Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the λ-tangent formalism. Some material from pensieve://Projects/UEA.

In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Prolog

In[•]:=
```
BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
    algebras and PBW algebras. It is in the public domain, available
    at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
    Dror Bar-Natan is committed to support it within
    reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
  Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule, δ},
  "."];
Begin["`Private`"];
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror
  Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, δ, $Basis, $PBWRule}.
```

Utilities

In[•]:=
```
δ_,_ := If[i == j, 1, 0];
```

```
SSQ[x_] := ¬ NilQ[x]; (* Semi-Simple Q *)
```

Implementing general universal enveloping algebras

In[•]:=
```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
In[•]:= x_ ⪯ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :> U_i[xs],
    x_ /; MemberQ[$Basis, x] :> U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
In[•]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

```
In[•]:= m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ⪯ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

```
In[•]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
   Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :> i, ∞];
```

```
In[•]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

Epilog

```
In[•]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

In[•]:=
```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
   NilQ[e] = NilQ[f] = True; NilQ[h] = False;
  );
```

$gl_{n,\epsilon}$

In[•]:=
```
Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

```
UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}.
```

```
SetAlgebra[gl_{n_,ϵ}] := (
   $Basis = Flatten@{
      Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
      Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
     };
   NilQ[y__] = True; NilQ[x_{α_,β_}] = (α =!= β); (* Nilpotent Q *)
   $PBWRule = Thread[$Basis → Range@Length@$Basis];
   B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
   B[y_{i_,j_}, y_{k_,l_}] := ϵ δ_{j,k} y_{i,l} - ϵ δ_{l,i} y_{k,j};
   B[x_{i_,j_}, y_{l_,k_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :↦ If[α ≤ β, ϵ x_{α,β}, y_{α,β}];
  );
```

```
In[•]:= SetAlgebra[gl_{3,ϵ}];
        $PBWRule
        MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

Out[•]=

$$\{y_{1,2} \to 1,\ y_{1,3} \to 2,\ y_{2,3} \to 3,\ x_{1,1} \to 4,\ x_{1,2} \to 5,\ x_{2,2} \to 6,\ x_{1,3} \to 7,\ x_{2,3} \to 8,\ x_{3,3} \to 9\}$$

Out[•]//MatrixForm=

$$\begin{pmatrix}
\{y_{1,2}, y_{1,2}\} \to 0 & \{y_{1,2}, y_{1,3}\} \to 0 & \{y_{1,2}, y_{2,3}\} \to \in y_{1,3} & \{y_{1,2}, x_{1,1}\} \to y_{\cdots} \\
\{y_{1,3}, y_{1,2}\} \to 0 & \{y_{1,3}, y_{1,3}\} \to 0 & \{y_{1,3}, y_{2,3}\} \to 0 & \{y_{1,3}, x_{1,1}\} \to y_{\cdots} \\
\{y_{2,3}, y_{1,2}\} \to -\in y_{1,3} & \{y_{2,3}, y_{1,3}\} \to 0 & \{y_{2,3}, y_{2,3}\} \to 0 & \{y_{2,3}, x_{1,1}\} \to \cdots \\
\{x_{1,1}, y_{1,2}\} \to -y_{2,1} & \{x_{1,1}, y_{1,3}\} \to -y_{3,1} & \{x_{1,1}, y_{2,3}\} \to 0 & \{x_{1,1}, x_{1,1}\} \to \cdots \\
\{x_{1,2}, y_{1,2}\} \to \in x_{1,1} - \in x_{2,2} & \{x_{1,2}, y_{1,3}\} \to -y_{3,2} & \{x_{1,2}, y_{2,3}\} \to 0 & \{x_{1,2}, x_{1,1}\} \to -x_{\cdots} \\
\{x_{2,2}, y_{1,2}\} \to y_{2,1} & \{x_{2,2}, y_{1,3}\} \to 0 & \{x_{2,2}, y_{2,3}\} \to -y_{3,2} & \{x_{2,2}, x_{1,1}\} \to \cdots \\
\{x_{1,3}, y_{1,2}\} \to -\in x_{2,3} & \{x_{1,3}, y_{1,3}\} \to \in x_{1,1} - \in x_{3,3} & \{x_{1,3}, y_{2,3}\} \to \in x_{1,2} & \{x_{1,3}, x_{1,1}\} \to -x_{\cdots} \\
\{x_{2,3}, y_{1,2}\} \to 0 & \{x_{2,3}, y_{1,3}\} \to y_{2,1} & \{x_{2,3}, y_{2,3}\} \to \in x_{2,2} - \in x_{3,3} & \{x_{2,3}, x_{1,1}\} \to \cdots \\
\{x_{3,3}, y_{1,2}\} \to 0 & \{x_{3,3}, y_{1,3}\} \to y_{3,1} & \{x_{3,3}, y_{2,3}\} \to y_{3,2} & \{x_{3,3}, x_{1,1}\} \to \cdots
\end{pmatrix}$$

```
In[•]:= $k = 2;
        x0 = ($Basis /. {x → ξ0, y → η0}).$Basis + O[ϵ]^($k+1)
        x1 = ($Basis /. {x → ξ1, y → η1}).$Basis + O[ϵ]^($k+1)
```

Out[•]=

$$(y_{1,2}\,\eta 0_{1,2} + y_{1,3}\,\eta 0_{1,3} + y_{2,3}\,\eta 0_{2,3} + x_{1,1}\,\xi 0_{1,1} + x_{1,2}\,\xi 0_{1,2} + x_{1,3}\,\xi 0_{1,3} + x_{2,2}\,\xi 0_{2,2} + x_{2,3}\,\xi 0_{2,3} + x_{3,3}\,\xi 0_{3,3}) + O[\epsilon]^3$$

Out[•]=

$$(y_{1,2}\,\eta 1_{1,2} + y_{1,3}\,\eta 1_{1,3} + y_{2,3}\,\eta 1_{2,3} + x_{1,1}\,\xi 1_{1,1} + x_{1,2}\,\xi 1_{1,2} + x_{1,3}\,\xi 1_{1,3} + x_{2,2}\,\xi 1_{2,2} + x_{2,3}\,\xi 1_{2,3} + x_{3,3}\,\xi 1_{3,3}) + O[\epsilon]^3$$

```
In[•]:= A = Table[Coefficient[B[b, Normal@x0], c] + O[ϵ]^($k+1), {c, $Basis}, {b, $Basis}]
```

Out[•]=

$$\begin{Bmatrix}
\{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\}, \\
\{\eta 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,2} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\}, \\
\{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\}, \\
\{-\xi 0_{1,2} \in + O[\epsilon]^3, -\xi 0_{1,3} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \eta 0_{1,2} \in + O[\epsilon]^3, O[\epsilon]^3, \\
\eta 0_{1,3} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\}, \{O[\epsilon]^3, O[\epsilon]^3, -\xi 0_{1,3} \in + O[\epsilon]^3, \xi 0_{1,2} + O[\epsilon]^3, \\
(-\xi 0_{1,1} + \xi 0_{2,2}) + O[\epsilon]^3, -\xi 0_{1,2} + O[\epsilon]^3, \eta 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\}, \\
\{\xi 0_{1,2} \in + O[\epsilon]^3, O[\epsilon]^3, -\xi 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,2} \in + O[\epsilon]^3, O[\epsilon]^3, \\
O[\epsilon]^3, \eta 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3\}, \{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \xi 0_{1,3} + O[\epsilon]^3, \\
\xi 0_{2,3} + O[\epsilon]^3, O[\epsilon]^3, (-\xi 0_{1,1} + \xi 0_{3,3}) + O[\epsilon]^3, -\xi 0_{1,2} + O[\epsilon]^3, -\xi 0_{1,3} + O[\epsilon]^3\}, \\
\{\xi 0_{1,3} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \xi 0_{2,3} + O[\epsilon]^3, -\eta 0_{1,2} \in + O[\epsilon]^3, \\
(-\xi 0_{2,2} + \xi 0_{3,3}) + O[\epsilon]^3, -\xi 0_{2,3} + O[\epsilon]^3\}, \{O[\epsilon]^3, \xi 0_{1,3} \in + O[\epsilon]^3, \\
\xi 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,3} \in + O[\epsilon]^3, -\eta 0_{2,3} \in + O[\epsilon]^3, O[\epsilon]^3\}
\end{Bmatrix}$$

In[•]:=
```
A // Normal // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon\,\eta\theta_{2,3} & 0 & -\epsilon\,\eta\theta_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\epsilon\,\xi\theta_{1,2} & -\epsilon\,\xi\theta_{1,3} & 0 & 0 & \epsilon\,\eta\theta_{1,2} & 0 & \epsilon\,\eta\theta_{1,3} & 0 & 0 \\
0 & 0 & -\epsilon\,\xi\theta_{1,3} & \xi\theta_{1,2} & -\xi\theta_{1,1}+\xi\theta_{2,2} & -\xi\theta_{1,2} & \epsilon\,\eta\theta_{2,3} & 0 & 0 \\
\epsilon\,\xi\theta_{1,2} & 0 & -\epsilon\,\xi\theta_{2,3} & 0 & -\epsilon\,\eta\theta_{1,2} & 0 & 0 & \epsilon\,\eta\theta_{2,3} & 0 \\
0 & 0 & 0 & \xi\theta_{1,3} & \xi\theta_{2,3} & 0 & -\xi\theta_{1,1}+\xi\theta_{3,3} & -\xi\theta_{1,2} & -\xi\theta_{1,3} \\
\epsilon\,\xi\theta_{1,3} & 0 & 0 & 0 & 0 & \xi\theta_{2,3} & -\epsilon\,\eta\theta_{1,2} & -\xi\theta_{2,2}+\xi\theta_{3,3} & -\xi\theta_{2,3} \\
0 & \epsilon\,\xi\theta_{1,3} & \epsilon\,\xi\theta_{2,3} & 0 & 0 & 0 & -\epsilon\,\eta\theta_{1,3} & -\epsilon\,\eta\theta_{2,3} & 0
\end{pmatrix}$$

In[•]:=
```
MatrixExp[(A // Normal) /. ϵ → 0] // Simplify
```

Out[•]=

$$\Bigg\{\{1,0,0,0,0,0,0,0,0\},\{0,1,0,0,0,0,0,0,0\},$$
$$\{0,0,1,0,0,0,0,0,0\},\{0,0,0,1,0,0,0,0,0\},$$
$$\Bigg\{0,0,0,\frac{\left(1-e^{-\xi\theta_{1,1}+\xi\theta_{2,2}}\right)\xi\theta_{1,2}}{\xi\theta_{1,1}-\xi\theta_{2,2}},e^{-\xi\theta_{1,1}+\xi\theta_{2,2}},-\frac{\left(1-e^{-\xi\theta_{1,1}+\xi\theta_{2,2}}\right)\xi\theta_{1,2}}{\xi\theta_{1,1}-\xi\theta_{2,2}},0,0,0\Bigg\},$$
$$\{0,0,0,0,0,1,0,0,0\},\Bigg\{0,0,0,$$
$$\Big(e^{-\xi\theta_{1,1}}\Big(\xi\theta_{1,1}\Big(-\Big(\left(e^{\xi\theta_{2,2}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,2}\,\xi\theta_{2,3}\Big)+\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,3}\,(\xi\theta_{2,2}-\xi\theta_{3,3})\Big)-\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)$$
$$\xi\theta_{1,3}\,\xi\theta_{2,2}\,(\xi\theta_{2,2}-\xi\theta_{3,3})+\xi\theta_{1,2}\,\xi\theta_{2,3}\Big(\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{2,2}-\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{2,2}}\right)\xi\theta_{3,3}\Big)\Big)\Big)\Big/$$
$$\left((\xi\theta_{1,1}-\xi\theta_{2,2})\,(\xi\theta_{1,1}-\xi\theta_{3,3})\,(\xi\theta_{2,2}-\xi\theta_{3,3})\right),\frac{e^{-\xi\theta_{1,1}}\left(e^{\xi\theta_{2,2}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{2,3}}{\xi\theta_{2,2}-\xi\theta_{3,3}},$$
$$\frac{e^{-\xi\theta_{1,1}-\xi\theta_{2,2}}\left(-e^{\xi\theta_{1,1}}+e^{\xi\theta_{2,2}}\right)\left(e^{\xi\theta_{2,2}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,2}\,\xi\theta_{2,3}}{(\xi\theta_{1,1}-\xi\theta_{2,2})\,(\xi\theta_{2,2}-\xi\theta_{3,3})},e^{-\xi\theta_{1,1}+\xi\theta_{3,3}},$$
$$\frac{e^{\xi\theta_{3,3}}\left(e^{-\xi\theta_{1,1}}-e^{-\xi\theta_{2,2}}\right)\xi\theta_{1,2}}{\xi\theta_{1,1}-\xi\theta_{2,2}},\frac{1}{(\xi\theta_{1,1}-\xi\theta_{2,2})\,(\xi\theta_{1,1}-\xi\theta_{3,3})\,(\xi\theta_{2,2}-\xi\theta_{3,3})}$$
$$e^{-\xi\theta_{1,1}-\xi\theta_{2,2}}\Big(\xi\theta_{1,1}\Big(e^{\xi\theta_{1,1}}\left(e^{\xi\theta_{2,2}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,2}\,\xi\theta_{2,3}-e^{\xi\theta_{2,2}}\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,3}\,(\xi\theta_{2,2}-\xi\theta_{3,3})\Big)+$$
$$e^{\xi\theta_{2,2}}\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{1,3}\,\xi\theta_{2,2}\,(\xi\theta_{2,2}-\xi\theta_{3,3})-$$
$$\xi\theta_{1,2}\,\xi\theta_{2,3}\Big(e^{\xi\theta_{2,2}}\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{3,3}}\right)\xi\theta_{2,2}-e^{\xi\theta_{3,3}}\left(e^{\xi\theta_{1,1}}-e^{\xi\theta_{2,2}}\right)\xi\theta_{3,3}\Big)\Big)\Bigg\},$$
$$\Bigg\{0,0,0,0,0,\frac{\left(1-e^{-\xi\theta_{2,2}+\xi\theta_{3,3}}\right)\xi\theta_{2,3}}{\xi\theta_{2,2}-\xi\theta_{3,3}},0,e^{-\xi\theta_{2,2}+\xi\theta_{3,3}},-\frac{\left(1-e^{-\xi\theta_{2,2}+\xi\theta_{3,3}}\right)\xi\theta_{2,3}}{\xi\theta_{2,2}-\xi\theta_{3,3}}\Bigg\},$$
$$\{0,0,0,0,0,0,0,0,1\}\Bigg\}$$

In[•]:=
```
MatrixExp[A // Normal]
```

Out[•]=

$Aborted

In[•]:=
```
MatrixExp[A]
```

Eigenvalues: Unable to find all roots of the characteristic polynomial.

Out[•]=

$$\text{MatrixExp}\Big[\Big\{\big\{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\big\},$$
$$\big\{\eta 0_{2,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,2}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\big\},$$
$$\big\{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\big\},$$
$$\big\{-\xi 0_{1,2}\,\epsilon + O[\epsilon]^3, -\xi 0_{1,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \eta 0_{1,2}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3,$$
$$\eta 0_{1,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\big\}, \big\{O[\epsilon]^3, O[\epsilon]^3, -\xi 0_{1,3}\,\epsilon + O[\epsilon]^3, \xi 0_{1,2} + O[\epsilon]^3,$$
$$(-\xi 0_{1,1} + \xi 0_{2,2}) + O[\epsilon]^3, -\xi 0_{1,2} + O[\epsilon]^3, \eta 0_{2,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3\big\},$$
$$\big\{\xi 0_{1,2}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, -\xi 0_{2,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,2}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3,$$
$$O[\epsilon]^3, \eta 0_{2,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3\big\}, \big\{O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \xi 0_{1,3} + O[\epsilon]^3,$$
$$\xi 0_{2,3} + O[\epsilon]^3, O[\epsilon]^3, (-\xi 0_{1,1} + \xi 0_{3,3}) + O[\epsilon]^3, -\xi 0_{1,2} + O[\epsilon]^3, -\xi 0_{1,3} + O[\epsilon]^3\big\},$$
$$\big\{\xi 0_{1,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, \xi 0_{2,3} + O[\epsilon]^3, -\eta 0_{1,2}\,\epsilon + O[\epsilon]^3,$$
$$(-\xi 0_{2,2} + \xi 0_{3,3}) + O[\epsilon]^3, -\xi 0_{2,3} + O[\epsilon]^3\big\}, \big\{O[\epsilon]^3, \xi 0_{1,3}\,\epsilon + O[\epsilon]^3, \xi 0_{2,3}\,\epsilon + O[\epsilon]^3,$$
$$O[\epsilon]^3, O[\epsilon]^3, O[\epsilon]^3, -\eta 0_{1,3}\,\epsilon + O[\epsilon]^3, -\eta 0_{2,3}\,\epsilon + O[\epsilon]^3, O[\epsilon]^3\big\}\big\}\Big]$$

In[•]:=
```
adExp[x_][y_] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   $Basis.MatrixExp[A].Table[Coefficient[y, b], {b, $Basis}]
  ]
```

In[•]:=
```
adExp[x0][x1]
```

Out[•]=

$$x_{1,2}\,\eta_{1,2} + x_{1,3}\,\eta_{1,3} + x_{1,4}\,\eta_{1,4} + x_{1,5}\,\eta_{1,5} + x_{2,3}\,\eta_{2,3} + x_{2,4}\,\eta_{2,4} + x_{2,5}\,\eta_{2,5} + x_{3,4}\,\eta_{3,4} + x_{3,5}\,\eta_{3,5} + x_{4,5}\,\eta_{4,5} -$$
$$x_{1,3}\,\eta_{2,3}\,\xi_{1,2} - x_{1,4}\,\eta_{2,4}\,\xi_{1,2} - x_{1,5}\,\eta_{2,5}\,\xi_{1,2} - x_{1,4}\,\eta_{3,4}\,\xi_{1,3} - x_{1,5}\,\eta_{3,5}\,\xi_{1,3} - x_{1,5}\,\eta_{4,5}\,\xi_{1,4} +$$
$$x_{1,3}\,\eta_{1,2}\,\xi_{2,3} - x_{2,4}\,\eta_{3,4}\,\xi_{2,3} - x_{2,5}\,\eta_{3,5}\,\xi_{2,3} + \frac{1}{2}\,x_{1,4}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3} + \frac{1}{2}\,x_{1,5}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3} +$$
$$x_{1,4}\,\eta_{1,2}\,\xi_{2,4} - x_{2,5}\,\eta_{4,5}\,\xi_{2,4} + \frac{1}{2}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4} + x_{1,5}\,\eta_{1,2}\,\xi_{2,5} + x_{1,4}\,\eta_{1,3}\,\xi_{3,4} +$$
$$x_{2,4}\,\eta_{2,3}\,\xi_{3,4} - x_{3,5}\,\eta_{4,5}\,\xi_{3,4} - x_{1,4}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4} + \frac{1}{2}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4} + \frac{1}{2}\,x_{1,4}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4} +$$
$$\frac{1}{2}\,x_{2,5}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4} - \frac{1}{6}\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4} + x_{1,5}\,\eta_{1,3}\,\xi_{3,5} + x_{2,5}\,\eta_{2,3}\,\xi_{3,5} - x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,5} +$$
$$\frac{1}{2}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5} + x_{1,5}\,\eta_{1,4}\,\xi_{4,5} + x_{2,5}\,\eta_{2,4}\,\xi_{4,5} + x_{3,5}\,\eta_{3,4}\,\xi_{4,5} - x_{1,5}\,\eta_{2,4}\,\xi_{1,2}\,\xi_{4,5} -$$
$$x_{1,5}\,\eta_{3,4}\,\xi_{1,3}\,\xi_{4,5} - x_{2,5}\,\eta_{3,4}\,\xi_{2,3}\,\xi_{4,5} + \frac{1}{2}\,x_{1,5}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{4,5} + \frac{1}{2}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5} +$$
$$\frac{1}{2}\,x_{1,5}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5} + \frac{1}{2}\,x_{2,5}\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5} - \frac{1}{2}\,x_{1,5}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}\,\xi_{4,5} + \frac{1}{6}\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
Coefficient[adExp[x0][y0], x_{1,5}]
```
Out[•]=
$$\eta_{1,5}-\eta_{2,5}\,\xi_{1,2}-\eta_{3,5}\,\xi_{1,3}-\eta_{4,5}\,\xi_{1,4}+\frac{1}{2}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\frac{1}{2}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}+\eta_{1,2}\,\xi_{2,5}+\frac{1}{2}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}-\frac{1}{6}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\eta_{1,3}\,\xi_{3,5}-\eta_{2,3}\,\xi_{1,2}\,\xi_{3,5}+\frac{1}{2}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\eta_{1,4}\,\xi_{4,5}-\eta_{2,4}\,\xi_{1,2}\,\xi_{4,5}-\eta_{3,4}\,\xi_{1,3}\,\xi_{4,5}+\frac{1}{2}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{4,5}+\frac{1}{2}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\frac{1}{2}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\frac{1}{2}\,\eta_{2,3}\,\xi_{1,2}\,\xi_{3,4}\,\xi_{4,5}+\frac{1}{6}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λ x;
λTangent[xs_List] := λTangent @@ xs
```

In[•]:=
```
$Basis ($Basis /. x → ξ)
```
Out[•]=
$$\{x_{1,2}\,\xi_{1,2},\ x_{1,3}\,\xi_{1,3},\ x_{2,3}\,\xi_{2,3},\ x_{1,4}\,\xi_{1,4},\ x_{2,4}\,\xi_{2,4},\ x_{3,4}\,\xi_{3,4},\ x_{1,5}\,\xi_{1,5},\ x_{2,5}\,\xi_{2,5},\ x_{3,5}\,\xi_{3,5},\ x_{4,5}\,\xi_{4,5}\}$$

In[•]:=
```
λTangent@(λ $Basis ($Basis /. x → ξ))
```
Out[•]=
$$x_{1,2}\,\xi_{1,2}+x_{1,3}\,\xi_{1,3}+x_{1,4}\,\xi_{1,4}+x_{1,5}\,\xi_{1,5}+x_{2,3}\,\xi_{2,3}+\lambda\,x_{1,3}\,\xi_{1,2}\,\xi_{2,3}+x_{2,4}\,\xi_{2,4}+\lambda\,x_{1,4}\,\xi_{1,2}\,\xi_{2,4}+x_{2,5}\,\xi_{2,5}+\lambda\,x_{1,5}\,\xi_{1,2}\,\xi_{2,5}+x_{3,4}\,\xi_{3,4}+\lambda\,x_{1,4}\,\xi_{1,3}\,\xi_{3,4}+\lambda\,x_{2,4}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{1,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+x_{3,5}\,\xi_{3,5}+\lambda\,x_{1,5}\,\xi_{1,3}\,\xi_{3,5}+\lambda\,x_{2,5}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}+x_{4,5}\,\xi_{4,5}+\lambda\,x_{1,5}\,\xi_{1,4}\,\xi_{4,5}+\lambda\,x_{2,5}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda\,x_{3,5}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{2,5}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
lhs = Join[λ $Basis ($Basis /. x → ξ), λ $Basis ($Basis /. x → η)] // λTangent
```
Out[•]=
$$x_{1,2}\,\eta_{1,2}+x_{1,3}\,\eta_{1,3}+x_{1,4}\,\eta_{1,4}+x_{1,5}\,\eta_{1,5}+x_{2,3}\,\eta_{2,3}+\lambda\,x_{1,3}\,\eta_{1,2}\,\eta_{2,3}+x_{2,4}\,\eta_{2,4}+\lambda\,x_{1,4}\,\eta_{1,2}\,\eta_{2,4}+x_{2,5}\,\eta_{2,5}+\lambda\,x_{1,5}\,\eta_{1,2}\,\eta_{2,5}+x_{3,4}\,\eta_{3,4}+\lambda\,x_{1,4}\,\eta_{1,3}\,\eta_{3,4}+\lambda\,x_{2,4}\,\eta_{2,3}\,\eta_{3,4}+\lambda^2\,x_{1,4}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}+x_{3,5}\,\eta_{3,5}+\lambda\,x_{1,5}\,\eta_{1,3}\,\eta_{3,5}+\lambda\,x_{2,5}\,\eta_{2,3}\,\eta_{3,5}+\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,5}+x_{4,5}\,\eta_{4,5}+\lambda\,x_{1,5}\,\eta_{1,4}\,\eta_{4,5}+\lambda\,x_{2,5}\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{2,4}\,\eta_{4,5}+\lambda\,x_{3,5}\,\eta_{3,4}\,\eta_{4,5}+\lambda^2\,x_{1,5}\,\eta_{1,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda^2\,x_{2,5}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+x_{1,2}\,\xi_{1,2}+\lambda\,x_{1,3}\,\eta_{2,3}\,\xi_{1,2}+\lambda\,x_{1,4}\,\eta_{2,4}\,\xi_{1,2}+\lambda\,x_{1,5}\,\eta_{2,5}\,\xi_{1,2}+\lambda^2\,x_{1,4}\,\eta_{2,3}\,\eta_{3,4}\,\xi_{1,2}+\lambda^2\,x_{1,5}\,\eta_{2,3}\,\eta_{3,5}\,\xi_{1,2}+\lambda^2\,x_{1,5}\,\eta_{2,4}\,\eta_{4,5}\,\xi_{1,2}+\lambda^3\,x_{1,5}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}+x_{1,3}\,\xi_{1,3}+\lambda\,x_{1,4}\,\eta_{3,4}\,\xi_{1,3}+\lambda\,x_{1,5}\,\eta_{3,5}\,\xi_{1,3}+\lambda^2\,x_{1,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,3}+x_{1,4}\,\xi_{1,4}+\lambda\,x_{1,5}\,\eta_{4,5}\,\xi_{1,4}+x_{1,5}\,\xi_{1,5}+x_{2,3}\,\xi_{2,3}-\lambda\,x_{1,3}\,\eta_{1,2}\,\xi_{2,3}+\lambda\,x_{2,4}\,\eta_{3,4}\,\xi_{2,3}-\lambda^2\,x_{1,4}\,\eta_{1,2}\,\eta_{3,4}\,\xi_{2,3}+\lambda\,x_{2,5}\,\eta_{3,5}\,\xi_{2,3}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{3,5}\,\xi_{2,3}+\lambda^2\,x_{2,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+\lambda\,x_{1,3}\,\xi_{1,2}\,\xi_{2,3}+\lambda^2\,x_{1,4}\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}+\lambda^2\,x_{1,5}\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\lambda^3\,x_{1,5}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}+x_{2,4}\,\xi_{2,4}-\lambda\,x_{1,4}\,\eta_{1,2}\,\xi_{2,4}+\lambda\,x_{2,5}\,\eta_{4,5}\,\xi_{2,4}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,4}+\lambda\,x_{1,4}\,\xi_{1,2}\,\xi_{2,4}+\lambda^2\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}+x_{2,5}\,\xi_{2,5}-\lambda\,x_{1,5}\,\eta_{1,2}\,\xi_{2,5}+\lambda\,x_{1,5}\,\xi_{1,2}\,\xi_{2,5}+x_{3,4}\,\xi_{3,4}-\lambda\,x_{1,4}\,\eta_{1,3}\,\xi_{3,4}-\lambda\,x_{2,4}\,\eta_{2,3}\,\xi_{3,4}+\lambda\,x_{3,5}\,\eta_{4,5}\,\xi_{3,4}-\lambda^2\,x_{1,5}\,\eta_{1,3}\,\eta_{4,5}\,\xi_{3,4}-\lambda^2\,x_{2,5}\,\eta_{2,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda\,x_{1,4}\,\xi_{1,3}\,\xi_{3,4}+\lambda^2\,x_{1,5}\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}+\lambda\,x_{2,4}\,\xi_{2,3}\,\xi_{3,4}-\lambda^2\,x_{1,4}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{2,5}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}+\lambda^2\,x_{1,4}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\lambda^3\,x_{1,5}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}+x_{3,5}\,\xi_{3,5}-\lambda\,x_{1,5}\,\eta_{1,3}\,\xi_{3,5}-\lambda\,x_{2,5}\,\eta_{2,3}\,\xi_{3,5}+\lambda\,x_{1,5}\,\xi_{1,3}\,\xi_{3,5}+\lambda\,x_{2,5}\,\xi_{2,3}\,\xi_{3,5}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}+x_{4,5}\,\xi_{4,5}-\lambda\,x_{1,5}\,\eta_{1,4}\,\xi_{4,5}-\lambda\,x_{2,5}\,\eta_{2,4}\,\xi_{4,5}-\lambda\,x_{3,5}\,\eta_{3,4}\,\xi_{4,5}+\lambda\,x_{1,5}\,\xi_{1,4}\,\xi_{4,5}+\lambda\,x_{2,5}\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,x_{1,5}\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda\,x_{3,5}\,\xi_{3,4}\,\xi_{4,5}-\lambda^2\,x_{1,5}\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^2\,x_{2,5}\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{1,5}\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^2\,x_{2,5}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^3\,x_{1,5}\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,x_{1,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}$$

In[•]:=
```
rhs = λTangent@($Basis ($Basis /. x_{αβ__} → f_{αβ}[λ]))
```

Out[•]=

$$\begin{aligned}
&x_{1,2}\,f_{1,2}'[\lambda]+x_{1,3}\,f_{2,3}[\lambda]\,f_{1,2}'[\lambda]+x_{1,4}\,f_{2,4}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,5}[\lambda]\,f_{1,2}'[\lambda]+\\
&\quad x_{1,4}\,f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,3}[\lambda]\,f_{3,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,5}\,f_{2,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]+\\
&\quad x_{1,5}\,f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]+x_{1,3}\,f_{1,3}'[\lambda]+x_{1,4}\,f_{3,4}[\lambda]\,f_{1,3}'[\lambda]+x_{1,5}\,f_{3,5}[\lambda]\,f_{1,3}'[\lambda]+\\
&\quad x_{1,5}\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,3}'[\lambda]+x_{1,4}\,f_{1,4}'[\lambda]+x_{1,5}\,f_{4,5}[\lambda]\,f_{1,4}'[\lambda]+x_{1,5}\,f_{1,5}'[\lambda]+x_{2,3}\,f_{2,3}'[\lambda]+\\
&\quad x_{2,4}\,f_{3,4}[\lambda]\,f_{2,3}'[\lambda]+x_{2,5}\,f_{3,5}[\lambda]\,f_{2,3}'[\lambda]+x_{2,5}\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{2,3}'[\lambda]+x_{2,4}\,f_{2,4}'[\lambda]+\\
&\quad x_{2,5}\,f_{4,5}[\lambda]\,f_{2,4}'[\lambda]+x_{2,5}\,f_{2,5}'[\lambda]+x_{3,4}\,f_{3,4}'[\lambda]+x_{3,5}\,f_{4,5}[\lambda]\,f_{3,4}'[\lambda]+x_{3,5}\,f_{3,5}'[\lambda]+x_{4,5}\,f_{4,5}'[\lambda]
\end{aligned}$$

In[•]:=
```
(Coefficient[lhs - rhs, #] == 0) & /@ $Basis
```

Out[•]=

$$\begin{aligned}
&\{\eta_{1,2}+\xi_{1,2}-f_{1,2}'[\lambda]==0,\\
&\eta_{1,3}+\lambda\,\eta_{1,2}\,\eta_{2,3}+\lambda\,\eta_{2,3}\,\xi_{1,2}+\xi_{1,3}-\lambda\,\eta_{1,2}\,\xi_{2,3}+\lambda\,\xi_{1,2}\,\xi_{2,3}-f_{2,3}[\lambda]\,f_{1,2}'[\lambda]-f_{1,3}'[\lambda]==0,\\
&\eta_{2,3}+\xi_{2,3}-f_{2,3}'[\lambda]==0,\\
&\eta_{1,4}+\lambda\,\eta_{1,2}\,\eta_{2,4}+\lambda\,\eta_{1,3}\,\eta_{3,4}+\lambda^2\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}+\lambda\,\eta_{2,4}\,\xi_{1,2}+\lambda^2\,\eta_{2,3}\,\eta_{3,4}\,\xi_{1,2}+\lambda\,\eta_{3,4}\,\xi_{1,3}+\xi_{1,4}-\\
&\quad\lambda^2\,\eta_{1,2}\,\eta_{3,4}\,\xi_{2,3}+\lambda^2\,\eta_{3,4}\,\xi_{1,2}\,\xi_{2,3}-\lambda\,\eta_{1,2}\,\xi_{2,4}+\lambda\,\xi_{1,2}\,\xi_{2,4}-\lambda\,\eta_{1,3}\,\xi_{3,4}+\lambda\,\xi_{1,3}\,\xi_{3,4}-\lambda^2\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}+\\
&\quad\lambda^2\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}-f_{2,4}[\lambda]\,f_{1,2}'[\lambda]-f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{1,2}'[\lambda]-f_{3,4}[\lambda]\,f_{1,3}'[\lambda]-f_{1,4}'[\lambda]==0,\\
&\eta_{2,4}+\lambda\,\eta_{2,3}\,\eta_{3,4}+\lambda\,\eta_{3,4}\,\xi_{2,3}+\xi_{2,4}-\lambda\,\eta_{2,3}\,\xi_{3,4}+\lambda\,\xi_{2,3}\,\xi_{3,4}-f_{3,4}[\lambda]\,f_{2,3}'[\lambda]-f_{2,4}'[\lambda]==0,\\
&\eta_{3,4}+\xi_{3,4}-f_{3,4}'[\lambda]==0,\\
&\eta_{1,5}+\lambda\,\eta_{1,2}\,\eta_{2,5}+\lambda\,\eta_{1,3}\,\eta_{3,5}+\lambda^2\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,5}+\lambda\,\eta_{1,4}\,\eta_{4,5}+\lambda^2\,\eta_{1,2}\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,\eta_{1,3}\,\eta_{3,4}\,\eta_{4,5}+\\
&\quad\lambda^3\,\eta_{1,2}\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{2,5}\,\xi_{1,2}+\lambda^2\,\eta_{2,3}\,\eta_{3,5}\,\xi_{1,2}+\lambda^2\,\eta_{2,4}\,\eta_{4,5}\,\xi_{1,2}+\lambda^3\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}+\\
&\quad\lambda\,\eta_{3,5}\,\xi_{1,3}+\lambda^2\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,3}+\lambda\,\eta_{4,5}\,\xi_{1,4}+\xi_{1,5}-\lambda^2\,\eta_{1,2}\,\eta_{3,5}\,\xi_{2,3}-\lambda^3\,\eta_{1,2}\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+\\
&\quad\lambda^2\,\eta_{3,5}\,\xi_{1,2}\,\xi_{2,3}+\lambda^3\,\eta_{3,4}\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}-\lambda^2\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,4}+\lambda^2\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,4}-\\
&\quad\lambda\,\eta_{1,2}\,\xi_{2,5}+\lambda\,\xi_{1,2}\,\xi_{2,5}-\lambda^2\,\eta_{1,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda^2\,\eta_{4,5}\,\xi_{1,3}\,\xi_{3,4}-\lambda^3\,\eta_{1,2}\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}+\\
&\quad\lambda^3\,\eta_{4,5}\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}-\lambda\,\eta_{1,3}\,\xi_{3,5}+\lambda\,\xi_{1,3}\,\xi_{3,5}-\lambda^2\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,5}+\lambda^2\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,5}-\\
&\quad\lambda\,\eta_{1,4}\,\xi_{4,5}+\lambda\,\xi_{1,4}\,\xi_{4,5}-\lambda^2\,\eta_{1,2}\,\xi_{2,4}\,\xi_{4,5}+\lambda^2\,\xi_{1,2}\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,\eta_{1,3}\,\xi_{3,4}\,\xi_{4,5}+\\
&\quad\lambda^2\,\xi_{1,3}\,\xi_{3,4}\,\xi_{4,5}-\lambda^3\,\eta_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\lambda^3\,\xi_{1,2}\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-f_{2,5}[\lambda]\,f_{1,2}'[\lambda]-\\
&\quad f_{2,3}[\lambda]\,f_{3,5}[\lambda]\,f_{1,2}'[\lambda]-f_{2,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]-f_{2,3}[\lambda]\,f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,2}'[\lambda]-\\
&\quad f_{3,5}[\lambda]\,f_{1,3}'[\lambda]-f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{1,3}'[\lambda]-f_{4,5}[\lambda]\,f_{1,4}'[\lambda]-f_{1,5}'[\lambda]==0,\\
&\eta_{2,5}+\lambda\,\eta_{2,3}\,\eta_{3,5}+\lambda\,\eta_{2,4}\,\eta_{4,5}+\lambda^2\,\eta_{2,3}\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{3,5}\,\xi_{2,3}+\lambda^2\,\eta_{3,4}\,\eta_{4,5}\,\xi_{2,3}+\lambda\,\eta_{4,5}\,\xi_{2,4}+\xi_{2,5}-\\
&\quad\lambda^2\,\eta_{2,3}\,\eta_{4,5}\,\xi_{3,4}+\lambda^2\,\eta_{4,5}\,\xi_{2,3}\,\xi_{3,4}-\lambda\,\eta_{2,3}\,\xi_{3,5}+\lambda\,\xi_{2,3}\,\xi_{3,5}-\lambda\,\eta_{2,4}\,\xi_{4,5}+\lambda\,\xi_{2,4}\,\xi_{4,5}-\lambda^2\,\eta_{2,3}\,\xi_{3,4}\,\xi_{4,5}+\\
&\quad\lambda^2\,\xi_{2,3}\,\xi_{3,4}\,\xi_{4,5}-f_{3,5}[\lambda]\,f_{2,3}'[\lambda]-f_{3,4}[\lambda]\,f_{4,5}[\lambda]\,f_{2,3}'[\lambda]-f_{4,5}[\lambda]\,f_{2,4}'[\lambda]-f_{2,5}'[\lambda]==0,\\
&\eta_{3,5}+\lambda\,\eta_{3,4}\,\eta_{4,5}+\lambda\,\eta_{4,5}\,\xi_{3,4}+\xi_{3,5}-\lambda\,\eta_{3,4}\,\xi_{4,5}+\lambda\,\xi_{3,4}\,\xi_{4,5}-f_{4,5}[\lambda]\,f_{3,4}'[\lambda]-f_{3,5}'[\lambda]==0,\\
&\eta_{4,5}+\xi_{4,5}-f_{4,5}'[\lambda]==0\}
\end{aligned}$$

```
In[•]:= {sol} = DSolve[
          Join[
           (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
           $Basis /. x_αβ__ :> f_αβ[0] == 0
          ],
          $Basis /. x_αβ__ → f_αβ[λ],
          λ
         ]
```

```
Out[•]=
{{f_{1,2}[λ] → λ η_{1,2} + λ ξ_{1,2}, f_{1,3}[λ] → λ η_{1,3} + λ ξ_{1,3} - λ^2 η_{1,2} ξ_{2,3},
  f_{2,3}[λ] → λ η_{2,3} + λ ξ_{2,3}, f_{1,4}[λ] → -λ (-η_{1,4} - ξ_{1,4} + λ η_{1,2} ξ_{2,4} + λ η_{1,3} ξ_{3,4}),
  f_{2,4}[λ] → -λ (-η_{2,4} - ξ_{2,4} + λ η_{2,3} ξ_{3,4}), f_{3,4}[λ] → λ (η_{3,4} + ξ_{3,4}), f_{4,5}[λ] → λ η_{4,5} + λ ξ_{4,5},
  f_{3,5}[λ] → λ η_{3,5} + λ ξ_{3,5} - λ^2 η_{3,4} ξ_{4,5}, f_{2,5}[λ] → λ η_{2,5} + λ ξ_{2,5} - λ^2 η_{2,3} ξ_{3,5} - λ^2 η_{2,4} ξ_{4,5},
  f_{1,5}[λ] → λ η_{1,5} + λ ξ_{1,5} - λ^2 η_{1,2} ξ_{2,5} - λ^2 η_{1,3} ξ_{3,5} - λ^2 η_{1,4} ξ_{4,5}}}
```

```
In[•]:= sol /. λ → 1
```

```
Out[•]=
{f_{1,2}[1] → η_{1,2} + ξ_{1,2}, f_{1,3}[1] → η_{1,3} + ξ_{1,3} - η_{1,2} ξ_{2,3}, f_{2,3}[1] → η_{2,3} + ξ_{2,3},
 f_{1,4}[1] → η_{1,4} + ξ_{1,4} - η_{1,2} ξ_{2,4} - η_{1,3} ξ_{3,4}, f_{2,4}[1] → η_{2,4} + ξ_{2,4} - η_{2,3} ξ_{3,4},
 f_{3,4}[1] → η_{3,4} + ξ_{3,4}, f_{4,5}[1] → η_{4,5} + ξ_{4,5}, f_{3,5}[1] → η_{3,5} + ξ_{3,5} - η_{3,4} ξ_{4,5},
 f_{2,5}[1] → η_{2,5} + ξ_{2,5} - η_{2,3} ξ_{3,5} - η_{2,4} ξ_{4,5}, f_{1,5}[1] → η_{1,5} + ξ_{1,5} - η_{1,2} ξ_{2,5} - η_{1,3} ξ_{3,5} - η_{1,4} ξ_{4,5}}
```

```
In[•]:= osol = 𝔼_{{1,2}→{2}}[x_{1,1}[2] ξ_{1,1}[1] + x_{1,1}[2] ξ_{1,1}[2] + e^{-ξ_{1,1}[2]} x_{1,2}[2] ξ_{1,2}[1] + e^{-ξ_{2,2}[1]} x_{1,2}[2] ξ_{1,2}[2] +
          e^{-ξ_{1,1}[2]} x_{1,3}[2] ξ_{1,3}[1] + e^{-ξ_{3,3}[1]} x_{1,3}[2] ξ_{1,3}[2] + e^{-ξ_{1,1}[2]} x_{1,4}[2] ξ_{1,4}[1] +
          e^{-ξ_{4,4}[1]} x_{1,4}[2] ξ_{1,4}[2] + e^{-ξ_{1,1}[2]} x_{1,5}[2] ξ_{1,5}[1] + e^{-ξ_{5,5}[1]} x_{1,5}[2] ξ_{1,5}[2] +
          x_{2,2}[2] ξ_{2,2}[1] + x_{2,2}[2] ξ_{2,2}[2] + e^{-ξ_{2,2}[2]} x_{2,3}[2] ξ_{2,3}[1] - x_{1,3}[2] ξ_{1,2}[2] ξ_{2,3}[1] +
          e^{-ξ_{3,3}[1]} x_{2,3}[2] ξ_{2,3}[2] + e^{-ξ_{2,2}[2]} x_{2,4}[2] ξ_{2,4}[1] - x_{1,4}[2] ξ_{1,2}[2] ξ_{2,4}[1] +
          e^{-ξ_{4,4}[1]} x_{2,4}[2] ξ_{2,4}[2] + e^{-ξ_{2,2}[2]} x_{2,5}[2] ξ_{2,5}[1] - x_{1,5}[2] ξ_{1,2}[2] ξ_{2,5}[1] +
          e^{-ξ_{5,5}[1]} x_{2,5}[2] ξ_{2,5}[2] + x_{3,3}[2] ξ_{3,3}[1] + x_{3,3}[2] ξ_{3,3}[2] + e^{-ξ_{3,3}[2]} x_{3,4}[2] ξ_{3,4}[1] -
          x_{1,4}[2] ξ_{1,3}[2] ξ_{3,4}[1] - x_{2,4}[2] ξ_{2,3}[2] ξ_{3,4}[1] + e^{-ξ_{4,4}[1]} x_{3,4}[2] ξ_{3,4}[2] +
          e^{-ξ_{3,3}[2]} x_{3,5}[2] ξ_{3,5}[1] - x_{1,5}[2] ξ_{1,3}[2] ξ_{3,5}[1] - x_{2,5}[2] ξ_{2,3}[2] ξ_{3,5}[1] +
          e^{-ξ_{5,5}[1]} x_{3,5}[2] ξ_{3,5}[2] + x_{4,4}[2] ξ_{4,4}[1] + x_{4,4}[2] ξ_{4,4}[2] + e^{-ξ_{4,4}[2]} x_{4,5}[2] ξ_{4,5}[1] -
          x_{1,5}[2] ξ_{1,4}[2] ξ_{4,5}[1] - x_{2,5}[2] ξ_{2,4}[2] ξ_{4,5}[1] - x_{3,5}[2] ξ_{3,4}[2] ξ_{4,5}[1] +
          e^{-ξ_{5,5}[1]} x_{4,5}[2] ξ_{4,5}[2] + x_{5,5}[2] ξ_{5,5}[1] + x_{5,5}[2] ξ_{5,5}[2], 0, 0]⟦1⟧ /. ξ_{i_,i_}[_] → 0
```

```
Out[•]=
x_{1,2}[2] ξ_{1,2}[1] + x_{1,2}[2] ξ_{1,2}[2] + x_{1,3}[2] ξ_{1,3}[1] + x_{1,3}[2] ξ_{1,3}[2] +
 x_{1,4}[2] ξ_{1,4}[1] + x_{1,4}[2] ξ_{1,4}[2] + x_{1,5}[2] ξ_{1,5}[1] + x_{1,5}[2] ξ_{1,5}[2] + x_{2,3}[2] ξ_{2,3}[1] -
 x_{1,3}[2] ξ_{1,2}[2] ξ_{2,3}[1] + x_{2,3}[2] ξ_{2,3}[2] + x_{2,4}[2] ξ_{2,4}[1] - x_{1,4}[2] ξ_{1,2}[2] ξ_{2,4}[1] +
 x_{2,4}[2] ξ_{2,4}[2] + x_{2,5}[2] ξ_{2,5}[1] - x_{1,5}[2] ξ_{1,2}[2] ξ_{2,5}[1] + x_{2,5}[2] ξ_{2,5}[2] + x_{3,4}[2] ξ_{3,4}[1] -
 x_{1,4}[2] ξ_{1,3}[2] ξ_{3,4}[1] - x_{2,4}[2] ξ_{2,3}[2] ξ_{3,4}[1] + x_{3,4}[2] ξ_{3,4}[2] + x_{3,5}[2] ξ_{3,5}[1] -
 x_{1,5}[2] ξ_{1,3}[2] ξ_{3,5}[1] - x_{2,5}[2] ξ_{2,3}[2] ξ_{3,5}[1] + x_{3,5}[2] ξ_{3,5}[2] + x_{4,5}[2] ξ_{4,5}[1] -
 x_{1,5}[2] ξ_{1,4}[2] ξ_{4,5}[1] - x_{2,5}[2] ξ_{2,4}[2] ξ_{4,5}[1] - x_{3,5}[2] ξ_{3,4}[2] ξ_{4,5}[1] + x_{4,5}[2] ξ_{4,5}[2]
```

In[•]:=
```
Expand@Total[sol /. {ξ_αβ__ :> ξ_αβ[1], η_αβ__ :> ξ_αβ[2] , Rule → Times, λ → 1, f_αβ__[_] :> x_αβ[2]}] -
 osol
```

Out[•]=
```
0
```