Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the λ-tangent formalism. Some material from pensieve://Projects/UEA.

In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Prolog

In[•]:=
```
BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
    algebras and PBW algebras. It is in the public domain, available
    at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
    Dror Bar-Natan is committed to support it within
    reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
  Sort@{"**", ϵ, $k, CF, SSQ, NilQ, B, m, SetAlgebra, U,
    UB, UProducts, USimp, UU, $Basis, $PBWRule, δ, adPower, adExp},
  "."];
Begin["`Private`"];
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ,
  SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ, ϵ, $Basis, $k, $PBWRule}.
```

Utilities

In[•]:=
```
δ_,_ := If[i == j, 1, 0];
```

In[•]:=
```
SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

In[•]:=
```
$k = 1;
```

In[•]:=
```
CF[ℰ_] := Expand[ℰ] /. {ϵ^k_. /; k > $k → 0};
```

Implementing general universal enveloping algebras

```
In[•]:= B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
In[•]:= x_ ≼ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :> U_i[xs],
    x_ /; MemberQ[$Basis, x] :> U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
In[•]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

```
In[•]:= m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≼ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

```
In[•]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :> i, ∞];
```

```
In[•]:= Unprotect[NonCommutativeMultiply];
        NonCommutativeMultiply[x_] := x;
        x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
            z = x; Do[z = m_{i→σ@i}[z], {i, is}];
            z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
        UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
In[•]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[•]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
In[•]:= SetAlgebra["sl2"] := (
            Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
            B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
            $Basis = {e, h, f};
            $PBWRule = {e → 1, h → 2, f → 3};
            NilQ[e] = NilQ[f] = True; NilQ[h] = False;
           );
```

$gl_{n,\epsilon}$

```
In[•]:= Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

```
UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}.
```

In[•]:=
```
SetAlgebra[gl_{n_,ϵ}] := (
  $Basis = Flatten@{
     Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
     Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
    };
  NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
  B[y_{i_,j_}, y_{k_,l_}] := CF[ϵ δ_{j,k} y_{i,l} - ϵ δ_{l,i} y_{k,j}];
  B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :> If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
 );
```

In[•]:=
```
SetAlgebra[gl_{2,ϵ}];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

Out[•]=

$$\{y_{1,2} \to 1,\ x_{1,1} \to 2,\ x_{1,2} \to 3,\ x_{2,2} \to 4\}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \to 0 & \{y_{1,2}, x_{1,1}\} \to y_{1,2} & \{y_{1,2}, x_{1,2}\} \to -\epsilon\, x_{1,1} + \epsilon\, x_{2,2} & \{y_{1,2}, x_{2,2}\} \to -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \to -y_{1,2} & \{x_{1,1}, x_{1,1}\} \to 0 & \{x_{1,1}, x_{1,2}\} \to x_{1,2} & \{x_{1,1}, x_{2,2}\} \to 0 \\ \{x_{1,2}, y_{1,2}\} \to \epsilon\, x_{1,1} - \epsilon\, x_{2,2} & \{x_{1,2}, x_{1,1}\} \to -x_{1,2} & \{x_{1,2}, x_{1,2}\} \to 0 & \{x_{1,2}, x_{2,2}\} \to x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \to y_{1,2} & \{x_{2,2}, x_{1,1}\} \to 0 & \{x_{2,2}, x_{1,2}\} \to -x_{1,2} & \{x_{2,2}, x_{2,2}\} \to 0 \end{pmatrix}$$

In[•]:=
```
Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_, 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];
```

In[•]:=
```
x0 = ($Basis /. {x → ξ0, y → η0}).$Basis
x1 = ($Basis /. {x → ξ1, y → η1}).$Basis
```

Out[•]=

$$y_{1,2}\,\eta 0_{1,2} + x_{1,1}\,\xi 0_{1,1} + x_{1,2}\,\xi 0_{1,2} + x_{2,2}\,\xi 0_{2,2}$$

Out[•]=

$$y_{1,2}\,\eta 1_{1,2} + x_{1,1}\,\xi 1_{1,1} + x_{1,2}\,\xi 1_{1,2} + x_{2,2}\,\xi 1_{2,2}$$

In[•]:=
```
adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   CF[$Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]].
     Table[Coefficient[y, b], {b, $Basis}]]
  ]
```

In[•]:=
```
SSQ[ξ0_{1,1} x_{1,1}]
```
Out[•]=
```
True
```

In[•]:=
```
adExp[ξ0_{1,1} x_{1,1}][x1]
```
Out[•]=
$$e^{\xi 0_{1,1}}\, y_{1,2}\, \eta 1_{1,2} + x_{1,1}\, \xi 1_{1,1} + e^{-\xi 0_{1,1}}\, x_{1,2}\, \xi 1_{1,2} + x_{2,2}\, \xi 1_{2,2}$$

In[•]:=
```
adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

In[•]:=
```
adExp[ξ0_{1,2} x_{1,2}][x1]
```
Out[•]=
$$y_{1,2}\, \eta 1_{1,2} - \epsilon\, x_{1,1}\, \eta 1_{1,2}\, \xi 0_{1,2} + \epsilon\, x_{2,2}\, \eta 1_{1,2}\, \xi 0_{1,2} - \epsilon\, x_{1,2}\, \eta 1_{1,2}\, \xi 0_{1,2}^2 + x_{1,1}\, \xi 1_{1,1} + x_{1,2}\, \xi 0_{1,2}\, \xi 1_{1,1} + x_{1,2}\, \xi 1_{1,2} + x_{2,2}\, \xi 1_{2,2} - x_{1,2}\, \xi 0_{1,2}\, \xi 1_{2,2}$$

In[•]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λ x;
λTangent[xs_List] := λTangent @@ xs
```

In[•]:=
```
$Basis ($Basis /. {x → ξ, y → η})
```
Out[•]=
$$\{y_{1,2}\, \eta_{1,2},\ x_{1,1}\, \xi_{1,1},\ x_{1,2}\, \xi_{1,2},\ x_{2,2}\, \xi_{2,2}\}$$

In[•]:=
```
λTangent@(λ $Basis ($Basis /. {x → ξ, y → η}))
```
Out[•]=
$$e^{\lambda \xi_{1,1} - \lambda \xi_{2,2}}\, y_{1,2}\, \eta_{1,2} + x_{1,1}\, \xi_{1,1} + e^{\lambda \xi_{2,2}}\, x_{1,2}\, \xi_{1,2} - e^{\lambda \xi_{1,1}}\, \epsilon\, \lambda\, x_{1,1}\, \eta_{1,2}\, \xi_{1,2} + e^{\lambda \xi_{1,1}}\, \epsilon\, \lambda\, x_{2,2}\, \eta_{1,2}\, \xi_{1,2} + e^{\lambda \xi_{2,2}}\, \lambda\, x_{1,2}\, \xi_{1,1}\, \xi_{1,2} - e^{\lambda \xi_{1,1} + \lambda \xi_{2,2}}\, \epsilon\, \lambda^2\, x_{1,2}\, \eta_{1,2}\, \xi_{1,2}^2 + x_{2,2}\, \xi_{2,2}$$

In[•]:=
```
$Basis
```
Out[•]=
$$\{y_{1,2},\ x_{1,1},\ x_{1,2},\ x_{2,2}\}$$

In[•]:=
```
lhs =
 Join[λ$Basis ($Basis /. {x → ξ1, y → η1}), λ$Basis ($Basis /. {x → ξ2, y → η2})] // λTangent
```

Out[•]=

$$\begin{aligned}
&e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,y_{1,2}\,\eta1_{1,2}+e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,y_{1,2}\,\eta2_{1,2}+x_{1,1}\,\xi1_{1,1}-\\
&e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\lambda\,y_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}+e^{\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,x_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{1,1}}\in\lambda\,x_{1,1}\,\eta1_{1,2}\,\xi1_{1,2}+\\
&e^{\lambda\,\xi1_{1,1}}\in\lambda\,x_{2,2}\,\eta1_{1,2}\,\xi1_{1,2}+e^{\lambda\,\xi1_{2,2}}\in\lambda\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}}\in\lambda\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,2}+\\
&2\,e^{\lambda\,\xi1_{1,1}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\in\lambda^2\,y_{1,2}\,\eta1_{1,2}\,\eta2_{1,2}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\in\lambda^2\,y_{1,2}\,\eta2_{1,2}^2\,\xi1_{1,2}+\\
&e^{\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}+e^{\lambda\,\xi1_{2,2}}\in\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}-e^{\lambda\,\xi1_{2,2}}\in\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}-\\
&e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\in\lambda^3\,y_{1,2}\,\eta2_{1,2}^2\,\xi1_{1,1}\,\xi1_{1,2}-e^{\lambda\,\xi1_{1,1}+\lambda\,\xi1_{2,2}-\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi1_{1,2}^2+x_{2,2}\,\xi1_{2,2}+\\
&e^{\lambda\,\xi2_{1,1}-\lambda\,\xi2_{2,2}}\,\lambda\,y_{1,2}\,\eta2_{1,2}\,\xi1_{2,2}+x_{1,1}\,\xi2_{1,1}+e^{\lambda\,\xi2_{2,2}}\,x_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}}\in\lambda\,x_{1,1}\,\eta1_{1,2}\,\xi2_{1,2}+\\
&e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}}\in\lambda\,x_{2,2}\,\eta1_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi2_{1,1}}\in\lambda\,x_{1,1}\,\eta2_{1,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\in\lambda\,x_{2,2}\,\eta2_{1,2}\,\xi2_{1,2}+\\
&e^{\lambda\,\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\in\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}-e^{\lambda\,\xi2_{1,1}}\in\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}-\\
&2\,e^{\lambda\,\xi1_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+2\,e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{2,2}}\in\lambda^2\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,2}\,\xi2_{1,2}+\\
&2\,e^{\lambda\,\xi1_{2,2}+\lambda\,\xi2_{2,2}}\in\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi1_{1,2}\,\xi2_{1,2}-e^{\lambda\,\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}-\\
&e^{\lambda\,\xi2_{1,1}}\in\lambda^2\,x_{1,1}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{1,1}}\in\lambda^2\,x_{2,2}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}+e^{\lambda\,\xi2_{2,2}}\,\lambda\,x_{1,2}\,\xi2_{1,1}\,\xi2_{1,2}-\\
&e^{\lambda\,\xi1_{1,1}-\lambda\,\xi1_{2,2}+\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^2\,x_{1,2}\,\eta1_{1,2}\,\xi2_{1,2}^2-e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^2\,x_{1,2}\,\eta2_{1,2}\,\xi2_{1,2}^2+\\
&e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{1,1}\,\xi2_{1,2}^2-e^{\lambda\,\xi2_{1,1}+\lambda\,\xi2_{2,2}}\in\lambda^3\,x_{1,2}\,\eta2_{1,2}\,\xi1_{2,2}\,\xi2_{1,2}^2+x_{2,2}\,\xi2_{2,2}
\end{aligned}$$

In[•]:=
```
rhs = λTangent@
  ($Basis ($Basis /. {x_α_ :> CF[λ (ξ1_{α,α} + ξ2_{α,α}) + ϵ f_{α,α}[λ]], x_{αβ__} → f_{αβ}[λ], y_{αβ__} → g_{αβ}[λ]}))
```

Out[•]=

$$\begin{aligned}
&x_{1,1}\,\xi1_{1,1}+x_{1,1}\,\xi2_{1,1}+e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\,x_{1,2}\,\xi1_{1,1}\,f_{1,2}[\lambda]+e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\,x_{1,2}\,\xi2_{1,1}\,f_{1,2}[\lambda]+\\
&e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in x_{1,2}\,\xi1_{1,1}\,f_{1,2}[\lambda]\,f_{2,2}[\lambda]+e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in x_{1,2}\,\xi2_{1,1}\,f_{1,2}[\lambda]\,f_{2,2}[\lambda]+\\
&\in x_{1,1}\,f_{1,1}'[\lambda]+e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in x_{1,2}\,f_{1,2}[\lambda]\,f_{1,1}'[\lambda]+e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\,x_{1,2}\,f_{1,2}'[\lambda]+\\
&e^{\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in x_{1,2}\,f_{2,2}[\lambda]\,f_{1,2}'[\lambda]+x_{2,2}\,(\xi1_{2,2}+\xi2_{2,2}+\in f_{2,2}'[\lambda])+\\
&e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})-\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\,y_{1,2}\,g_{1,2}'[\lambda]+e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})-\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in y_{1,2}\,f_{1,1}[\lambda]\,g_{1,2}'[\lambda]-\\
&e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})}\in x_{1,1}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]+e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})}\in x_{2,2}\,f_{1,2}[\lambda]\,g_{1,2}'[\lambda]-\\
&e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})+\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in x_{1,2}\,f_{1,2}[\lambda]^2\,g_{1,2}'[\lambda]-e^{\lambda\,(\xi1_{1,1}+\xi2_{1,1})-\lambda\,(\xi1_{2,2}+\xi2_{2,2})}\in y_{1,2}\,f_{2,2}[\lambda]\,g_{1,2}'[\lambda]
\end{aligned}$$

In[•]:=
```
SSBasis = Select[$Basis, SSQ]
```

Out[•]=

$$\{x_{1,1},\ x_{2,2}\}$$

In[•]:=
```
eqns = (Coefficient[lhs - rhs, #] == 0) & /@ $Basis
```

Out[•]=

$$\Big\{ e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \eta 1_{1,2} + e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \eta 2_{1,2} -
e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \lambda\, \eta 2_{1,2}\, \xi 1_{1,1} + 2\, e^{\lambda \xi 1_{1,1}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \eta 2_{1,2}\, \xi 1_{1,2} -
e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}^2\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}^2\, \xi 1_{1,1}\, \xi 1_{1,2} +
e^{\lambda \xi 2_{1,1}-\lambda \xi 2_{2,2}}\, \lambda\, \eta 2_{1,2}\, \xi 1_{2,2} - e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})-\lambda (\xi 1_{2,2}+\xi 2_{2,2})}\, g_{1,2}'[\lambda] -
e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})-\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,1}[\lambda]\, g_{1,2}'[\lambda] + e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})-\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{2,2}[\lambda]\, g_{1,2}'[\lambda] == 0,$$

$$- e^{\lambda \xi 1_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}} \in \lambda\, \eta 2_{1,2}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2} -
e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 2_{1,2} - e^{\lambda \xi 2_{1,1}} \in \lambda\, \eta 2_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2} -
e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2} - \in f_{1,1}'[\lambda] + e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]\, g_{1,2}'[\lambda] == 0,$$

$$e^{\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}}\, \xi 1_{1,2} + e^{\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{1,1}\, \xi 1_{1,2} - e^{\lambda \xi 1_{1,1}+\lambda \xi 1_{2,2}-\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}^2 +
e^{\lambda \xi 2_{2,2}}\, \xi 2_{1,2} + e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{1,1}\, \xi 2_{1,2} - 2\, e^{\lambda \xi 1_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}\, \xi 2_{1,2} +
2\, e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2}\, \xi 2_{1,2} + 2\, e^{\lambda \xi 1_{2,2}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2}\, \xi 2_{1,2} -
e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 1_{2,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{2,2}}\, \lambda\, \xi 2_{1,1}\, \xi 2_{1,2} - e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 1_{1,2}\, \xi 2_{1,2}^2 -
e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}^2 + e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2}^2 - e^{\lambda \xi 2_{1,1}+\lambda \xi 2_{2,2}} \in \lambda^3\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2}^2 -
e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 1_{1,1}\, f_{1,2}[\lambda] - e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 2_{1,1}\, f_{1,2}[\lambda] - e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in \xi 1_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] -
e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in \xi 2_{1,1}\, f_{1,2}[\lambda]\, f_{2,2}[\lambda] - e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,2}[\lambda]\, f_{1,1}'[\lambda] - e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})}\, f_{1,2}'[\lambda] -
e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{2,2}[\lambda]\, f_{1,2}'[\lambda] + e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})+\lambda (\xi 1_{2,2}+\xi 2_{2,2})} \in f_{1,2}[\lambda]^2\, g_{1,2}'[\lambda] == 0,$$

$$e^{\lambda \xi 1_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}} \in \lambda\, \eta 2_{1,2}\, \xi 1_{1,2} - e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 1_{1,2} +
e^{\lambda \xi 1_{1,1}-\lambda \xi 1_{2,2}+\lambda \xi 2_{1,1}} \in \lambda\, \eta 1_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 2_{1,1}} \in \lambda\, \eta 2_{1,2}\, \xi 2_{1,2} - e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,1}\, \xi 2_{1,2} +
e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{2,2}\, \xi 2_{1,2} - \in f_{2,2}'[\lambda] - e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]\, g_{1,2}'[\lambda] == 0\Big\}$$

In[•]:=
```
Simplify[eqns]
```

Out[•]=

$$\Big\{ e^{-\lambda (\xi 1_{2,2}-\xi 2_{1,1}+\xi 2_{2,2})} \left(- e^{2\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}^2\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 1_{1,1}}\, \eta 1_{1,2} \left(1 + 2\, e^{\lambda \xi 1_{2,2}} \in \lambda^2\, \eta 2_{1,2}\, \xi 1_{1,2}\right) +
e^{\lambda \xi 1_{2,2}}\, \eta 2_{1,2}\, (1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2}) - e^{\lambda \xi 1_{1,1}}\, (1 + \in f_{1,1}[\lambda] - \in f_{2,2}[\lambda])\, g_{1,2}'[\lambda]\right) == 0,$$

$$\in \left(\lambda\, \eta 1_{1,2} \left(e^{\lambda \xi 1_{1,1}}\, \xi 1_{1,2} + e^{\lambda (\xi 1_{1,1}-\xi 1_{2,2}+\xi 2_{1,1})}\, \xi 2_{1,2}\right) -
\lambda\, \eta 2_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, (-1 + \lambda\, \xi 1_{1,1} - \lambda\, \xi 1_{2,2})\, \xi 2_{1,2}\right) +
f_{1,1}'[\lambda] - e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})}\, f_{1,2}[\lambda]\, g_{1,2}'[\lambda]\right) == 0,$$

$$e^{-\lambda (\xi 1_{2,2}+\xi 2_{1,1})} \left(- e^{\lambda (\xi 1_{1,1}+2\, \xi 1_{2,2}+\xi 2_{2,2})} \in \lambda^2\, \eta 1_{1,2}\, \xi 1_{1,2}^2 - e^{\lambda (\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}\, (-1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2} - \lambda\, \xi 2_{1,1})\,
\xi 2_{1,2} - e^{\lambda (2\, \xi 2_{1,1}+\xi 2_{2,2})} \in \lambda^2 \left(e^{\lambda \xi 1_{1,1}}\, \eta 1_{1,2} + e^{\lambda \xi 1_{2,2}}\, \eta 2_{1,2}\, (1 - \lambda\, \xi 1_{1,1} + \lambda\, \xi 1_{2,2})\right) \xi 2_{1,2}^2 +
e^{\lambda (\xi 1_{2,2}+\xi 2_{2,2})}\, \xi 1_{1,2} \left(-2\, e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})} \in \lambda^2\, \eta 1_{1,2}\, \xi 2_{1,2} + e^{\lambda \xi 1_{2,2}} \left(1 + 2\, e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}\right) +
e^{\lambda \xi 1_{2,2}}\, \lambda\, \xi 1_{1,1} \left(1 + 2\, e^{\lambda \xi 2_{1,1}} \in \lambda^2\, \eta 2_{1,2}\, \xi 2_{1,2}\right)\right) -
e^{\lambda (2\, \xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})} \left(\xi 1_{1,1}\, f_{1,2}[\lambda]\, (1 + \in f_{2,2}[\lambda]) + \xi 2_{1,1}\, f_{1,2}[\lambda]\, (1 + \in f_{2,2}[\lambda]) +
\in f_{1,2}[\lambda]\, f_{1,1}'[\lambda] + f_{1,2}'[\lambda] + \in f_{2,2}[\lambda]\, f_{1,2}'[\lambda] - e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})} \in f_{1,2}[\lambda]^2\, g_{1,2}'[\lambda]\right)\right) == 0,$$

$$e^{\lambda (\xi 1_{1,1}-\xi 1_{2,2})} \in \lambda\, \eta 1_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, \xi 2_{1,2}\right) ==
\in \left(\lambda\, \eta 2_{1,2} \left(e^{\lambda \xi 1_{2,2}}\, (1 + \lambda\, \xi 1_{1,1})\, \xi 1_{1,2} + e^{\lambda \xi 2_{1,1}}\, (-1 + \lambda\, \xi 1_{1,1} - \lambda\, \xi 1_{2,2})\, \xi 2_{1,2}\right) +
f_{2,2}'[\lambda] + e^{\lambda (\xi 1_{1,1}+\xi 2_{1,1})}\, f_{1,2}[\lambda]\, g_{1,2}'[\lambda]\right)\Big\}$$

In[•]:=
```
Length@eqns
```

Out[•]=

4

In[•]:=
```
unknowns = $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]}
```

Out[•]=
$$\{g_{1,2}[\lambda],\ f_{1,1}[\lambda],\ f_{1,2}[\lambda],\ f_{2,2}[\lambda]\}$$

In[•]:=
```
$Basis /. {x_αβ__ :> f_αβ[0] == 0, y_αβ__ :> g_αβ[0] == 0}
```

Out[•]=
$$\{g_{1,2}[0] == 0,\ f_{1,1}[0] == 0,\ f_{1,2}[0] == 0,\ f_{2,2}[0] == 0\}$$

In[•]:=
```
{sol1} = DSolve[
  Join[
   Simplify[(Coefficient[lhs - rhs, #] == 0) & /@ $Basis],
   $Basis /. {x_αβ__ :> f_αβ[0] == 0, y_αβ__ :> g_αβ[0] == 0}
  ],
  $Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]},
  λ
 ]
```

Set: Lists {sol1} and
DSolve[{ℯ^-λ (Subscript[≪3≫]+Times[≪2≫]+Subscript[≪3≫]) (-ℯ^Times[≪3≫] ϵ λ^2 Subscript[≪3≫]^2 (1 + Times[≪2≫]) ξ1_1,2 + ℯ^Times[≪2≫] η1_1,2 (1 + Times[≪6≫]) + ℯ^Times[≪2≫] η2_1,2 (1 + Times[≪3≫] + Times[≪2≫]) - ℯ^Times[≪2≫] (1 + Times[≪2≫] + Times[≪3≫]) Subscript[≪3≫]'[λ]) == 0, ϵ (λ η1_1,2 (Times[≪2≫] + Times[≪2≫]) - λ η2_1,2 (Times[≪3≫] + Times[≪3≫]) + f_1,1'[λ] - ℯ^Times[≪2≫] f_1,2[λ] Subscript[≪3≫]'[λ]) == 0, ℯ^-λ (Subscript[≪3≫]+Subscript[≪3≫]) (≪1≫) == 0, ℯ^λ (Subscript[≪3≫]+Times[≪2≫]) ϵ λ η1_1,2 (ℯ^Times[≪2≫] ξ1_1,2 + ℯ^Times[≪2≫] ξ2_1,2) == ϵ (≪1≫), g_1,2[0] == 0, f_1,1[0] == 0, f_1,2[0] == 0, f_2,2[0] == 0}, {g_1,2[λ], f_1,1[λ], f_≪1≫[λ], f_2,2[λ]}, λ] are not the same shape.

Out[•]=
$$\begin{aligned}
&\mathrm{DSolve}\Big[\\
&\Big\{e^{-\lambda(\xi 1_{2,2}-\xi 2_{1,1}+\xi 2_{2,2})}\Big(-e^{2\lambda\xi 1_{2,2}}\epsilon\lambda^2\eta 2_{1,2}^2(1+\lambda\xi 1_{1,1})\xi 1_{1,2}+e^{\lambda\xi 1_{1,1}}\eta 1_{1,2}\Big(1+2e^{\lambda\xi 1_{2,2}}\epsilon\lambda^2\eta 2_{1,2}\xi 1_{1,2}\Big)+\\
&\quad e^{\lambda\xi 1_{2,2}}\eta 2_{1,2}(1-\lambda\xi 1_{1,1}+\lambda\xi 1_{2,2})-e^{\lambda\xi 1_{1,1}}(1+\epsilon f_{1,1}[\lambda]-\epsilon f_{2,2}[\lambda])g_{1,2}'[\lambda]\Big)==0,\\
&\epsilon\Big(\lambda\eta 1_{1,2}\Big(e^{\lambda\xi 1_{1,1}}\xi 1_{1,2}+e^{\lambda(\xi 1_{1,1}-\xi 1_{2,2}+\xi 2_{1,1})}\xi 2_{1,2}\Big)-\lambda\eta 2_{1,2}\Big(e^{\lambda\xi 1_{2,2}}(1+\lambda\xi 1_{1,1})\xi 1_{1,2}+\\
&\quad e^{\lambda\xi 2_{1,1}}(-1+\lambda\xi 1_{1,1}-\lambda\xi 1_{2,2})\xi 2_{1,2}\Big)+f_{1,1}'[\lambda]-e^{\lambda(\xi 1_{1,1}+\xi 2_{1,1})}f_{1,2}[\lambda]g_{1,2}'[\lambda]\Big)==0,\\
&e^{-\lambda(\xi 1_{2,2}+\xi 2_{1,1})}\Big(-e^{\lambda(\xi 1_{1,1}+2\xi 1_{2,2}+\xi 2_{2,2})}\epsilon\lambda^2\eta 1_{1,2}\xi 1_{1,2}^2-e^{\lambda(\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}(-1-\lambda\xi 1_{1,1}+\lambda\xi 1_{2,2}-\lambda\xi 2_{1,1})\\
&\quad \xi 2_{1,2}-e^{\lambda(2\xi 2_{1,1}+\xi 2_{2,2})}\epsilon\lambda^2\Big(e^{\lambda\xi 1_{1,1}}\eta 1_{1,2}+e^{\lambda\xi 1_{2,2}}\eta 2_{1,2}(1-\lambda\xi 1_{1,1}+\lambda\xi 1_{2,2})\Big)\xi 2_{1,2}^2+\\
&\quad e^{\lambda(\xi 1_{2,2}+\xi 2_{2,2})}\xi 1_{1,2}\Big(-2e^{\lambda(\xi 1_{1,1}+\xi 2_{1,1})}\epsilon\lambda^2\eta 1_{1,2}\xi 2_{1,2}+e^{\lambda\xi 1_{2,2}}\Big(1+2e^{\lambda\xi 2_{1,1}}\epsilon\lambda^2\eta 2_{1,2}\xi 2_{1,2}\Big)+\\
&\quad e^{\lambda\xi 1_{2,2}}\lambda\xi 1_{1,1}\Big(1+2e^{\lambda\xi 2_{1,1}}\epsilon\lambda^2\eta 2_{1,2}\xi 2_{1,2}\Big)\Big)-\\
&\quad e^{\lambda(2\xi 1_{2,2}+\xi 2_{1,1}+\xi 2_{2,2})}\Big(\xi 1_{1,1}f_{1,2}[\lambda](1+\epsilon f_{2,2}[\lambda])+\xi 2_{1,1}f_{1,2}[\lambda](1+\epsilon f_{2,2}[\lambda])+\\
&\quad \epsilon f_{1,2}[\lambda]f_{1,1}'[\lambda]+f_{1,2}'[\lambda]+\epsilon f_{2,2}[\lambda]f_{1,2}'[\lambda]-e^{\lambda(\xi 1_{1,1}+\xi 2_{1,1})}\epsilon f_{1,2}[\lambda]^2g_{1,2}'[\lambda]\Big)\Big)==0,\\
&e^{\lambda(\xi 1_{1,1}-\xi 1_{2,2})}\epsilon\lambda\eta 1_{1,2}\Big(e^{\lambda\xi 1_{2,2}}\xi 1_{1,2}+e^{\lambda\xi 2_{1,1}}\xi 2_{1,2}\Big)==\\
&\quad \epsilon\Big(\lambda\eta 2_{1,2}\Big(e^{\lambda\xi 1_{2,2}}(1+\lambda\xi 1_{1,1})\xi 1_{1,2}+e^{\lambda\xi 2_{1,1}}(-1+\lambda\xi 1_{1,1}-\lambda\xi 1_{2,2})\xi 2_{1,2}\Big)+\\
&\quad f_{2,2}'[\lambda]+e^{\lambda(\xi 1_{1,1}+\xi 2_{1,1})}f_{1,2}[\lambda]g_{1,2}'[\lambda]\Big),\\
&g_{1,2}[0]==0,\ f_{1,1}[0]==0,\ f_{1,2}[0]==0,\ f_{2,2}[0]==0\Big\},\ \{g_{1,2}[\lambda],\\
&f_{1,1}[\lambda],\\
&f_{1,2}[\lambda],\\
&f_{2,2}[\lambda]\},\ \lambda\Big]
\end{aligned}$$

In[•]:=
```
$k
```
Out[•]=
```
1
```

In[•]:=
```
sol1 /. λ → 1
```
Out[•]=
```
sol1
```

In[•]:=
```
osol = 𝔼{1,2}→{2}[x1,1[2] ξ1,1[1] + x1,1[2] ξ1,1[2] + e^-ξ1,1[2] x1,2[2] ξ1,2[1] + e^-ξ2,2[1] x1,2[2] ξ1,2[2] +
    e^-ξ1,1[2] x1,3[2] ξ1,3[1] + e^-ξ3,3[1] x1,3[2] ξ1,3[2] + e^-ξ1,1[2] x1,4[2] ξ1,4[1] +
    e^-ξ4,4[1] x1,4[2] ξ1,4[2] + e^-ξ1,1[2] x1,5[2] ξ1,5[1] + e^-ξ5,5[1] x1,5[2] ξ1,5[2] +
    x2,2[2] ξ2,2[1] + x2,2[2] ξ2,2[2] + e^-ξ2,2[2] x2,3[2] ξ2,3[1] - x1,3[2] ξ1,2[2] ξ2,3[1] +
    e^-ξ3,3[1] x2,3[2] ξ2,3[2] + e^-ξ2,2[2] x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
    e^-ξ4,4[1] x2,4[2] ξ2,4[2] + e^-ξ2,2[2] x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] +
    e^-ξ5,5[1] x2,5[2] ξ2,5[2] + x3,3[2] ξ3,3[1] + x3,3[2] ξ3,3[2] + e^-ξ3,3[2] x3,4[2] ξ3,4[1] -
    x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + e^-ξ4,4[1] x3,4[2] ξ3,4[2] +
    e^-ξ3,3[2] x3,5[2] ξ3,5[1] - x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] +
    e^-ξ5,5[1] x3,5[2] ξ3,5[2] + x4,4[2] ξ4,4[1] + x4,4[2] ξ4,4[2] + e^-ξ4,4[2] x4,5[2] ξ4,5[1] -
    x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] +
    e^-ξ5,5[1] x4,5[2] ξ4,5[2] + x5,5[2] ξ5,5[1] + x5,5[2] ξ5,5[2], 0, 0][[1]] /. ξi_,i_[_] → 0
```
Out[•]=
```
x1,2[2] ξ1,2[1] + x1,2[2] ξ1,2[2] + x1,3[2] ξ1,3[1] + x1,3[2] ξ1,3[2] +
 x1,4[2] ξ1,4[1] + x1,4[2] ξ1,4[2] + x1,5[2] ξ1,5[1] + x1,5[2] ξ1,5[2] + x2,3[2] ξ2,3[1] -
 x1,3[2] ξ1,2[2] ξ2,3[1] + x2,3[2] ξ2,3[2] + x2,4[2] ξ2,4[1] - x1,4[2] ξ1,2[2] ξ2,4[1] +
 x2,4[2] ξ2,4[2] + x2,5[2] ξ2,5[1] - x1,5[2] ξ1,2[2] ξ2,5[1] + x2,5[2] ξ2,5[2] + x3,4[2] ξ3,4[1] -
 x1,4[2] ξ1,3[2] ξ3,4[1] - x2,4[2] ξ2,3[2] ξ3,4[1] + x3,4[2] ξ3,4[2] + x3,5[2] ξ3,5[1] -
 x1,5[2] ξ1,3[2] ξ3,5[1] - x2,5[2] ξ2,3[2] ξ3,5[1] + x3,5[2] ξ3,5[2] + x4,5[2] ξ4,5[1] -
 x1,5[2] ξ1,4[2] ξ4,5[1] - x2,5[2] ξ2,4[2] ξ4,5[1] - x3,5[2] ξ3,4[2] ξ4,5[1] + x4,5[2] ξ4,5[2]
```

In[•]:=
```
Expand@Total[sol1 /. {ξαβ__ :↦ ξαβ[1], ηαβ__ :↦ ξαβ[2] , Rule → Times, λ → 1, fαβ__[_] :↦ xαβ[2]}] -
 osol
```
Out[•]=
```
Total[sol1] - x1,2[2] ξ1,2[1] - x1,2[2] ξ1,2[2] - x1,3[2] ξ1,3[1] - x1,3[2] ξ1,3[2] -
 x1,4[2] ξ1,4[1] - x1,4[2] ξ1,4[2] - x1,5[2] ξ1,5[1] - x1,5[2] ξ1,5[2] - x2,3[2] ξ2,3[1] +
 x1,3[2] ξ1,2[2] ξ2,3[1] - x2,3[2] ξ2,3[2] - x2,4[2] ξ2,4[1] + x1,4[2] ξ1,2[2] ξ2,4[1] -
 x2,4[2] ξ2,4[2] - x2,5[2] ξ2,5[1] + x1,5[2] ξ1,2[2] ξ2,5[1] - x2,5[2] ξ2,5[2] - x3,4[2] ξ3,4[1] +
 x1,4[2] ξ1,3[2] ξ3,4[1] + x2,4[2] ξ2,3[2] ξ3,4[1] - x3,4[2] ξ3,4[2] - x3,5[2] ξ3,5[1] +
 x1,5[2] ξ1,3[2] ξ3,5[1] + x2,5[2] ξ2,3[2] ξ3,5[1] - x3,5[2] ξ3,5[2] - x4,5[2] ξ4,5[1] +
 x1,5[2] ξ1,4[2] ξ4,5[1] + x2,5[2] ξ2,4[2] ξ4,5[1] + x3,5[2] ξ3,4[2] ξ4,5[1] - x4,5[2] ξ4,5[2]
```