Pensieve header: An isolated attempt to compute the multiplication tensor of QU.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Initialization / Utilities

```
In[∘]:= $k = 1; $U = QU; $E := {$k};
$trim := {ϵ^k_. /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
qℏ = e^(γ ϵ ℏ);
(* Upper to lower and lower to Upper: *)
U2l = {B_i_^p_. → e^(-p ℏ γ b_i), B^p_. → e^(-p ℏ γ b), T_i_^p_. → e^(p ℏ t_i), T^p_. → e^(p ℏ t), 𝒜_i_^p_. → e^(p γ α_i), 𝒜^p_. → e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^(-c/(ℏ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ℏ γ)) e^d,
   e^(c_. t_i_ + d_.) :→ T_i^(c/ℏ) e^d, e^(c_. t + d_.) :→ T^(c/ℏ) e^d,
   e^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
   e^ℰ_ :→ e^(Expand@ℰ)};
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. U2l, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
c_Integer_{k_Integer} := c + O[ϵ]^(k+1);
```

```
In[∘]:= CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^(CF[x])];
```

```
In[∘]:= Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs__] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

```
In[∘]:= SP_{}[P_] := P; SP_{ξ_→x_,ps___}[P_] := Expand[P // SP_{ps}] /. f_. ξ^d_. :→ ∂_{x,d} f
```

DeclareAlgebra

```
In[•]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ⊠ y) ⊠ z;
0 ⊠ _ = _ ⊠ 0 = 0;
(x_Plus) ⊠ y_ := (# ⊠ y) & /@ x; x_ ⊠ (y_Plus) := (x ⊠ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⊠ y - y ⊠ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

```
In[•]:= DeclareAlgebra[U_Symbol, opts__Rule] := Module[{gp, sr, g, cp, M, pow, k = 0,
     gs = Generators /. {opts},
     cs = Centrals /. {opts} /. Centrals → {} },
   (#_U = U@#) & /@ gs;
   gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
   sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
   cp = Alternatives @@ cs; (* cents *)
   SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
   CE[ℰ_] := Collect[ℰ, _U, Expand] /. 𝕖^x_ :→ 𝕖^Expand[x] /. $trim;
   U_i_ [ℰ_] := ℰ /. {t : cp :→ t_i, u_U :→ (#_i &) /@ u};
   U_i_ [NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
   B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
   B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
   B[U@y_, U@x_] := CE[-B[U@x, U@y]];
   x_ ⊠ (c_. 1_U) := CE[c x]; (c_. 1_U) ⊠ x_ := CE[c x];
   (a_. U[xx___, x_]) ⊠ (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
     CE@M[a b /. $trim, U[xx, x, y, yy]],
     U@xx ⊠ CE@M[a b /. $trim, U@y ⊠ U@x + B[U@x, U@y, $E]] ⊠ U@yy ];
   U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⊠ U@{r}];
   U@{c_. * l : gp, r___} := CE[c U[l] ⊠ U@{r}];
   U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
   U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
   U@{l_, r___} := U@{Expand[l], r};
   U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
   𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
     sp = Replace[{specs}, l_List :→ l_null, {1}];
     vs = Join @@ (First /@ sp);
     us = Join @@ (sp /. l_s_ :→ (l /. x_i_ :→ x_s));
     CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :→ c U@(us^p)
       ]] /. x_null :→ x];
   𝕆_U[specs___, 𝔼[L_, Q_, P_]] := 𝕆_U[specs, SS@Normal[P 𝕖^(L+Q)]];
   pow[ℰ_, n_] := pow[ℰ, n - 1] ⊠ ℰ;
   𝕊_U[ℰ_, ss___Rule] := CE@Total[
```

```
    CoefficientRules[ℰ, First /@ {ss}] /.
     (p_ → c_) ⧴ c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
σrs___[c_. * u_U] := (c /. (t : cp)j_ ⧴ tj/.{rs}) U[List @@ (u /. v_j_ ⧴ vj/.{rs})];
mj_→k_[c_. * u_U] := CE[((c /. (t : cp)j → tk) DeleteCases[u, _j|k]) ⊠
    U @@ Cases[u, w_j ⧴ wk] ⊠ U @@ Cases[u, _k]];
U /: c_. * u_U * v_U := CE[c u ⊠ v];
Si_[c_. * u_U] := CE[((c /. Si[U, Centrals]) DeleteCases[u, _i]) ⊠
    Ui[NCM @@ Reverse@Cases[u, x_i ⧴ S@U@x]]];
Δi_→j_,k_[c_. * u_U] := CE[((c /. Δi→j,k[U, Centrals]) DeleteCases[u, _i]) ⊠
    (NCM @@ Cases[u, x_i ⧴ σ1→j,2→k@Δ@U@x] /. NCM[] → U[])]; ]
```

Meta-Operations

```
In[∘]:= σrs___[ℰ_Plus] := σrs /@ ℰ;
mj_→j_ = Identity; mj_→k_[0] = 0;
mj_→k_[ℰ_Plus] := Simp[mj→k /@ ℰ];
mis___,i_,j_→k_[ℰ_] := mj→k@mis,i→j@ℰ;
Si_[ℰ_Plus] := Simp[Si /@ ℰ];
Δis___[ℰ_Plus] := Simp[Δis /@ ℰ];
```

Implementing CU = $\mathcal{U}(\mathrm{sl}_2^{\gamma\epsilon})$

Verify σ and Δ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\ldots}$.

```
In[∘]:= DeclareAlgebra[CU, Generators → {y, a, x}, Centrals → {t}];
B[aCU, yCU] = -γ yCU; B[xCU, aCU] = -γ xCU;
B[xCU, yCU] = 2 ϵ aCU - t 1CU;
(S@yCU = -yCU; S@aCU = -aCU; S@xCU = -xCU;)
Si_[CU, Centrals] = {ti → -ti};
Δ@yCU = CU@y1 + CU@y2; Δ@aCU = CU@a1 + CU@a2; Δ@xCU = CU@x1 + CU@x2;
Δi_→j_,k_[CU, Centrals] = {ti → tj + tk};
```

Implementing QU = $\mathcal{U}_q(\mathrm{sl}_2^{\gamma\epsilon})$

```
In[∘]:= DeclareAlgebra[QU, Generators → {y, a, x}, Centrals → {t, T}];
B[aQU, yQU] = -γ yQU; B[xQU, aQU] = -γ QU@x;
B[xQU, yQU] := SS[qℏ - 1] QU@{y, x} + 𝕆QU[{a}, SS[(1 - T ⅇ^(-2 ϵ a ℏ)) / ℏ]];
(S@yQU := 𝕆QU[{a, y}, SS[-T^-1 ⅇ^(ℏ ϵ a) y]]; S@aQU = -aQU; S@xQU := 𝕆QU[{a, x}, SS[-ⅇ^(ℏ ϵ a) x]];)
Si_[QU, Centrals] = {ti → -ti, Ti → Ti^-1};
Δ@yQU := 𝕆QU[{y1, a1}1, {y2}2, SS[y1 + T1 ⅇ^(-ℏ ϵ a1) y2]];
Δ@aQU = QU@a1 + QU@a2; Δ@xQU := 𝕆QU[{a1, x1}1, {x2}2, SS[x1 + ⅇ^(-ℏ ϵ a1) x2]];
Δi_→j_,k_[QU, Centrals] = {ti → tj + tk, Ti → Tj Tk};
```

tSW

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Goal. In either U, compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta(e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
In[•]:= SW_xy[U_, kk_] :=
   SW_xy[U, kk] = Block[{$U = U, $k = kk, $p = kk}, Module[{G, F, fs, f, bs, e, b, es},
      G = Simp[Table[ξ^k / k!, {k, 0, $k + 1}].NestList[Simp[B[x_U, #]] &, y_U, $k + 1]];
      fs = Flatten@Table[f_{l,i,j,k}[η], {l, 0, $k}, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
      F = fs.(bs = fs /. f_{l_,i_,j_,k_}[η] :> ϵ^l U@{y^i, a^j, x^k});
      es = Flatten[
        Table[Coefficient[e, b] == 0, {e, {F - 1_U /. η → 0, F ⊠ G - y_U ⊠ F - ∂_η F}}, {b, bs}]];
      F = F /. DSolve[es, fs, η][[1]];
      𝔼[0,
        ξ x + η y + (U /. {CU → -t η ξ, QU → η ξ (1 - T) / ℏ}),
        F + O_$k /. {e- → 1, U → Times}
       ] /. (v : η | ξ | t | T | y | a | x) → v_1
     ]];
tSW_{xy_,i_,j_→k_} := SW_xy[$U, $k] /. {ξ_1 → ξ_i, η_1 → η_j, (v : t | T | y | a | x)_1 → v_k};
tSW_{xa,i_,j_→k_} := 𝔼[α_j a_k, e^{-γ α_j} ξ_i x_k, 1];
tSW_{ay,i_,j_→k_} := 𝔼[α_i a_k, e^{-γ α_i} η_j y_k, 1];
```

Exponentials as needed.

Task. Define $\mathrm{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi\,\mathbb{O}(P)}$ to ϵ^k in the algebra U_i, where ξ is a scalar, X is x_i or y_i, and P is an ϵ-dependent near-docile element, giving the answer in $\mathbb{E}$-form. Should satisfy
$U@\mathrm{Exp}_{U_i,k}[\xi, P] == \mathbb{S}_U[e^{\xi x},\ x \to \mathbb{O}(P)]$.
Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\xi\,\mathbb{O}(P)} = \mathbb{O}(e^{\xi P_0} F(\xi))$, then $F(\xi = 0) = 1$ and we have:
$\mathbb{O}(e^{\xi P_0}(P_0 F(\xi) + \partial_\xi F)) = \mathbb{O}(\partial_\xi e^{\xi P_0} F(\xi)) = \partial_\xi \mathbb{O}(e^{\xi P_0} F(\xi)) = \partial_\xi e^{\xi\,\mathbb{O}(P)} = e^{\xi\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}(e^{\xi P_0} F(\xi))\,\mathbb{O}(P)$.
This is an ODE for F. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ.

In[∘]:=
```
(* Bug: The first line is valid only if 𝕆(ⅇ^P0)==ⅇ^𝕆(P0). *)
(* Bug: ξ must be a symbol. *)
Exp_{U_i_,0}[ξ_, P_] := Module[{LQ = Normal@P /. ϵ → 0},
  𝔼[ξ LQ /. (x | y)_i → 0, ξ LQ /. (t | a)_i → 0, 1]];
Exp_{U_i_,k_}[ξ_, P_] := Block[{$U = U, $k = k},
  Module[{P0, φ, φs, F, j, rhs, at0, atξ},
   P0 = Normal@P /. ϵ → 0;
   φs =
    Flatten@Table[φ_{j1,j2,j3}[ξ], {j2, 0, k}, {j1, 0, 2 k + 1 - j2}, {j3, 0, 2 k + 1 - j2 - j1}];
   F = Normal@Last@Exp_{U_i,k-1}[ξ, P] + ϵ^k φs.(φs /. φ_{js__}[ξ] :> Times @@ {y_i, a_i, x_i}^{js});
   rhs = Normal@
     Last@m_{i,j→i}[𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + 0_k] m_{i→j}@𝔼[0, 0, P + 0_k]];
   at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /. ξ → 0, {y_i, a_i, x_i}];
   atξ = (# == 0) & /@ Flatten@CoefficientList[(∂_ξ F) + P0 F - rhs, {y_i, a_i, x_i}];
   𝔼[ξ P0 /. (x | y)_i → 0, ξ P0 /. (t | a)_i → 0, F + 0_k] /.
    DSolve[And @@ (at0 ⋃ atξ), φs, ξ][[1]] ]]
```

In[∘]:=
```
$k = 1
```

Out[∘]=

1

In[∘]:=
```
$Basis = {a_QU, x_QU, y_QU};
Module[{a, b}, Table[{a, b} → CE[B[a, b]], {a, $Basis}, {b, $Basis}]] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \{QU[a], QU[a]\} \to 0 & \{QU[a], QU[x]\} \to \gamma\, QU[x] & \\ \{QU[x], QU[a]\} \to -\gamma\, QU[x] & \{QU[x], QU[x]\} \to 0 & \{QU[x], \\ \{QU[y], QU[a]\} \to \gamma\, QU[y] & \{QU[y], QU[x]\} \to \left(-\frac{1}{\hbar} + \frac{T}{\hbar}\right) QU[] - 2\, T\, \epsilon\, QU[a] - \gamma\, \epsilon\, \hbar\, QU[y, x] & \end{pmatrix}$$

In[ ]:=
```
$Basis = {a_QU, x_QU, y_QU};
Module[{a, b},
 DeleteCases[
  Flatten[Table[{a, b, c} → CE[B[a, b] + B[b, a]], {a, $Basis}, {b, $Basis}]], _ → 0]
]
Module[{a, b, c},
 DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
    {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
]
$ABasis = {a_QU, x_QU, y_QU, QU[y, y], QU[y, a], QU[y, x], QU[a, a], QU[a, x], QU[x, x]};
Module[{a, b, c},
 DeleteCases[Flatten[Table[{a, b, c} → CE[a ⊠ CE[b ⊠ c] - CE[a ⊠ b] ⊠ c],
    {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]], _ → 0]
]
```

Out[ ]=
{}

Out[ ]=
{}

Out[ ]=
{}

In[ ]:=
```
Module[{a, b, c},
  Table[{a, b, c} → CE[a ⊠ CE[b ⊠ c]], {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]
 ];
```

In[ ]:= `B[x_QU, y_QU] // CE`

Out[ ]=
$$\left(\frac{1}{\hbar} - \frac{T}{\hbar}\right) \mathrm{QU}[\,] + 2\,T\,\epsilon\,\mathrm{QU}[a] + \gamma\,\epsilon\,\hbar\,\mathrm{QU}[y, x]$$

In[ ]:= `B[x_QU, B[x_QU, y_QU]] // CE`

Out[ ]=
$$(\gamma\,\epsilon - 3\,T\,\gamma\,\epsilon)\,\mathrm{QU}[x]$$

In[ ]:= `B[x_QU, B[x_QU, B[x_QU, y_QU]]] // CE`

Out[ ]=
0

In[ ]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]] // CE`

Out[ ]=
0

In[ ]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]]] // Expand`

Out[ ]=
0

In[ ]:= `B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, B[x_QU, y_QU]]]]]] // Expand`

Out[ ]=
0

In[∘]:=
```
adPower[x_, k_, Env_][c_ * y_QU] := CE[c adPower[x, k, Env][y]];
adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_][y_] := y;
adPower[x_, k_, Env_][0] = 0;
adPower[x_, k_, Env_][y_QU] :=
  adPower[x, k, Env][y] = CE@B[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

In[∘]:= adPower[x_{QU}, 1][y_{QU}]

Out[∘]=

$$\left(-\frac{1}{\hbar}+\frac{T}{\hbar}\right) QU[\,] - 2\,T\,\epsilon\,QU[a] - \gamma\,\epsilon\,\hbar\,QU[y, x]$$

In[∘]:= adPower[x_{QU}, 2][y_{QU}]

Out[∘]=

$$(\gamma\,\epsilon - 3\,T\,\gamma\,\epsilon)\;QU[x]$$

In[∘]:= adPower[x_{QU}, 3][y_{QU}]

Out[∘]=

0

In[∘]:= adPower[x_{QU}, 4][y_{QU}]

Out[∘]=

0

In[∘]:= adPower[x_{QU}, 5][y_{QU}]

Out[∘]=

0

In[∘]:=
```
NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
PotQ[c_. * x_QU] := ¬ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

In[∘]:= Ad[c_. a_{QU}][z_] := z /. w_QU :→ $e^{c\,\gamma\,(\text{Count}[w,y]-\text{Count}[w,x])}$ w

In[∘]:=
```
Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

In[∘]:= Ad[x_{QU}][y_{QU}]

Out[∘]=

$$-\frac{QU[\,]}{\hbar}+\frac{T\,QU[\,]}{\hbar} - 2\,T\,\epsilon\,QU[a] + \frac{1}{2}\,\gamma\,\epsilon\,QU[x] - \frac{3}{2}\,T\,\gamma\,\epsilon\,QU[x] + QU[y] - \gamma\,\epsilon\,\hbar\,QU[y, x]$$

In[•]:=
```
Ad[c a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]}
```
Out[•]=

$$\{QU[],\ e^{-c\,\gamma}\,QU[x],\ e^{c\,\gamma}\,QU[y],\ QU[x,y]\}$$

In[•]:=
```
Ad[(α1 + α2 + ϵ fa[λ]) a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]}
```
Out[•]=

$$\left\{QU[],\ e^{-\gamma\,(\alpha1+\alpha2+\epsilon\,fa[\lambda])}\,QU[x],\ e^{\gamma\,(\alpha1+\alpha2+\epsilon\,fa[\lambda])}\,QU[y],\ QU[x,y]\right\}$$

In[•]:=
```
Normal@Series[e^(-γ (α1+α2+ϵ fa[λ])), {ϵ, 0, $k}]
```
Out[•]=

$$e^{-((\alpha1+\alpha2)\,\gamma)} - e^{-((\alpha1+\alpha2)\,\gamma)}\,\gamma\,\epsilon\,fa[\lambda]$$

In[•]:=
```
CE[
 Ad[(α1 + α2 + ϵ fa[λ]) a_QU] /@ {QU[], x_QU, y_QU, QU[x, y]} /. e^ℰ_ :→ Normal@Series[e^ℰ, {ϵ, 0, $k}]]
```
Out[•]=

$$\left\{QU[],\ \left(e^{-\alpha1\,\gamma-\alpha2\,\gamma} - e^{-\alpha1\,\gamma-\alpha2\,\gamma}\,\gamma\,\epsilon\,fa[\lambda]\right)QU[x],\ \left(e^{\alpha1\,\gamma+\alpha2\,\gamma} + e^{\alpha1\,\gamma+\alpha2\,\gamma}\,\gamma\,\epsilon\,fa[\lambda]\right)QU[y],\ QU[x,y]\right\}$$

In[•]:=
```
Ad[a_QU][Ad[x_QU][y_QU]]
```
Out[•]=

$$-\frac{QU[]}{\hbar} + \frac{T\,QU[]}{\hbar} - 2\,T\,\epsilon\,QU[a] + \frac{1}{2}\,e^{-\gamma}\,\gamma\,\epsilon\,QU[x] - \frac{3}{2}\,e^{-\gamma}\,T\,\gamma\,\epsilon\,QU[x] + e^{\gamma}\,QU[y] - \gamma\,\epsilon\,\hbar\,QU[y,x]$$

In[•]:=
```
λTangent[] = 0;
λTangent[xs___, x_] := CE[Ad[x][λTangent[xs]] + ∂_λ x];
λTangent[xs_List] := λTangent @@ xs
```

In[•]:=
```
λTangent[λ y_QU, λ a_QU, λ x_QU]
```
Out[•]=

$$\left(-\frac{e^{\gamma\,\lambda}\,\lambda}{\hbar} + \frac{e^{\gamma\,\lambda}\,T\,\lambda}{\hbar}\right)QU[] + \left(1 - 2\,e^{\gamma\,\lambda}\,T\,\epsilon\,\lambda\right)QU[a] +$$
$$\left(1 + \gamma\,\lambda + \frac{1}{2}\,e^{\gamma\,\lambda}\,\gamma\,\epsilon\,\lambda^2 - \frac{3}{2}\,e^{\gamma\,\lambda}\,T\,\gamma\,\epsilon\,\lambda^2\right)QU[x] + e^{\gamma\,\lambda}\,QU[y] - e^{\gamma\,\lambda}\,\gamma\,\epsilon\,\lambda\,\hbar\,QU[y,x]$$

In[•]:=
```
$k = 1
```
Out[•]=

1

In[∘]:= **lhs = λTangent[λ η1 y_QU, α1 a_QU, λ ξ1 x_QU, λ η2 y_QU, α2 a_QU, λ ξ2 x_QU]**

Out[∘]=

$$\left(-\frac{e^{\alpha 1\,\gamma}\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,T\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{T\,\eta 2\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}-\frac{2\,e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\eta 1\,\lambda\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,T\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{4\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}+\frac{2\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{3\,e^{\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}\right)\mathrm{QU}[\,]+$$
$$\left(-2\,e^{\alpha 1\,\gamma}\,T\,\epsilon\,\eta 1\,\lambda\,\xi 1+2\,T\,\epsilon\,\eta 2\,\lambda\,\xi 1-2\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\eta 1\,\lambda\,\xi 2-2\,e^{\alpha 2\,\gamma}\,T\,\epsilon\,\eta 2\,\lambda\,\xi 2\right)\mathrm{QU}[a]+$$
$$\left(e^{-\alpha 2\,\gamma}\,\xi 1+\frac{1}{2}\,e^{\alpha 1\,\gamma-\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1^2-\frac{3}{2}\,e^{\alpha 1\,\gamma-\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1^2+\xi 2+e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1\,\xi 2-3\,e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1\,\xi 2-\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2+3\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2+\frac{1}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 2^2-\frac{3}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 2^2+\frac{1}{2}\,e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 2^2-\frac{3}{2}\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 2^2\right)\mathrm{QU}[x]+$$
$$\left(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1+e^{\alpha 2\,\gamma}\,\eta 2-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+\frac{1}{2}\,e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1-\frac{3}{2}\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1\right)\mathrm{QU}[y]+$$
$$\left(-e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 1\,\hbar+\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 1\,\hbar-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 2\,\hbar-e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 2\,\hbar\right)\mathrm{QU}[y,x]$$

In[∘]:= **rhs = CE[(ft0[λ] + ϵ ft1[λ]) QU[] + λTangent[(fy0[λ] + ϵ fy1[λ]) y_QU, (α1 + α2 + ϵ fa1[λ]) a_QU, (fx0[λ] + ϵ fx1[λ]) x_QU] /. e^{δ_} :→ Normal@Series[e^δ, {ϵ, 0, $k}]]**

Out[∘]=

$$-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\hbar\,\mathrm{fx0}[\lambda]\,\mathrm{QU}[y,x]\,\mathrm{fy0}'[\lambda]+$$
$$\mathrm{QU}[a]\left(\epsilon\,\mathrm{fa1}'[\lambda]-2\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\right)+\mathrm{QU}[x]\left(\gamma\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fa1}'[\lambda]+\mathrm{fx0}'[\lambda]+\epsilon\,\mathrm{fx1}'[\lambda]+\frac{1}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]-\frac{3}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]\right)+$$
$$\mathrm{QU}[y]\left(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fy0}'[\lambda]+e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fy0}'[\lambda]+e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fy1}'[\lambda]\right)+$$
$$\mathrm{QU}[\,]\left(\mathrm{ft0}[\lambda]+\epsilon\,\mathrm{ft1}[\lambda]-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}\right)$$

In[∘]:= **diff = CE[lhs − rhs]**

Out[∘]=

$$
\begin{aligned}
&\mathrm{QU}[a]\,\Big(-2\,e^{\alpha 1\,\gamma}\,T\,\epsilon\,\eta 1\,\lambda\,\xi 1+2\,T\,\epsilon\,\eta 2\,\lambda\,\xi 1-2\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\eta 1\,\lambda\,\xi 2-\\
&\quad 2\,e^{\alpha 2\,\gamma}\,T\,\epsilon\,\eta 2\,\lambda\,\xi 2-\epsilon\,\mathrm{fa1}'[\lambda]+2\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[y,x]\,\Big(-e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 1\,\hbar+\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 1\,\hbar-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 2\,\hbar-\\
&\quad e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 2\,\hbar+e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\hbar\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[x]\,\Big(e^{-\alpha 2\,\gamma}\,\xi 1+\frac{1}{2}\,e^{\alpha 1\,\gamma-\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1^2-\frac{3}{2}\,e^{\alpha 1\,\gamma-\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1^2+\xi 2+e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1\,\xi 2-\\
&\quad 3\,e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 1\,\xi 2-\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2+3\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2+\frac{1}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 2^2-\\
&\quad \frac{3}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda^2\,\xi 2^2+\frac{1}{2}\,e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 2^2-\frac{3}{2}\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 2^2-\gamma\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fa1}'[\lambda]-\\
&\quad \mathrm{fx0}'[\lambda]-\epsilon\,\mathrm{fx1}'[\lambda]-\frac{1}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]+\frac{3}{2}\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fx0}[\lambda]^2\,\mathrm{fy0}'[\lambda]\Big)+\mathrm{QU}[y]\\
&\Big(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1+e^{\alpha 2\,\gamma}\,\eta 2-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+\frac{1}{2}\,e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1-\\
&\quad \frac{3}{2}\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fy0}'[\lambda]-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fy0}'[\lambda]-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fy1}'[\lambda]\Big)+\\
&\mathrm{QU}[\,]\,\Big(-\frac{e^{\alpha 1\,\gamma}\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,T\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{T\,\eta 2\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}-\\
&\quad \frac{2\,e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1\,\lambda\,\xi 2}{\hbar}+\\
&\quad \frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\eta 1\,\lambda\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,T\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\\
&\quad \frac{4\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}+\\
&\quad \frac{2\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{3\,e^{\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}-\mathrm{ft0}[\lambda]-\epsilon\,\mathrm{ft1}[\lambda]+\\
&\quad \frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\\
&\quad \frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\mathrm{fa1}[\lambda]\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\\
&\quad \frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx1}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\mathrm{fx0}[\lambda]\,\mathrm{fy1}'[\lambda]}{\hbar}\Big)
\end{aligned}
$$

In[∘]:= **diff /. ϵ → 0**

Out[∘]=

$$
\begin{aligned}
&\mathrm{QU}[x]\,\Big(e^{-\alpha 2\,\gamma}\,\xi 1+\xi 2-\mathrm{fx0}'[\lambda]\Big)+\mathrm{QU}[y]\,\Big(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1+e^{\alpha 2\,\gamma}\,\eta 2-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fy0}'[\lambda]\Big)+\\
&\mathrm{QU}[\,]\,\Big(-\frac{e^{\alpha 1\,\gamma}\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,T\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{T\,\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\eta 1\,\lambda\,\xi 2}{\hbar}-\\
&\quad \frac{e^{\alpha 2\,\gamma}\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,T\,\eta 2\,\lambda\,\xi 2}{\hbar}-\mathrm{ft0}[\lambda]+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\mathrm{fx0}[\lambda]\,\mathrm{fy0}'[\lambda]}{\hbar}\Big)
\end{aligned}
$$

In[◦]:=
```
{sol1} = DSolve[Coefficient[diff /. ϵ → 0, x_QU] == 0 ∧ fx0[0] == 0, fx0, λ]
```
Out[◦]=

$$\left\{\left\{\text{fx0}\to\text{Function}\left[\{\lambda\},\ e^{-\alpha 2\,\gamma}\,\lambda\,\left(\xi 1+e^{\alpha 2\,\gamma}\,\xi 2\right)\right]\right\}\right\}$$

In[◦]:=
```
CE[diff /. sol1 /. ϵ → 0]
```
Out[◦]=

$$\text{QU}[y]\,\left(e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1+e^{\alpha 2\,\gamma}\,\eta 2-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\text{fy0}'[\lambda]\right)+$$
$$\text{QU}[\,]\left(-\frac{e^{\alpha 1\,\gamma}\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,T\,\eta 1\,\lambda\,\xi 1}{\hbar}+\frac{\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{T\,\eta 2\,\lambda\,\xi 1}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\eta 1\,\lambda\,\xi 2}{\hbar}+\right.$$
$$\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\eta 1\,\lambda\,\xi 2}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,\eta 2\,\lambda\,\xi 2}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,T\,\eta 2\,\lambda\,\xi 2}{\hbar}-\text{ft0}[\lambda]+\frac{e^{\alpha 1\,\gamma}\,\lambda\,\xi 1\,\text{fy0}'[\lambda]}{\hbar}-$$
$$\left.\frac{e^{\alpha 1\,\gamma}\,T\,\lambda\,\xi 1\,\text{fy0}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\lambda\,\xi 2\,\text{fy0}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\lambda\,\xi 2\,\text{fy0}'[\lambda]}{\hbar}\right)$$

In[◦]:=
```
{sol2} = DSolve[Coefficient[CE[diff /. sol1 /. ϵ → 0], y_QU] == 0 ∧ fy0[0] == 0, fy0, λ]
```
Out[◦]=

$$\left\{\left\{\text{fy0}\to\text{Function}\left[\{\lambda\},\ e^{-\alpha 1\,\gamma}\,\left(e^{\alpha 1\,\gamma}\,\eta 1+\eta 2\right)\,\lambda\right]\right\}\right\}$$

In[◦]:=
```
{sol3} = Solve[CE[diff /. sol1 /. sol2 /. ϵ → 0] == 0, ft0[λ]]
```
Out[◦]=

$$\left\{\left\{\text{ft0}[\lambda]\to-\frac{2\,\left(-\eta 2\,\lambda\,\xi 1+T\,\eta 2\,\lambda\,\xi 1\right)}{\hbar}\right\}\right\}$$

In[*]:= **CE[diff /. sol1 /. sol2 /. sol3]**

Out[*]=

$$2\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 1\,\hbar\,\mathrm{QU}[y,x]+\mathrm{QU}[a]\,\left(4\,T\,\epsilon\,\eta 2\,\lambda\,\xi 1-\epsilon\,\mathrm{fa1}'[\lambda]\right)+$$
$$\mathrm{QU}[x]\,\left(-\frac{1}{2}\,e^{-\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1^2+\frac{3}{2}\,e^{-\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1^2-2\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2+\right.$$
$$\left.6\,T\,\gamma\,\epsilon\,\eta 2\,\lambda^2\,\xi 1\,\xi 2-e^{-\alpha 2\,\gamma}\,\gamma\,\epsilon\,\lambda\,\xi 1\,\mathrm{fa1}'[\lambda]-\gamma\,\epsilon\,\lambda\,\xi 2\,\mathrm{fa1}'[\lambda]-\epsilon\,\mathrm{fx1}'[\lambda]\right)+$$
$$\mathrm{QU}[y]\,\left(-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^2\,\xi 1+\frac{1}{2}\,e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1-\right.$$
$$\left.\frac{3}{2}\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^2\,\xi 1-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\mathrm{fa1}[\lambda]-e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\mathrm{fa1}[\lambda]-e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\mathrm{fy1}'[\lambda]\right)+$$
$$\mathrm{QU}[\,]\,\left(\frac{e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}-\frac{2\,e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1^2}{2\,\hbar}+\right.$$
$$\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{4\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}+\frac{3\,e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 1\,\eta 2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-$$
$$\frac{e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}+\frac{2\,e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{\hbar}-\frac{3\,e^{\alpha 2\,\gamma}\,T^2\,\gamma\,\epsilon\,\eta 2^2\,\lambda^3\,\xi 1\,\xi 2}{2\,\hbar}+$$
$$\frac{e^{\alpha 1\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}+\frac{\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}-$$
$$\frac{T\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 1\,\mathrm{fa1}[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 1\,\lambda\,\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}+$$
$$\frac{e^{\alpha 2\,\gamma}\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,T\,\gamma\,\epsilon\,\eta 2\,\lambda\,\xi 2\,\mathrm{fa1}[\lambda]}{\hbar}-\epsilon\,\mathrm{ft1}[\lambda]+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\eta 1\,\mathrm{fx1}[\lambda]}{\hbar}-$$
$$\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\eta 1\,\mathrm{fx1}[\lambda]}{\hbar}+\frac{e^{\alpha 2\,\gamma}\,\epsilon\,\eta 2\,\mathrm{fx1}[\lambda]}{\hbar}-\frac{e^{\alpha 2\,\gamma}\,T\,\epsilon\,\eta 2\,\mathrm{fx1}[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma}\,\epsilon\,\lambda\,\xi 1\,\mathrm{fy1}'[\lambda]}{\hbar}-$$
$$\left.\frac{e^{\alpha 1\,\gamma}\,T\,\epsilon\,\lambda\,\xi 1\,\mathrm{fy1}'[\lambda]}{\hbar}+\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,\epsilon\,\lambda\,\xi 2\,\mathrm{fy1}'[\lambda]}{\hbar}-\frac{e^{\alpha 1\,\gamma+\alpha 2\,\gamma}\,T\,\epsilon\,\lambda\,\xi 2\,\mathrm{fy1}'[\lambda]}{\hbar}\right)$$