Hidden TEX

A Timing Game

Headers

pdf

Pensieve header: An isolated attempt to compute the multiplication tensor of ${\mathcal U}_q(sl^+_{3,\epslion})$. Continues SjabboQUSL3Seepage.nb.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

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In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

pdf

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

nb2tex

In[•]:=
```
Get["../nb2tex/nb2tex.m"];
nb2tex["Qsl3+eps", PDFWidth → 7.5, PDFFolder → "Snips/Qsl3+eps"];
texfile = OpenAppend["Qsl3+eps.tex"];
WriteLine[texfile, "\\end{document}"]; Close[texfile];
Run@
  "\"C:\\Users\\drorb\\AppData\\Local\\Programs\\MiKTeX\\miktex\\bin\\x64\\pdflatex.exe\"
    Qsl3+eps.tex";
```

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Initialization / Utilities

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```
In[•]:= $k = 1; $U = QU; $E := {$k};
$trim := {ϵ^(k_·) /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
q_ħ_ := e^(γ ϵ ħ);
(* Upper to lower and lower to Upper: *)
U2l = {B_i_^p_· → e^(-p ħ γ b_i), B^p_· → e^(-p ħ γ b), T_i_^p_· → e^(p ħ t_i), T^p_· → e^(p ħ t), 𝒜_i_^p_· → e^(p γ α_i), 𝒜^p_· → e^(p γ α)};
l2U = {e^(c_· b_i_ + d_·) :> B_i^(-c/(ħ γ)) e^d, e^(c_· b + d_·) :> B^(-c/(ħ γ)) e^d,
    e^(c_· t_i_ + d_·) :> T_i^(c/ħ) e^d, e^(c_· t + d_·) :> T^(c/ħ) e^d,
    e^(c_· α_i_ + d_·) :> 𝒜_i^(c/γ) e^d, e^(c_· α + d_·) :> 𝒜^(c/γ) e^d,
    e^(ℰ_) :> e^(Expand@ℰ)};
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. U2l, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
c_Integer_k_Integer := c + O[ϵ]^(k+1);
```

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DeclareAlgebra

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```
In[•]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ⊠ y) ⊠ z;
0 ⊠ _ = _ ⊠ 0 = 0;
(x_Plus) ⊠ y_ := (# ⊠ y) & /@ x; x_ ⊠ (y_Plus) := (x ⊠ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⊠ y - y ⊠ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

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```
In[•]:= DeclareAlgebra[U_Symbol, opts__Rule] := Module[
  {gp, sr, g, cp, M, pow, k = 0,
   gs = Generators /. {opts},
   cs = Centrals /. {opts} /. Centrals → {}
  },
  U[Generators] = gs;
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
```

```
SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
CE[ℰ_] := Collect[ℰ, _U, Expand] /. e^x_ :> e^Expand[x] /. $trim;
U_i_[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> (#_i &) /@ u};
U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
B[U@y_, U@x_] := CE[-B[U@x, U@y]];
x_ ⊠ (c_ . 1_U) := CE[c x]; (c_ . 1_U) ⊠ x_ := CE[c x];
(a_ . U[xx___, x_]) ⊠ (b_ . U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
  CE@M[a b /. $trim, U[xx, x, y, yy]],
  U@xx ⊠ CE@M[a b /. $trim, U@y ⊠ U@x + B[U@x, U@y, $E]] ⊠ U@yy ];
U@{c_ . * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⊠ U@{r}];
U@{c_ . * l : gp, r___} := CE[c U[l] ⊠ U@{r}];
U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
U@{l_, r___} := U@{Expand[l], r};
U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
𝒪_U[specs___, poly_] := Module[{sp, null, vs, us},
  sp = Replace[{specs}, l_List :> l_null, {1}];
  vs = Join @@ (First /@ sp);
  us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
  CE[Total[
     CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
    ]] /. x_null :> x];
𝒪_U[specs___, 𝔼[L_, Q_, P_]] := 𝒪_U[specs, SS@Normal[P e^(L+Q)]];
pow[ℰ_, n_] := pow[ℰ, n - 1] ⊠ ℰ;
𝒮_U[ℰ_, ss___Rule] := CE@Total[
   CoefficientRules[ℰ, First /@ {ss}] /.
    (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
σ_rs___[c_ . * u_U] := (c /. (t : cp)_j_ :> t_j/.{rs}) U[List @@ (u /. v_j_ :> v_j/.{rs})];
m_j_→k_[c_ . * u_U] := CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) ⊠
   U @@ Cases[u, w_j :> w_k] ⊠ U @@ Cases[u, _k]];
U /: c_ . * u_U * v_U := CE[c u ⊠ v];
S_i_[c_ . * u_U] := CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) ⊠
   U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]]];
Δ_i_→j_,k_[c_ . * u_U] := CE[((c /. Δ_i→j,k[U, Centrals]) DeleteCases[u, _i]) ⊠
   (NCM @@ Cases[u, x_i :> σ_1→j,2→k@Δ@U@x] /. NCM[] → U[])];
```

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```
In[•]:= Invert_U[ℰ_] := Module[{A, B, β}, (* ℰ=A U[]+∈ B *)
  A = Coefficient[ℰ, 1_U] /. ∈ → 0;
  B = CE[∈^-1 (ℰ - A 1_U)];
  𝒮_U[SS[(A + ∈ β)^-1], β → B]
 ]
```

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Implementing QU = $\mathcal{U}_q\left(\mathrm{sl}^+_{3,\gamma\epsilon}\right)$

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Greek - Latin duality:

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In[•]:=
```
{X*, Y*, Z*, b*, a*, z*, y*, x*, Ξ*, ϒ*, Ω*, β*, α*, ζ*, η*, ξ*} =
  {Ξ, ϒ, Ω, β, α, ζ, η, ξ, X, Y, Z, b, a, z, y, x};
(u_αβ__)* := (u*)_αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

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In[•]:=
```
ℏ = γ = 1;
DeclareAlgebra[QU, Generators → {X, Y, Z, b, a, z, y, x}, Centrals → {s, t}];
B[a_QU, x_QU] = -2 QU@x;
B[b_QU, x_QU] = QU@x;
B[b_QU, y_QU] = -2 QU@y;
B[a_QU, y_QU] = QU@y;
B[a_QU, z_QU] = -QU@z;
B[b_QU, z_QU] = -QU@z;
B[y_QU, x_QU] = -z_QU + ϵ ℏ QU[y, x];
B[z_QU, x_QU] = -ϵ ℏ QU[z, x];
B[b_QU, a_QU] = 0;
B[z_QU, y_QU] = ϵ ℏ QU[z, y];
B[a_QU, X_QU] = 2 X_QU;
B[b_QU, X_QU] = -X_QU;
B[a_QU, Y_QU] = -Y_QU;
B[b_QU, Y_QU] = 2 Y_QU;
B[a_QU, Z_QU] = Z_QU;
B[b_QU, Z_QU] = Z_QU;
B[x_QU, X_QU] = -2 ϵ ℏ QU@{X, x} + (1 - s) QU[]/ℏ + 2 s ϵ QU[a];
B[y_QU, Y_QU] = -2 ϵ ℏ QU@{Y, y} + (1 - t) QU[]/ℏ + 2 t ϵ QU[b];
B[z_QU, Z_QU] = -2 ϵ ℏ QU@{Z, z} + (1 - s t) QU[]/ℏ + 2 s t ϵ QU[a] + 2 s t ϵ QU[b];
B[y_QU, Z_QU] = X_QU - ϵ ℏ QU@{Z, y};
B[x_QU, Z_QU] = -ϵ ℏ QU@{Z, x} - (s + s ϵ ℏ) QU[Y] + 2 s ϵ ℏ QU[Y, a];
B[z_QU, X_QU] = 2 ϵ y_QU - ϵ ℏ QU@{X, z};
B[z_QU, Y_QU] = -2 t ϵ QU[x] - ϵ ℏ QU[Y, z];
B[y_QU, X_QU] = ϵ ℏ QU[X, y];
B[x_QU, Y_QU] = ϵ ℏ QU[Y, x];
B[X_QU, Y_QU] = 2 ϵ Z_QU - ϵ ℏ QU[X, Y];
B[X_QU, Z_QU] = ϵ ℏ QU[X, Z];
B[Z_QU, Y_QU] = ϵ ℏ QU[Y, Z];
```

In[•]:=
```
QU@Generators
```

Time: 0.02 seconds Out[•]=
```
{X, Y, Z, b, a, z, y, x}
```

```
In[•]:= $Basis = QU /@ QU@Generators;
        Module[{a, b}, Table[{a, b} → CE[B[a, b]], {a, $Basis}, {b, $Basis}]] // MatrixForm
```

Time: 0.02 seconds Out[•]=

$$\begin{pmatrix} \{QU[X], QU[X]\} \to 0 & \{QU[X], QU[Y]\} \to 2 \in QU[Z] - \in QU \\ \{QU[Y], QU[X]\} \to -2 \in QU[Z] + \in QU[X, Y] & \{QU[Y], QU[Y]\} \to 0 \\ \{QU[Z], QU[X]\} \to -\in QU[X, Z] & \{QU[Z], QU[Y]\} \to \in QU[Y, Z \\ \{QU[b], QU[X]\} \to -QU[X] & \{QU[b], QU[Y]\} \to 2\,QU[Y] \\ \{QU[a], QU[X]\} \to 2\,QU[X] & \{QU[a], QU[Y]\} \to -QU[Y] \\ \{QU[z], QU[X]\} \to 2 \in QU[y] - \in QU[X, z] & \{QU[z], QU[Y]\} \to -2\,t \in QU[x] - \in \\ \{QU[y], QU[X]\} \to \in QU[X, y] & \{QU[y], QU[Y]\} \to (1 - t)\,QU[] + 2\,t \in QU[t \\ \{QU[x], QU[X]\} \to (1 - s)\,QU[] + 2\,s \in QU[a] - 2 \in QU[X, x] & \{QU[x], QU[Y]\} \to \in QU[Y, x \end{pmatrix}$$

```
In[•]:= Module[{a, b, c},
         DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
             {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
        ]
```

Time: 0.80 seconds Out[•]=

```
{}
```

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The Multiplication Tensor

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```
In[•]:= adPower[x_, k_, Env_][c_ * y_QU] := CE[c adPower[x, k, Env][y]];
        adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
        adPower[_, 0, Env_][y_] := y;
        adPower[x_, k_, Env_][0] = 0;
        adPower[x_, k_, Env_][y_QU] :=
          adPower[x, k, Env][y] = CE@B[adPower[x, k - 1, Env][y], x];
        adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

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```
In[•]:= NilQ[zQU] = NilQ[ZQU] = NilQ[XQU] = NilQ[YQU] = NilQ[xQU] = NilQ[yQU] = True; NilQ[aQU] = False;
        NilQ[bQU] = False;
        PotQ[c_. * x_QU] := ¬ NilQ[x]; (* Potent Q *)
        NilQ[c_ * x_QU] := NilQ[x];
        PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
        NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

pdf

```
In[•]:= Ad[0] = Identity;
        Ad[c_. aQU][z_] := z /. w_QU :↦ e^(c γ (Sum[B[gQU, aQU]/gQU Count[w, g], {g, QU@Generators}])) w;
        Ad[c_. bQU][z_] := z /. w_QU :↦ e^(c γ (Sum[B[gQU, bQU]/gQU Count[w, g], {g, QU@Generators}])) w;
        Ad[c_. QU[]][z_] := z;
```

```
In[•]:= Table[SeriesCoefficient[Ad[c a_QU][g_QU], {c, 0, 1}] -
          B[g_QU, a_QU], {g, Generators // QU}]
        Table[SeriesCoefficient[Ad[c b_QU][g_QU], {c, 0, 1}] -
          B[g_QU, b_QU], {g, Generators // QU}]
```

Time: 0.06 seconds Out[•]=

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

Out[•]=

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

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```
In[•]:= Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
          While[adPower[x, k][y] =!= 0,
           s += adPower[x, k][y] / k!;
           ++k
          ];
          s
         ]
```

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```
In[•]:= TT_λ_[] = 0; (* Translated Tangent *)
        TT_λ_[xs___, x_] := CE[Ad[x][TT_λ[xs]] + ∂_λ x];
        TT_λ_[xs_List] := TT_λ @@ xs
```

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```
In[•]:= Seepage_U_[{}, c_] = c;
        Seepage_U_[ξs_List, P_] := Module[{gs = U[Generators], lx},
          lx = gs[[Length@ξs]];
          Collect[P, lx, (Ad[Last[ξs] lx_U][Θ_U[gs, Seepage_U[Most@ξs, #]]] /. U → Times) &]
         ]
```

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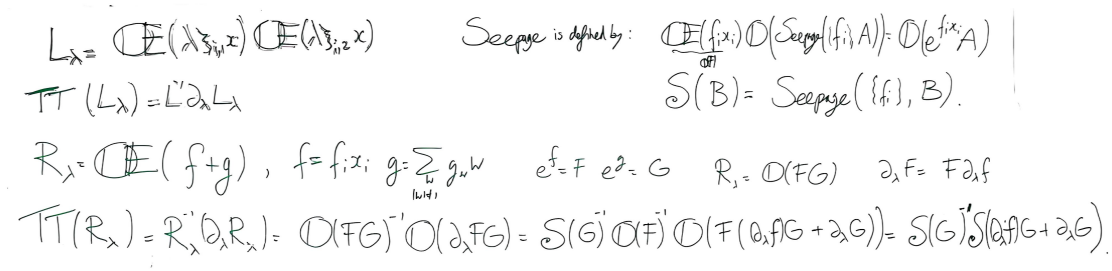


```
In[•]:= $k = 1
```

Time: 0.00 seconds Out[•]=

```
1
```

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In[•]:=
```
TTL = CE[TTλ @@ (Flatten@Table[
        (If[PotQ[#QU], 1, λ] #*[k] #QU) & /@ QU@Generators,
        {k, {i, j}}
       ])]
```

Time: 12.36 seconds Out[•]=

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In[•]:=
```
mons[k_] := Union@Cases[Coefficient[TTL, ϵ, k], _QU, ∞];
mons /@ Range[0, $k]
```

Time: 0.01 seconds Out[•]=

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```
{{QU[], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z]},
 {QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X],
  QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y],
  QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]}}
```

pdf

In[•]:=
```
ff = Table[
   Sum[ϵ^k f_{k,u}[λ], {k, 0, $k}],
   {u, QU@Generators}
  ] /. f_{0,u_}[λ] /; PotQ[u_QU] :> u*[i] + u*[j]
```

Time: 0.00 seconds Out[•]=

pdf

$$\{f_{0,X}[\lambda]+\epsilon f_{1,X}[\lambda],\ f_{0,Y}[\lambda]+\epsilon f_{1,Y}[\lambda],\ f_{0,Z}[\lambda]+\epsilon f_{1,Z}[\lambda],\ \beta[i]+\beta[j]+\epsilon f_{1,b}[\lambda],$$
$$\alpha[i]+\alpha[j]+\epsilon f_{1,a}[\lambda],\ f_{0,z}[\lambda]+\epsilon f_{1,z}[\lambda],\ f_{0,y}[\lambda]+\epsilon f_{1,y}[\lambda],\ f_{0,x}[\lambda]+\epsilon f_{1,x}[\lambda]\}$$

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In[•]:=
```
Short[fs = Flatten[Table[
    f_{k,Times@@m},
    {k, 0, $k}, {m, mons[k]}
   ]], 10]
```

Time: 0.05 seconds Out[•]=

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$$\{f_{0,1},\ f_{0,x},\ f_{0,X},\ f_{0,y},\ f_{0,Y},\ f_{0,z},\ f_{0,Z},\ f_{1,1},\ f_{1,a},\ f_{1,b},\ f_{1,x},\ f_{1,X},$$
$$f_{1,y},\ f_{1,Y},\ f_{1,z},\ f_{1,Z},\ f_{1,xx},\ f_{1,X^2},\ f_{1,Xy},\ f_{1,XY},\ f_{1,Xz},\ f_{1,XZ},\ f_{1,xy},\ f_{1,aY},$$
$$f_{1,xY},\ f_{1,yY},\ f_{1,Y^2},\ f_{1,Yz},\ f_{1,YZ},\ f_{1,xz},\ f_{1,yz},\ f_{1,z^2},\ f_{1,xZ},\ f_{1,yZ},\ f_{1,zZ}\}$$

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In[•]:=
```
gg = Sum[
   ϵ^k f_{k,Times@@u}[λ] Times @@ u,
   {k, 0, $k}, {u, Complement[mons[k], QU /@ QU@Generators]}
  ]
```

Time: 0.03 seconds Out[•]=

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$$f_{0,1}[\lambda]+\epsilon f_{1,1}[\lambda]+x X\epsilon f_{1,xX}[\lambda]+X^2\epsilon f_{1,X^2}[\lambda]+x y\epsilon f_{1,xy}[\lambda]+X y\epsilon f_{1,Xy}[\lambda]+$$
$$a Y\epsilon f_{1,aY}[\lambda]+x Y\epsilon f_{1,xY}[\lambda]+X Y\epsilon f_{1,XY}[\lambda]+y Y\epsilon f_{1,yY}[\lambda]+Y^2\epsilon f_{1,Y^2}[\lambda]+$$
$$x z\epsilon f_{1,xz}[\lambda]+X z\epsilon f_{1,Xz}[\lambda]+y z\epsilon f_{1,yz}[\lambda]+Y z\epsilon f_{1,Yz}[\lambda]+z^2\epsilon f_{1,z^2}[\lambda]+$$
$$x Z\epsilon f_{1,xZ}[\lambda]+X Z\epsilon f_{1,XZ}[\lambda]+y Z\epsilon f_{1,yZ}[\lambda]+Y Z\epsilon f_{1,YZ}[\lambda]+z Z\epsilon f_{1,zZ}[\lambda]$$

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In[•]:= `G = SS[𝕖^gg]`

Time: 0.11 seconds Out[•]=

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$$\mathbb{e}^{f_{0,1}[\lambda]} + \mathbb{e}^{f_{0,1}[\lambda]} \in \left(f_{1,1}[\lambda] + x\,X\,f_{1,xx}[\lambda] + X^2\,f_{1,X^2}[\lambda] + x\,y\,f_{1,xy}[\lambda] + X\,y\,f_{1,xy}[\lambda] + a\,Y\,f_{1,aY}[\lambda] + x\,Y\,f_{1,xY}[\lambda] + X\,Y\,f_{1,XY}[\lambda] + y\,Y\,f_{1,yY}[\lambda] + Y^2\,f_{1,Y^2}[\lambda] + x\,z\,f_{1,xz}[\lambda] + X\,z\,f_{1,xz}[\lambda] + y\,z\,f_{1,yz}[\lambda] + Y\,z\,f_{1,Yz}[\lambda] + z^2\,f_{1,z^2}[\lambda] + x\,Z\,f_{1,xZ}[\lambda] + X\,Z\,f_{1,xZ}[\lambda] + y\,Z\,f_{1,yZ}[\lambda] + Y\,Z\,f_{1,YZ}[\lambda] + z\,Z\,f_{1,zZ}[\lambda]\right)$$

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In[•]:= `𝒮[B_] := CE[SS@𝒪QU[QU@Generators, Seepage_QU[ff, SS@CE@B]]];`

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In[•]:= `Short[𝒮G = 𝒮[G], 10]`

Time: 23.88 seconds Out[•]=

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$$\mathbb{e}^{f_{0,1}[\lambda]} \in \mathrm{QU}[y, x]\; f_{1,xy}[\lambda] + \ll 25 \gg + \mathrm{QU}[z]$$
$$\left(\mathbb{e}^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]+f_{0,1}[\lambda]} \in f_{0,x}[\lambda]^2\, f_{1,xy}[\lambda] - \mathbb{e}^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]+f_{0,1}[\lambda]}\; s \in f_{0,x}[\lambda]^2\, f_{1,xy}[\lambda] + \mathbb{e}^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]+f_{0,1}[\lambda]} \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]^2\, f_{1,aY}[\lambda] - \mathbb{e}^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]+f_{0,1}[\lambda]}\; t \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]^2\, f_{1,aY}[\lambda] + \ll 18 \gg + \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]} \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]\, f_{1,zZ}[\lambda] - \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]}\; s \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]\, f_{1,zZ}[\lambda] - \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]} \in f_{0,z}[\lambda]\, f_{1,zZ}[\lambda] + \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]+f_{0,1}[\lambda]}\; s\,t \in f_{0,z}[\lambda]\, f_{1,zZ}[\lambda]\right)$$

In[•]:= `Short[Invert_QU[𝒮G], 10]`

Time: 0.03 seconds Out[•]=

$$-\mathbb{e}^{-f_{0,1}[\lambda]} \in \mathrm{QU}[y, x]\; f_{1,xy}[\lambda] - \mathbb{e}^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]-f_{0,1}[\lambda]} \in \mathrm{QU}[Y, a]\; f_{1,aY}[\lambda] + \ll 28 \gg + \mathrm{QU}[X, z]\; (\ll 1 \gg) + \mathrm{QU}[z]\left(-\mathbb{e}^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]-f_{0,1}[\lambda]} \in f_{0,x}[\lambda]^2\, f_{1,xy}[\lambda] + \mathbb{e}^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]-f_{0,1}[\lambda]}\; s \in f_{0,x}[\lambda]^2\, f_{1,xy}[\lambda] - \mathbb{e}^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]-f_{0,1}[\lambda]} \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]^2\, f_{1,aY}[\lambda] + \ll 21 \gg + \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]}\; s \in f_{0,x}[\lambda]\, f_{0,y}[\lambda]\, f_{1,zZ}[\lambda] + \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]} \in f_{0,z}[\lambda]\, f_{1,zZ}[\lambda] - \mathbb{e}^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]-f_{0,1}[\lambda]}\; s\,t \in f_{0,z}[\lambda]\, f_{1,zZ}[\lambda]\right)$$

In[•]:= `CE[Invert_QU[𝒮G] ⊠ 𝒮G]`

Time: 1.95 seconds Out[•]=

QU[]

$L_\lambda = \mathbb{E}(\lambda \xi_{ij} x)\,\mathbb{E}(\lambda \xi_{ji} x)$ Seepage is defined by: $\mathbb{E}(f_{ij} x_i)\,\mathbb{D}(\mathrm{Seepage}(\{f_{ij}\}, A)) = \mathbb{D}(e^{f_{ij} x_i} A)$ (diff)

$\mathcal{T}(L_\lambda) = L^{-1} \partial_\lambda L_\lambda$ $S(B) = \mathrm{Seepage}(\{f_{ij}\}, B)$.

$R_\lambda = \mathbb{E}(f+g)$, $f = f_{ij} x_i$, $g = \sum_{|w|\ge 1} g_w w$, $e^f = F$, $e^g = G$, $R_\lambda = \mathbb{D}(FG)$, $\partial_\lambda F = F \partial_\lambda f$

$\mathcal{T}(R_\lambda) = R_\lambda^{-1}(\partial_\lambda R_\lambda) = \mathbb{D}(FG)^{-1}\mathbb{D}(\partial_\lambda FG) = S(G)^{-1}\mathbb{D}(F)^{-1}\mathbb{D}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)^{-1}S(\partial_\lambda f G + \partial_\lambda G)$

```
In[•]:= Short[CE[(∂λ (ff.QU@Generators) G + ∂λG)], 10]
```

Time: 0.03 seconds Out[•]=

$$e^{f_{0,1}[\lambda]}\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]} \in f_{1,1}[\lambda]\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]}\, x\, X \in f_{1,x\,X}[\lambda]\, f_{0,1}'[\lambda] +$$
$$e^{f_{0,1}[\lambda]}\, X^2 \in f_{1,X^2}[\lambda]\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]}\, x\, y \in f_{1,x\,y}[\lambda]\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]}\, X\, y \in f_{1,X\,y}[\lambda]\, f_{0,1}'[\lambda] +$$
$$a\, e^{f_{0,1}[\lambda]}\, Y \in f_{1,a\,Y}[\lambda]\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]}\, x\, Y \in f_{1,x\,Y}[\lambda]\, f_{0,1}'[\lambda] + e^{f_{0,1}[\lambda]}\, X\, Y \in f_{1,X\,Y}[\lambda]\, f_{0,1}'[\lambda] +$$
$$\ll 158 \gg + e^{f_{0,1}[\lambda]}\, Y\, z \in f_{1,Y\,z}'[\lambda] + e^{f_{0,1}[\lambda]}\, z^2 \in f_{1,z^2}'[\lambda] + e^{f_{0,1}[\lambda]}\, Z \in f_{1,Z}'[\lambda] + e^{f_{0,1}[\lambda]}\, x\, Z \in f_{1,x\,Z}'[\lambda] +$$
$$e^{f_{0,1}[\lambda]}\, X\, Z \in f_{1,X\,Z}'[\lambda] + e^{f_{0,1}[\lambda]}\, y\, Z \in f_{1,y\,Z}'[\lambda] + e^{f_{0,1}[\lambda]}\, Y\, Z \in f_{1,Y\,Z}'[\lambda] + e^{f_{0,1}[\lambda]}\, z\, Z \in f_{1,z\,Z}'[\lambda]$$

```
In[•]:= Length@CE[(∂λ (ff.QU@Generators) G + ∂λG)]
```

Time: 0.02 seconds Out[•]=

175

```
In[•]:= 𝒮[CE[(∂λ (ff.QU@Generators) G + ∂λG)]]
```

Time: 140.48 seconds Out[•]=

$Aborted

pdf

```
In[•]:= TTR = CE[Invert_QU[𝒮G] ⊠ 𝒮[(∂λ (ff.QU@Generators) G + ∂λG)]]
```

Time: 90.61 seconds Out[•]=

pdf

```
In[•]:= Cases[TTR, x_QU, Infinity] // Sort
        Cases[TTL, x_QU, Infinity] // Sort
```

Time: 0.01 seconds Out[•]=

```
{QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X],
 QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y],
 QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]}
```

Out[•]=

```
{QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X],
 QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y],
 QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]}
```

pdf

```
In[•]:= Short[diff = CE[TTL - TTR], 10];
```

pdf

```
In[•]:= Short[eqns = Flatten[Table[
            Coefficient[Coefficient[diff, m], ϵ, k] == 0,
            {k, 0, $k}, {m, mons[k]}
           ]], 10];
```

pdf

```
In[•]:= Short[inits = (#[0] == 0) & /@ fs, 10];
```

pdf

```
In[•]:= DSolve[eqns ⋃ inits, fs, λ]
```

Time: 101.66 seconds Out[•]=

pdf

$$\Big\{\Big\{f_{0,x} \to \text{Function}\Big[\{\lambda},\ e^{-\beta[j]}\,\lambda\,\Big(e^{2\,\alpha[j]}\,\xi[i] + e^{\beta[j]}\,\xi[j]\Big)\Big],$$
$$f_{0,y} \to \text{Function}\Big[\{\lambda},\ e^{-\alpha[j]}\,\lambda\,\Big(e^{2\,\beta[j]}\,\eta[i] + e^{\alpha[j]}\,\eta[j]\Big)\Big],$$
$$f_{0,x} \to \text{Function}\Big[\{\lambda},\ e^{-\beta[i]}\,\lambda\,\Big(e^{\beta[i]}\,\Xi[i] + e^{2\,\alpha[i]}\,\Xi[j] + e^{2\,\alpha[i]}\,\lambda\,\eta[i]\,\Omega[j]\Big)\Big],$$

$$\mathbf{f}_{0,Y} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{-\alpha[\mathbf{i}]}\ \lambda\ \left(\mathbb{e}^{\alpha[\mathbf{i}]}\ \Upsilon[\mathbf{i}] + \mathbb{e}^{2\beta[\mathbf{i}]}\ \Upsilon[\mathbf{j}] - \mathbb{e}^{2\beta[\mathbf{i}]}\ \mathbf{s}\ \lambda\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]\right)\right],$$

$$\mathbf{f}_{0,Z} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{-\beta[\mathbf{j}]}\ \lambda\ \left(\mathbb{e}^{\alpha[\mathbf{j}]+2\beta[\mathbf{j}]}\ \zeta[\mathbf{i}] + \mathbb{e}^{\beta[\mathbf{j}]}\ \zeta[\mathbf{j}] + \mathbb{e}^{2\alpha[\mathbf{j}]}\ \lambda\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\right)\right],$$

$$\mathbf{f}_{0,1} \to \text{Function}\left[\{\lambda},\ \lambda^2\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}] - \mathbf{s}\ \lambda^2\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}] + \lambda^2\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}] - \mathbf{t}\ \lambda^2\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \lambda^2\ \zeta[\mathbf{i}]\ \Omega[\mathbf{j}] - \mathbf{s}\ \mathbf{t}\ \lambda^2\ \zeta[\mathbf{i}]\ \Omega[\mathbf{j}] - \mathbf{s}\ \lambda^3\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \mathbf{t}\ \lambda^3\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]\right],$$

$$\mathbf{f}_{0,Z} \to \text{Function}\left[\{\lambda},\ \lambda\ \left(\Omega[\mathbf{i}] + \mathbb{e}^{\alpha[\mathbf{i}]+\beta[\mathbf{i}]}\ \Omega[\mathbf{j}]\right)\right],\ \mathbf{f}_{1,a} \to$$

$$\text{Function}\left[\{\lambda},\ -2\ \mathbf{s}\ \lambda^2\ (-\xi[\mathbf{i}]\ \Xi[\mathbf{j}] - \mathbf{t}\ \zeta[\mathbf{i}]\ \Omega[\mathbf{j}] - \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{t}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}])\right],$$

$$\mathbf{f}_{1,b} \to \text{Function}\left[\{\lambda},\ -2\ \mathbf{t}\ \lambda^2\ (-\eta[\mathbf{i}]\ \Upsilon[\mathbf{j}] - \mathbf{s}\ \zeta[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}])\right],$$

$$\mathbf{f}_{1,x\,y} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{2\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \lambda^2\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\right],$$

$$\mathbf{f}_{1,x\,X} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{2\alpha[\mathbf{i}]+2\alpha[\mathbf{j}]-\beta[\mathbf{i}]-\beta[\mathbf{j}]}\ \lambda^2\ \xi[\mathbf{i}]\ (2\ \Xi[\mathbf{j}] + \lambda\ \eta[\mathbf{i}]\ \Omega[\mathbf{j}])\right],$$

$$\mathbf{f}_{1,x\,y} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{2\alpha[\mathbf{i}]-\alpha[\mathbf{j}]-\beta[\mathbf{i}]+2\beta[\mathbf{j}]}\ \lambda^2\ \eta[\mathbf{i}]\ \Xi[\mathbf{j}]\right],$$

$$\mathbf{f}_{1,X^2} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{4\alpha[\mathbf{i}]-2\beta[\mathbf{i}]}\ \lambda^3\ \eta[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}]\right],$$

$$\mathbf{f}_{1,a\,Y} \to \text{Function}\left[\{\lambda},\ 2\ \mathbb{e}^{-\alpha[\mathbf{i}]+2\beta[\mathbf{i}]}\ \mathbf{s}\ \lambda^2\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]\right],$$

$$\mathbf{f}_{1,x\,Y} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{-\alpha[\mathbf{i}]+2\alpha[\mathbf{j}]+2\beta[\mathbf{i}]-\beta[\mathbf{j}]}\ \lambda^2\ \xi[\mathbf{i}]\ (\Upsilon[\mathbf{j}] + 2\ \mathbf{s}\ \lambda\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}])\right],$$

$$\mathbf{f}_{1,y\,Y} \to \text{Function}\left[\{\lambda},\ -2\ \mathbb{e}^{-\alpha[\mathbf{i}]-\alpha[\mathbf{j}]+2\beta[\mathbf{i}]+2\beta[\mathbf{j}]}\ \lambda^2\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}]\right],$$

$$\mathbf{f}_{1,X\,Y} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{\alpha[\mathbf{i}]-\beta[\mathbf{i}]}\ \lambda^2\ \left(\mathbb{e}^{\alpha[\mathbf{i}]}\ \Xi[\mathbf{j}]\ \Upsilon[\mathbf{i}] + 2\ \mathbb{e}^{2\beta[\mathbf{i}]}\ \mathbf{s}\ \lambda\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{i}]}\ \lambda\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{i}]\ \Omega[\mathbf{j}] - \mathbb{e}^{2\beta[\mathbf{i}]}\ \lambda\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] + 2\ \mathbb{e}^{2\beta[\mathbf{i}]}\ \mathbf{s}\ \lambda^2\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]^2\right)\right],$$

$$\mathbf{f}_{1,Y^2} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{-2\alpha[\mathbf{i}]+4\beta[\mathbf{i}]}\ \mathbf{s}\ \lambda^3\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}]\right],$$

$$\mathbf{f}_{1,x\,z} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{2\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \lambda^2\ \zeta[\mathbf{j}]\ \xi[\mathbf{i}]\right],$$

$$\mathbf{f}_{1,x} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{2\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \lambda^2\ \left(\lambda\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}] - 3\ \mathbf{s}\ \lambda\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}] + 2\ \mathbf{t}\ \zeta[\mathbf{i}]\ \Upsilon[\mathbf{j}] - \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \mathbf{t}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \lambda\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] - 3\ \mathbf{s}\ \mathbf{t}\ \lambda\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] - 2\ \mathbf{s}\ \lambda^2\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}] + 2\ \mathbf{s}\ \mathbf{t}\ \lambda^2\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}]\right)\right],\ \mathbf{f}_{1,y\,z} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{-\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \lambda^2\ \left(\mathbb{e}^{3\beta[\mathbf{j}]}\ \zeta[\mathbf{j}]\ \eta[\mathbf{i}] + \mathbb{e}^{2\alpha[\mathbf{j}]+2\beta[\mathbf{j}]}\ \lambda\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}] + \mathbb{e}^{3\alpha[\mathbf{j}]}\ \lambda\ \eta[\mathbf{j}]^2\ \xi[\mathbf{i}]\right)\right],\ \mathbf{f}_{1,y} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{-\alpha[\mathbf{j}]+2\beta[\mathbf{j}]}\ \lambda^2\ \left(-2\ \zeta[\mathbf{i}]\ \Xi[\mathbf{j}] + 2\ \mathbf{s}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}] + \lambda\ \eta[\mathbf{i}]^2\ \Upsilon[\mathbf{j}] - 3\ \mathbf{t}\ \lambda\ \eta[\mathbf{i}]^2\ \Upsilon[\mathbf{j}] - \lambda\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \Omega[\mathbf{j}] - \mathbf{s}\ \mathbf{t}\ \lambda\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \lambda^2\ \eta[\mathbf{i}]^2\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \mathbf{t}\ \lambda^2\ \eta[\mathbf{i}]^2\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]\right)\right],$$

$$\mathbf{f}_{1,x\,Z} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{2\alpha[\mathbf{i}]+\alpha[\mathbf{j}]-\beta[\mathbf{i}]-\beta[\mathbf{j}]}\ \lambda^2\ \left(\mathbb{e}^{2\beta[\mathbf{j}]}\ \zeta[\mathbf{i}]\ \Xi[\mathbf{j}] + 2\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}] + \mathbb{e}^{2\beta[\mathbf{j}]}\ \lambda\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda^2\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]\right)\right],$$

$$\mathbf{f}_{1,X} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{2\alpha[\mathbf{i}]-\beta[\mathbf{i}]}\ \lambda^3\ \left(\xi[\mathbf{i}]\ \Xi[\mathbf{j}]^2 - 3\ \mathbf{s}\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}]^2 - \eta[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Upsilon[\mathbf{j}] + \mathbf{t}\ \eta[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Upsilon[\mathbf{j}] - \zeta[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] - \mathbf{s}\ \mathbf{t}\ \zeta[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] - 3\ \mathbf{s}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \mathbf{t}\ \lambda\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] + \lambda\ \eta[\mathbf{i}]^2\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] - \mathbf{t}\ \lambda\ \eta[\mathbf{i}]^2\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] - 2\ \mathbf{s}\ \mathbf{t}\ \lambda\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \Omega[\mathbf{j}]^2 - 2\ \mathbf{s}\ \lambda^2\ \eta[\mathbf{i}]^2\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]^2 + 2\ \mathbf{s}\ \mathbf{t}\ \lambda^2\ \eta[\mathbf{i}]^2\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]^2\right)\right],$$

$$\mathbf{f}_{1,Y\,z} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{-\alpha[\mathbf{i}]+\alpha[\mathbf{j}]+2\beta[\mathbf{i}]-\beta[\mathbf{j}]}\ \lambda^2\ \left(-\mathbb{e}^{2\beta[\mathbf{j}]}\ \zeta[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}] + 3\ \mathbb{e}^{2\beta[\mathbf{j}]}\ \mathbf{s}\ \lambda\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + 2\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{s}\ \lambda^2\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}]\right)\right],$$

$$\mathbf{f}_{1,Y} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{-\alpha[\mathbf{i}]+2\beta[\mathbf{i}]}\ \lambda^2\ \left(-2\ \mathbf{s}\ \lambda\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}]\ \Upsilon[\mathbf{j}] - \lambda\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}]^2 + 3\ \mathbf{t}\ \lambda\ \eta[\mathbf{i}]\ \Upsilon[\mathbf{j}]^2 - \mathbf{s}\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbf{s}\ \lambda^2\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] - \mathbf{s}^2\ \lambda^2\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}]\ \Omega[\mathbf{j}] - \lambda\ \zeta[\mathbf{i}]\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] + 3\ \mathbf{s}\ \mathbf{t}\ \lambda\ \zeta[\mathbf{i}]\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] - 4\ \mathbf{s}\ \mathbf{t}\ \lambda^2\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}]\ \Omega[\mathbf{j}] + 2\ \mathbf{s}\ \lambda^2\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]^2 - 4\ \mathbf{s}^2\ \mathbf{t}\ \lambda^2\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}]^2 - \mathbf{s}^2\ \lambda^3\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}]^2 + 3\ \mathbf{s}^2\ \mathbf{t}\ \lambda^3\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}]^2\right)\right],$$

$$\mathbf{f}_{1,z^2} \to \text{Function}\left[\{\lambda},\ \mathbb{e}^{2\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \lambda^3\ \zeta[\mathbf{j}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\right],\ \mathbf{f}_{1,z} \to \text{Function}\left[\{\lambda},\ -\mathbb{e}^{\alpha[\mathbf{j}]-\beta[\mathbf{j}]}\ \left(-2\ \mathbb{e}^{2\beta[\mathbf{j}]}\ \mathbf{s}\ \lambda^3\ \zeta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Xi[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda^4\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}] - 3\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{s}\ \lambda^4\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]^2\ \Xi[\mathbf{j}] + 2\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{t}\ \lambda^3\ \zeta[\mathbf{i}]\ \eta[\mathbf{j}]\ \Upsilon[\mathbf{j}] - \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda^4\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{t}\ \lambda^4\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Upsilon[\mathbf{j}] + \mathbb{e}^{2\beta[\mathbf{j}]}\ \lambda^3\ \zeta[\mathbf{i}]^2\ \Omega[\mathbf{j}] - 3\ \mathbb{e}^{2\beta[\mathbf{j}]}\ \mathbf{s}\ \mathbf{t}\ \lambda^3\ \zeta[\mathbf{i}]^2\ \Omega[\mathbf{j}] - 2\ \mathbb{e}^{2\beta[\mathbf{j}]}\ \mathbf{s}\ \lambda^4\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + 2\ \mathbb{e}^{2\beta[\mathbf{j}]}\ \mathbf{s}\ \mathbf{t}\ \lambda^4\ \zeta[\mathbf{i}]\ \eta[\mathbf{i}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] + \mathbb{e}^{\alpha[\mathbf{j}]}\ \lambda^4\ \zeta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] - 3\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{s}\ \mathbf{t}\ \lambda^4\ \zeta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]\ \Omega[\mathbf{j}] - 2\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{s}\ \lambda^5\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}] + 2\ \mathbb{e}^{\alpha[\mathbf{j}]}\ \mathbf{s}\ \mathbf{t}\ \lambda^5\ \eta[\mathbf{i}]\ \eta[\mathbf{j}]\ \xi[\mathbf{i}]^2\ \Omega[\mathbf{j}]\right)\right],$$

$$\mathtt{f_{1,1}} \to \mathtt{Function}\Big[\{\lambda\},\ \frac{1}{2}\ \Big(-\lambda^4\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]^2+4\,\mathtt{s}\,\lambda^4\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]^2-3\,\mathtt{s}^2\,\lambda^4\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]^2+ 4\,\lambda^3\,\zeta[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Upsilon[\mathtt{j}]-4\,\mathtt{t}\,\lambda^3\,\zeta[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Upsilon[\mathtt{j}]-4\,\mathtt{s}\,\lambda^4\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Upsilon[\mathtt{j}]+ 4\,\mathtt{s}\,\mathtt{t}\,\lambda^4\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Upsilon[\mathtt{j}]-\lambda^4\,\eta[\mathtt{i}]^2\,\Upsilon[\mathtt{j}]^2+4\,\mathtt{t}\,\lambda^4\,\eta[\mathtt{i}]^2\,\Upsilon[\mathtt{j}]^2-3\,\mathtt{t}^2\,\lambda^4\,\eta[\mathtt{i}]^2\,\Upsilon[\mathtt{j}]^2- 2\,\mathtt{s}\,\lambda^3\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]+2\,\mathtt{s}\,\mathtt{t}\,\lambda^3\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]+4\,\mathtt{s}\,\mathtt{t}\,\lambda^4\,\zeta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]- 4\,\mathtt{s}^2\,\mathtt{t}\,\lambda^4\,\zeta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]+2\,\mathtt{s}\,\lambda^5\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]-2\,\mathtt{s}^2\,\lambda^5\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]- 2\,\mathtt{s}\,\mathtt{t}\,\lambda^5\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]+2\,\mathtt{s}^2\,\mathtt{t}\,\lambda^5\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]^2\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]+ 4\,\mathtt{s}\,\mathtt{t}\,\lambda^4\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]-4\,\mathtt{s}\,\mathtt{t}^2\,\lambda^4\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]-4\,\mathtt{s}\,\mathtt{t}\,\lambda^5\,\eta[\mathtt{i}]^2\,\xi[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]+ 4\,\mathtt{s}\,\mathtt{t}^2\,\lambda^5\,\eta[\mathtt{i}]^2\,\xi[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]-\lambda^4\,\zeta[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2+4\,\mathtt{s}\,\mathtt{t}\,\lambda^4\,\zeta[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2- 3\,\mathtt{s}^2\,\mathtt{t}^2\,\lambda^4\,\zeta[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2+2\,\mathtt{s}\,\lambda^5\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2-2\,\mathtt{s}\,\mathtt{t}\,\lambda^5\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2- 6\,\mathtt{s}^2\,\mathtt{t}\,\lambda^5\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2+6\,\mathtt{s}^2\,\mathtt{t}^2\,\lambda^5\,\zeta[\mathtt{i}]\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2- \mathtt{s}^2\,\lambda^6\,\eta[\mathtt{i}]^2\,\xi[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2+4\,\mathtt{s}^2\,\mathtt{t}\,\lambda^6\,\eta[\mathtt{i}]^2\,\xi[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2-3\,\mathtt{s}^2\,\mathtt{t}^2\,\lambda^6\,\eta[\mathtt{i}]^2\,\xi[\mathtt{i}]^2\,\Omega[\mathtt{j}]^2\Big)\Big],$$

$$\mathtt{f_{1,x\,z}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{\alpha[\mathtt{i}]+2\,\alpha[\mathtt{j}]+\beta[\mathtt{i}]-\beta[\mathtt{j}]}\,\lambda^2\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]\Big],$$

$$\mathtt{f_{1,y\,z}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{\alpha[\mathtt{i}]-\alpha[\mathtt{j}]+\beta[\mathtt{i}]+2\,\beta[\mathtt{j}]}\,\lambda^2\,\eta[\mathtt{i}]\,\Omega[\mathtt{j}]\Big],$$

$$\mathtt{f_{1,x\,z}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{2\,\alpha[\mathtt{i}]-\beta[\mathtt{i}]}\,\lambda^2\,\Big(\Xi[\mathtt{j}]\,\Omega[\mathtt{i}]+\lambda\,\eta[\mathtt{i}]\,\Omega[\mathtt{i}]\,\Omega[\mathtt{j}]+e^{\alpha[\mathtt{i}]+\beta[\mathtt{i}]}\,\lambda\,\eta[\mathtt{i}]\,\Omega[\mathtt{j}]^2\Big)\Big],$$

$$\mathtt{f_{1,y\,z}} \to \mathtt{Function}\Big[\{\lambda\},\ e^{-\alpha[\mathtt{i}]+2\,\beta[\mathtt{i}]}\,\lambda^2\,\Big(\Upsilon[\mathtt{j}]\,\Omega[\mathtt{i}]-\mathtt{s}\,\lambda\,\xi[\mathtt{i}]\,\Omega[\mathtt{i}]\,\Omega[\mathtt{j}]+e^{\alpha[\mathtt{i}]+\beta[\mathtt{i}]}\,\mathtt{s}\,\lambda\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2\Big)\Big],$$

$$\mathtt{f_{1,z\,z}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{\alpha[\mathtt{i}]+\alpha[\mathtt{j}]+\beta[\mathtt{i}]-\beta[\mathtt{j}]}\,\lambda^2\,\Big(2\,e^{2\,\beta[\mathtt{j}]}\,\zeta[\mathtt{i}]+e^{\alpha[\mathtt{j}]}\,\lambda\,\eta[\mathtt{j}]\,\xi[\mathtt{i}]\Big)\,\Omega[\mathtt{j}]\Big],$$

$$\mathtt{f_{1,z}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{\alpha[\mathtt{i}]-\beta[\mathtt{i}]}\,\lambda^2\,\Big(2\,e^{\alpha[\mathtt{i}]}\,\Xi[\mathtt{j}]\,\Upsilon[\mathtt{i}]-2\,e^{2\,\beta[\mathtt{i}]}\,\mathtt{s}\,\lambda\,\xi[\mathtt{i}]\,\Xi[\mathtt{j}]\,\Omega[\mathtt{j}]+2\,e^{\alpha[\mathtt{i}]}\,\lambda\,\eta[\mathtt{i}]\,\Upsilon[\mathtt{i}]\,\Omega[\mathtt{j}]+ 2\,e^{2\,\beta[\mathtt{i}]}\,\lambda\,\eta[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]-2\,e^{2\,\beta[\mathtt{i}]}\,\mathtt{t}\,\lambda\,\eta[\mathtt{i}]\,\Upsilon[\mathtt{j}]\,\Omega[\mathtt{j}]+e^{2\,\beta[\mathtt{i}]}\,\lambda\,\zeta[\mathtt{i}]\,\Omega[\mathtt{j}]^2- 3\,e^{2\,\beta[\mathtt{i}]}\,\mathtt{s}\,\mathtt{t}\,\lambda\,\zeta[\mathtt{i}]\,\Omega[\mathtt{j}]^2-3\,e^{2\,\beta[\mathtt{i}]}\,\mathtt{s}\,\lambda^2\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2+3\,e^{2\,\beta[\mathtt{i}]}\,\mathtt{s}\,\mathtt{t}\,\lambda^2\,\eta[\mathtt{i}]\,\xi[\mathtt{i}]\,\Omega[\mathtt{j}]^2\Big)\Big]\Big\}\Big\}$$