \xi_{2,1,1}^2)) \right) / (\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})^2 + \\
 & \left. \frac{1}{(\xi_{1,1,1} - \xi_{1,2,2} + 2 \xi_{2,1,1})^3} 4 \lambda \eta_{2,1,2} (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})^3 \right. \\
 & \left. (2 \xi_{1,1,1}^4 + 2 \xi_{1,2,2}^4 - 13 \xi_{1,2,2}^3 \xi_{2,1,1} + 30 \xi_{1,2,2}^2 \xi_{2,1,1}^2 - 20 \xi_{1,2,2} \xi_{2,1,1}^3 + 4 \xi_{2,1,1}^4 + \right. \\
 & \left. \xi_{1,1,1}^3 (-8 \xi_{1,2,2} + 13 \xi_{2,1,1}) + 3 \xi_{1,1,1}^2 (4 \xi_{1,2,2}^2 - 13 \xi_{1,2,2} \xi_{2,1,1} + 10 \xi_{2,1,1}^2) + \right.
 \end{aligned}$$





$$\begin{aligned}
 & e^{\lambda (\xi_{1,1} - \xi_{2,2})} \eta_{1,2} (-\xi_{1,1} + \xi_{2,2} + \xi_{2,1})^4 (\xi_{2,1}^2 (-2 + \lambda \xi_{2,1}) + \xi_{2,2} \\
 & \quad \xi_{2,1} (-2 + \lambda \xi_{2,1})^2 - \xi_{2,2}^2 (2 - 2\lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2) + \xi_{1,1} \\
 & \quad (-\xi_{2,1} (3 - 3\lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2) + \xi_{2,2} (2 - 2\lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2))) \xi_{2,2} / \\
 & (\xi_{2,1}^4 (\xi_{1,1} - \xi_{2,2} + \xi_{2,1})^4 (-\xi_{1,1} + \xi_{2,2} + \xi_{2,1})^4) - (4 \\
 & \quad \eta_{1,2}^2 \\
 & \quad (3 \xi_{1,1} \xi_{1,2} (\xi_{2,2} - \xi_{2,1})^2 - \\
 & \quad \quad \xi_{1,2} (\xi_{2,2} - \xi_{2,1})^3 + \\
 & \quad \quad 3 \xi_{1,1}^2 \xi_{1,2} (-\xi_{2,2} + \xi_{2,1}) + \\
 & \quad \quad \xi_{1,1}^3 (\xi_{1,2} + \xi_{2,2}))) / \\
 & (\xi_{1,1}^3 (\xi_{1,1} - \xi_{2,2} + \xi_{2,1})^3) + \frac{1}{\xi_{2,2}^4 \xi_{2,1}^4}
 \end{aligned}$$

$$\begin{aligned}
 & 4 \\
 & \eta_{1,2} \\
 & \eta_{2,2} \\
 & (\xi_{1,2} \xi_{2,2} \xi_{2,1}^4 + \\
 & \quad \xi_{2,2}^4 (3 \xi_{2,2} - 2 \xi_{2,1}) \xi_{2,2} - \\
 & \quad 3 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^4 + \xi_{2,2}^4 \xi_{2,2})) - \\
 & \quad \quad \quad 1 \\
 & \hline
 & (\xi_{1,1} - 2 \xi_{2,2})^4 (-\xi_{1,1} + \xi_{2,2} + \xi_{2,1})^4
 \end{aligned}$$

$$\begin{aligned}
 & 4 \\
 & \eta_{2,2}^2 \\
 & (2 \xi_{1,1}^5 (\xi_{1,2} + \xi_{2,2}) - \\
 & \quad 2 \xi_{1,1}^2 (\xi_{1,2} (\xi_{2,2} + \xi_{2,1})^2 (7 \xi_{2,2} + 4 \xi_{2,1}) + 4 \xi_{2,2}^2 (14 \xi_{2,2} - 3 \xi_{2,1}) \xi_{2,2}) - \\
 & \quad \xi_{2,2} (\xi_{1,2} (\xi_{2,2} + \xi_{2,1})^4 + 16 \xi_{2,2}^3 (2 \xi_{2,2} - \xi_{2,1}) \xi_{2,2}) + \\
 & \quad 2 \xi_{1,1} (\xi_{1,2} (\xi_{2,2} + \xi_{2,1})^3 (3 \xi_{2,2} + \xi_{2,1}) + 16 \xi_{2,2}^3 (3 \xi_{2,2} - \xi_{2,1}) \xi_{2,2}) + \\
 & \quad 4 \xi_{1,1}^3 (\xi_{1,2} (4 \xi_{2,2}^2 + 7 \xi_{2,2} \xi_{2,1} + 3 \xi_{2,1}^2) + 2 \xi_{2,2} (8 \xi_{2,2} - \xi_{2,1}) \xi_{2,2}) + \\
 & \quad \xi_{1,1}^4 (-\xi_{1,2} (9 \xi_{2,2} + 8 \xi_{2,1}) + (-18 \xi_{2,2} + \xi_{2,1}) \xi_{2,2})) + \\
 & 2 \lambda \eta_{1,2} \left( - \left( (\eta_{1,2} (\xi_{1,2} (\xi_{2,2} - \xi_{2,1})^2 + 2 \xi_{1,1} \xi_{1,2} (-\xi_{2,2} + \xi_{2,1}) + \xi_{1,1}^2 (\xi_{1,2} + \xi_{2,2}))) \right) / \right. \\
 & \quad \left. (\xi_{1,1}^2 (\xi_{1,1} - \xi_{2,2} + \xi_{2,1})^2) \right) + \frac{1}{\xi_{2,2}^3 \xi_{2,1}^3} \\
 & \quad \eta_{2,2} (\xi_{1,2} \xi_{2,2} \xi_{2,1}^3 + \xi_{2,2}^3 (2 \xi_{2,2} - \xi_{2,1}) \xi_{2,2} - 2 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^3 + \xi_{2,2}^3 \xi_{2,2})) \left. \right) - \\
 & \quad \quad \quad 1
 \end{aligned}$$

$$\xi_{1,1}^2 \xi_{2,2}^4 \xi_{2,1}^3 (\xi_{1,1} - \xi_{2,2} + \xi_{2,1})^2$$

$$\begin{aligned}
 & 2 \\
 & e^{\lambda (-\xi_{1,1} + \xi_{2,2})} \\
 & (e^{2\lambda \xi_{1,1} - \lambda \xi_{2,2}} \eta_{1,2}^2 \xi_{1,2} \xi_{2,2}^4 \xi_{2,1}^3 \\
 & \quad (2 (\xi_{2,2} - \xi_{2,1}) + \xi_{1,1} (-1 - 2\lambda \xi_{2,2} + 2\lambda \xi_{2,1})) - \\
 & \quad e^{\lambda \xi_{1,1}} \eta_{1,2} \eta_{2,2} \xi_{1,2} \xi_{2,1}^3 (\xi_{1,1}^5 (6 - 4\lambda \xi_{2,2} + \lambda^2 \xi_{2,2}^2) + \\
 & \quad \quad \lambda \xi_{2,2}^4 (\xi_{2,2} - \xi_{2,1})^2 - \lambda \xi_{1,1} \xi_{2,2}^4 (\xi_{2,2} - \xi_{2,1}) (2 + \lambda \xi_{2,2} - \lambda \xi_{2,1}) +
 \end{aligned}$$

$$\begin{aligned} & \xi_{1,1}^2 \xi_{1,2} (\xi_{1,2} - \xi_{2,1}) (3 \lambda \xi_{2,2}^2 + 2 \xi_{2,1} - \xi_{1,2} (4 + \lambda \xi_{2,1})) + \\ & \xi_{1,1}^4 (-2 \lambda^2 \xi_{2,2}^3 + 12 \xi_{2,1} + \lambda \xi_{2,2}^2 (9 + 2 \lambda \xi_{2,1}) - 2 \xi_{1,2} (7 + 4 \lambda \xi_{2,1})) + \\ & \xi_{1,1}^3 (2 \lambda^2 \xi_{2,2}^4 + 6 \xi_{2,1}^2 - 4 \xi_{1,2} \xi_{2,1} (5 + \lambda \xi_{2,1}) - \\ & \quad \lambda \xi_{2,2}^3 (9 + 4 \lambda \xi_{2,1}) + \xi_{2,2}^2 (13 + 14 \lambda \xi_{2,1} + \lambda^2 \xi_{2,1}^2)) + \\ & \lambda \eta_{2,2} \xi_{1,2} (\eta_{1,2} \xi_{2,2}^3 \xi_{2,1}^3 (\xi_{1,2} (\xi_{1,2} - \xi_{2,1})^2 + 2 \xi_{1,1} \xi_{1,2} (-\xi_{1,2} + \xi_{2,1}) + \\ & \quad \xi_{1,1}^2 (\xi_{1,2} + \xi_{2,2})) + \eta_{2,2} \xi_{1,1}^2 (\xi_{1,1} - \xi_{1,2} + \xi_{2,1})^2 (-\xi_{1,2} \xi_{2,2} \xi_{2,1}^3 + \\ & \quad \xi_{2,2}^3 (-2 \xi_{1,2} + \xi_{2,1}) \xi_{2,2} + 2 \xi_{1,1} (\xi_{1,2} \xi_{2,1}^3 + \xi_{2,2}^3 \xi_{2,2}))) \} \end{aligned}$$

```
In[ ]:= SetAlgebra[gl2,ε];
```

```
$PBWRule
```

```
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[ ]:=
```

```
{y1,2 → 1, x1,1 → 2, x1,2 → 3, x2,2 → 4}
```

```
Out[ ]//MatrixForm=
```

$$\left( \begin{array}{cccc} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0 \end{array} \right)$$

```
In[ ]:= sol[-1] = 0 $Basis;
```

```
Do[
```

```
Block[{$k = k},
```

```
EchoLabel["$k"]@$k;
```

```
s0 = EchoLabel["s0"]@Thread[$Basis → sol[k - 1]];
```

```
lhs = EchoLabel["lhs"]@λTangent[
```

```
Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
```

```
];
```

```
rhs = EchoLabel["rhs"]@Simplify@λTangent@Echo[
```

```
$Basis ($Basis /. Echo@{xαβ → (xαβ /. s0) + εk fαβ[λ], yαβ → (yαβ /. s0) + εk gαβ[λ]}]
```

```
];
```

```
{s} = DSolve[
```

```
Join[
```

```
(Coefficient[Simplify@CF[lhs - rhs], #] == 0) & /@ $Basis,
```

```
$Basis /. {xαβ → fαβ[0], yαβ → gαβ[0]} == 0]
```

```
],
```

```
$Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]},
```

```
λ
```

```
];
```

```
sol[k] = EchoLabel["solk"] [
```

```
sol[k - 1] + Simplify[εk ($Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]} /. s)]]
```

```
],
```

```
{k, 0, $k}
```

```
];
```

```
sol[$k]
```

```
» $k 0
```

- »  $s_0 \{y_{1,2} \rightarrow 0, x_{1,1} \rightarrow 0, x_{1,2} \rightarrow 0, x_{2,2} \rightarrow 0\}$
- »  $lhs e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + x_{2,2} \xi_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} + x_{2,2} \xi_{2,2}$
- »  $\{x_{\alpha\beta} \_ \Rightarrow (x_{\alpha\beta} / . s_0) + \epsilon^k f_{\alpha\beta}[\lambda], y_{\alpha\beta} \_ \Rightarrow (y_{\alpha\beta} / . s_0) + \epsilon^k g_{\alpha\beta}[\lambda]\}$
- »  $\{y_{1,2} g_{1,2}[\lambda], x_{1,1} f_{1,1}[\lambda], x_{1,2} f_{1,2}[\lambda], x_{2,2} f_{2,2}[\lambda]\}$
- »  $rhs x_{1,1} f_{1,1}'[\lambda] + e^{f_{2,2}[\lambda]} x_{1,2} (f_{1,2}[\lambda] f_{1,1}'[\lambda] + f_{1,2}'[\lambda]) + x_{2,2} f_{2,2}'[\lambda] + e^{f_{1,1}[\lambda] - f_{2,2}[\lambda]} y_{1,2} g_{1,2}'[\lambda]$
- »  $solk \{\lambda (\eta_{1,2} + e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \eta_{2,2}), \lambda (\xi_{1,1} + \xi_{2,1}), e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2}, \lambda (\xi_{1,2} + \xi_{2,2})\}$
- »  $\$k 1$
- »  $s_0 \{y_{1,2} \rightarrow \lambda (\eta_{1,2} + e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \eta_{2,2}), x_{1,1} \rightarrow \lambda (\xi_{1,1} + \xi_{2,1}), x_{1,2} \rightarrow e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2}, x_{2,2} \rightarrow \lambda (\xi_{1,2} + \xi_{2,2})\}$
- »  $lhs e^{\lambda \xi_{1,1} - \lambda \xi_{1,2} + \lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} y_{1,2} \eta_{2,2} + x_{1,1} \xi_{1,1} - e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,1} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} x_{1,2} \xi_{1,2} + e^{\lambda \xi_{1,2} - \lambda \xi_{2,1} + \lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{1,2} + x_{2,2} \xi_{1,2} + e^{\lambda \xi_{2,1} - \lambda \xi_{2,2}} \lambda y_{1,2} \eta_{2,2} \xi_{1,2} + x_{1,1} \xi_{2,1} + e^{\lambda \xi_{2,2}} x_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,1} \xi_{2,2} - e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{1,2} \xi_{2,2} + e^{\lambda \xi_{2,2}} \lambda x_{1,2} \xi_{2,1} \xi_{2,2} + x_{2,2} \xi_{2,2}$
- »  $\{x_{\alpha\beta} \_ \Rightarrow (x_{\alpha\beta} / . s_0) + \epsilon^k f_{\alpha\beta}[\lambda], y_{\alpha\beta} \_ \Rightarrow (y_{\alpha\beta} / . s_0) + \epsilon^k g_{\alpha\beta}[\lambda]\}$
- »  $\{y_{1,2} (\lambda (\eta_{1,2} + e^{\lambda (-\xi_{1,1} + \xi_{1,2})} \eta_{2,2}) + g_{1,2}[\lambda]), x_{1,1} (\lambda (\xi_{1,1} + \xi_{2,1}) + f_{1,1}[\lambda]), x_{1,2} (e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{1,2}} \lambda \xi_{2,2} + f_{1,2}[\lambda]), x_{2,2} (\lambda (\xi_{1,2} + \xi_{2,2}) + f_{2,2}[\lambda])\}$

» rhs 
$$\begin{aligned} & e^{-\lambda (3 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \left( e^{\lambda (4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} x_{1,2} \xi_{1,1,2} + e^{\lambda (\xi_{1,1,1} + 3 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda x_{2,2} \eta_{1,2} \xi_{1,1,2} + \right. \\ & e^{\lambda (4 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda x_{2,2} \eta_{2,1,2} \xi_{1,1,2} + e^{\lambda (4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \lambda x_{1,2} \xi_{1,1,1} \xi_{1,1,2} - \\ & e^{\lambda (4 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^2 x_{2,2} \eta_{2,1,2} \xi_{1,1,1} \xi_{1,1,2} - e^{\lambda (\xi_{1,1,1} + 4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^2 x_{1,2} \eta_{1,1,2} \xi_{1,1,2}^2 - \\ & e^{\lambda (5 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^2 x_{1,2} \eta_{2,1,2} \xi_{1,1,2}^2 + e^{\lambda (5 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,1,1} \xi_{1,1,2}^2 + \\ & e^{\lambda (3 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} x_{2,2} \xi_{1,2,2} + e^{\lambda (4 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^2 x_{2,2} \eta_{2,1,2} \xi_{1,1,2} \xi_{1,2,2} - \\ & e^{\lambda (5 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,1,2}^2 \xi_{1,2,2} + e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} x_{1,2} \xi_{2,1,2} + \\ & e^{\lambda (\xi_{1,1,1} + 2 \xi_{1,2,2} + 3 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda x_{2,2} \eta_{1,1,2} \xi_{2,1,2} + e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda x_{2,2} \eta_{2,1,2} \xi_{2,1,2} + \\ & e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \lambda x_{1,2} \xi_{1,1,1} \xi_{2,1,2} - e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^2 x_{2,2} \eta_{2,1,2} \xi_{1,1,1} \xi_{2,1,2} - \\ & 2 e^{\lambda (\xi_{1,1,1} + 3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in \lambda^2 x_{1,2} \eta_{1,1,2} \xi_{1,1,2} \xi_{2,1,2} - 2 e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^2 x_{1,2} \eta_{2,1,2} \xi_{1,1,2} \xi_{2,1,2} + \\ & 2 e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,1,1} \xi_{1,1,2} \xi_{2,1,2} - e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \lambda x_{1,2} \xi_{1,2,2} \xi_{2,1,2} + \\ & e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^2 x_{2,2} \eta_{2,1,2} \xi_{1,2,2} \xi_{2,1,2} - 2 e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,1,2} \xi_{1,2,2} \xi_{2,1,2} + \\ & e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \lambda x_{1,2} \xi_{2,1,1} \xi_{2,1,2} - e^{\lambda (\xi_{1,1,1} + 2 \xi_{1,2,2} + 3 \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^2 x_{1,2} \eta_{1,1,2} \xi_{2,1,2}^2 - \\ & e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^2 x_{1,2} \eta_{2,1,2} \xi_{2,1,2}^2 + e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,1,1} \xi_{2,1,2}^2 - \\ & e^{\lambda (3 \xi_{1,2,2} + 3 \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda^3 x_{1,2} \eta_{2,1,2} \xi_{1,2,2} \xi_{2,1,2}^2 + e^{\lambda (3 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} x_{2,2} \xi_{2,2,2} + \\ & e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in x_{1,2} \xi_{1,1,1} f_{1,2} [\lambda] + e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in x_{1,2} \xi_{2,1,1} f_{1,2} [\lambda] + \\ & e^{\lambda (4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in x_{1,2} \xi_{1,1,2} f_{2,2} [\lambda] + e^{\lambda (4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda x_{1,2} \xi_{1,1,1} \xi_{1,1,2} f_{2,2} [\lambda] + \\ & e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in x_{1,2} \xi_{2,1,2} f_{2,2} [\lambda] + e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in \lambda x_{1,2} \xi_{1,1,1} \xi_{2,1,2} f_{2,2} [\lambda] - \\ & e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in \lambda x_{1,2} \xi_{1,2,2} \xi_{2,1,2} f_{2,2} [\lambda] + e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in \lambda x_{1,2} \xi_{2,1,1} \xi_{2,1,2} f_{2,2} [\lambda] + \\ & e^{\lambda (4 \xi_{1,2,2} + \xi_{2,1,1} + 2 \xi_{2,2,2})} \in \lambda x_{1,2} \xi_{1,1,2} f_{1,1}' [\lambda] + e^{\lambda (3 \xi_{1,2,2} + 2 (\xi_{2,1,1} + \xi_{2,2,2}))} \in \lambda x_{1,2} \xi_{2,1,2} f_{1,1}' [\lambda] + \\ & e^{\lambda (2 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} x_{1,1} \left( -e^{\lambda \xi_{1,1,1}} \in \lambda \eta_{1,1,2} \left( e^{\lambda \xi_{1,2,2}} \xi_{1,1,2} + e^{\lambda \xi_{2,1,1}} \xi_{2,1,2} \right) + \right. \\ & \quad e^{\lambda \xi_{1,2,2}} \xi_{1,1,1} \left( 1 + \in \lambda^2 \eta_{2,1,2} \left( e^{\lambda \xi_{1,2,2}} \xi_{1,1,2} + e^{\lambda \xi_{2,1,1}} \xi_{2,1,2} \right) \right) + \\ & \quad \left. e^{\lambda \xi_{1,2,2}} \left( \xi_{2,1,1} - \in \lambda \eta_{2,1,2} \left( 1 + \lambda \xi_{1,2,2} \right) \left( e^{\lambda \xi_{1,2,2}} \xi_{1,1,2} + e^{\lambda \xi_{2,1,1}} \xi_{2,1,2} \right) + \in f_{1,1}' [\lambda] \right) \right) + \\ & e^{2 \lambda (2 \xi_{1,2,2} + \xi_{2,1,1} + \xi_{2,2,2})} \in x_{1,2} f_{1,2}' [\lambda] + e^{\lambda (3 \xi_{1,2,2} + 2 \xi_{2,1,1} + \xi_{2,2,2})} \in x_{2,2} f_{2,2}' [\lambda] + \\ & e^{2 \lambda \xi_{1,2,2} + 3 \lambda \xi_{2,1,1}} y_{1,2} \left( e^{\lambda \xi_{1,1,1}} \eta_{1,1,2} \left( 1 + \in f_{1,1} [\lambda] - \in f_{2,2} [\lambda] \right) + \right. \\ & \quad \left. e^{\lambda \xi_{1,2,2}} \eta_{2,1,2} \left( 1 - \lambda \xi_{1,1,1} + \lambda \xi_{1,2,2} \right) \left( 1 + \in f_{1,1} [\lambda] - \in f_{2,2} [\lambda] \right) + e^{\lambda \xi_{1,1,1}} \in g_{1,2}' [\lambda] \right) \end{aligned}$$

Out[\*]=

\$Aborted

Out[\*]=

sol[1]

In[\*]:= Simplify[ $e^{\lambda (-\xi_{1,1,1} - \xi_{2,1,1})} \lambda \left( e^{\lambda \xi_{1,1,1}} \xi_{1,1,2} + e^{\lambda (\xi_{1,1,1} - \xi_{1,2,2} + \xi_{2,1,1})} \xi_{2,1,2} \right)$ ]

Out[\*]=

$e^{-\lambda \xi_{2,1,1}} \lambda \xi_{1,1,2} + e^{-\lambda \xi_{1,2,2}} \lambda \xi_{2,1,2}$

In[\*]:= \$k

Out[\*]=

1

In[\*]:= sol1 /. λ → 1

Out[\*]=

sol1

$$\begin{aligned}
 \text{In[*]} := \text{osol} = & \mathbb{E}_{\{1,2\} \rightarrow \{2\}} \left[ x_{1,1}[2] \xi_{1,1}[1] + x_{1,1}[2] \xi_{1,1}[2] + e^{-\xi_{1,1}[2]} x_{1,2}[2] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[2] \xi_{1,2}[2] + \right. \\
 & e^{-\xi_{1,1}[2]} x_{1,3}[2] \xi_{1,3}[1] + e^{-\xi_{3,3}[1]} x_{1,3}[2] \xi_{1,3}[2] + e^{-\xi_{1,1}[2]} x_{1,4}[2] \xi_{1,4}[1] + \\
 & e^{-\xi_{4,4}[1]} x_{1,4}[2] \xi_{1,4}[2] + e^{-\xi_{1,1}[2]} x_{1,5}[2] \xi_{1,5}[1] + e^{-\xi_{5,5}[1]} x_{1,5}[2] \xi_{1,5}[2] + \\
 & x_{2,2}[2] \xi_{2,2}[1] + x_{2,2}[2] \xi_{2,2}[2] + e^{-\xi_{2,2}[2]} x_{2,3}[2] \xi_{2,3}[1] - x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + \\
 & e^{-\xi_{3,3}[1]} x_{2,3}[2] \xi_{2,3}[2] + e^{-\xi_{2,2}[2]} x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] + \\
 & e^{-\xi_{4,4}[1]} x_{2,4}[2] \xi_{2,4}[2] + e^{-\xi_{2,2}[2]} x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + \\
 & e^{-\xi_{5,5}[1]} x_{2,5}[2] \xi_{2,5}[2] + x_{3,3}[2] \xi_{3,3}[1] + x_{3,3}[2] \xi_{3,3}[2] + e^{-\xi_{3,3}[2]} x_{3,4}[2] \xi_{3,4}[1] - \\
 & x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + e^{-\xi_{4,4}[1]} x_{3,4}[2] \xi_{3,4}[2] + \\
 & e^{-\xi_{3,3}[2]} x_{3,5}[2] \xi_{3,5}[1] - x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + \\
 & e^{-\xi_{5,5}[1]} x_{3,5}[2] \xi_{3,5}[2] + x_{4,4}[2] \xi_{4,4}[1] + x_{4,4}[2] \xi_{4,4}[2] + e^{-\xi_{4,4}[2]} x_{4,5}[2] \xi_{4,5}[1] - \\
 & x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + \\
 & e^{-\xi_{5,5}[1]} x_{4,5}[2] \xi_{4,5}[2] + x_{5,5}[2] \xi_{5,5}[1] + x_{5,5}[2] \xi_{5,5}[2], \theta, \theta \text{ [[1]]} / . \xi_{i_,i}_[] \rightarrow \theta
 \end{aligned}$$

Out[\*]=

$$\begin{aligned}
 & x_{1,2}[2] \xi_{1,2}[1] + x_{1,2}[2] \xi_{1,2}[2] + x_{1,3}[2] \xi_{1,3}[1] + x_{1,3}[2] \xi_{1,3}[2] + \\
 & x_{1,4}[2] \xi_{1,4}[1] + x_{1,4}[2] \xi_{1,4}[2] + x_{1,5}[2] \xi_{1,5}[1] + x_{1,5}[2] \xi_{1,5}[2] + x_{2,3}[2] \xi_{2,3}[1] - \\
 & x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + x_{2,3}[2] \xi_{2,3}[2] + x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] + \\
 & x_{2,4}[2] \xi_{2,4}[2] + x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + x_{2,5}[2] \xi_{2,5}[2] + x_{3,4}[2] \xi_{3,4}[1] - \\
 & x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + x_{3,4}[2] \xi_{3,4}[2] + x_{3,5}[2] \xi_{3,5}[1] - \\
 & x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + x_{3,5}[2] \xi_{3,5}[2] + x_{4,5}[2] \xi_{4,5}[1] - \\
 & x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + x_{4,5}[2] \xi_{4,5}[2]
 \end{aligned}$$

$$\text{In[*]} := \text{Expand@Total[sol1 /. \{ \xi_{\alpha\beta} \Rightarrow \xi_{\alpha\beta}[1], \eta_{\alpha\beta} \Rightarrow \xi_{\alpha\beta}[2], \text{Rule} \rightarrow \text{Times}, \lambda \rightarrow 1, f_{\alpha\beta}[_] \Rightarrow x_{\alpha\beta}[2] \}}] - \text{osol}$$

Out[\*]=

$$\begin{aligned}
 \text{Total[sol1]} - & x_{1,2}[2] \xi_{1,2}[1] - x_{1,2}[2] \xi_{1,2}[2] - x_{1,3}[2] \xi_{1,3}[1] - x_{1,3}[2] \xi_{1,3}[2] - \\
 & x_{1,4}[2] \xi_{1,4}[1] - x_{1,4}[2] \xi_{1,4}[2] - x_{1,5}[2] \xi_{1,5}[1] - x_{1,5}[2] \xi_{1,5}[2] - x_{2,3}[2] \xi_{2,3}[1] + \\
 & x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] - x_{2,3}[2] \xi_{2,3}[2] - x_{2,4}[2] \xi_{2,4}[1] + x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] - \\
 & x_{2,4}[2] \xi_{2,4}[2] - x_{2,5}[2] \xi_{2,5}[1] + x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] - x_{2,5}[2] \xi_{2,5}[2] - x_{3,4}[2] \xi_{3,4}[1] + \\
 & x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] + x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] - x_{3,4}[2] \xi_{3,4}[2] - x_{3,5}[2] \xi_{3,5}[1] + \\
 & x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] + x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] - x_{3,5}[2] \xi_{3,5}[2] - x_{4,5}[2] \xi_{4,5}[1] + \\
 & x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] + x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] + x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] - x_{4,5}[2] \xi_{4,5}[2]
 \end{aligned}$$