

Pensieve header: An isolated attempt to compute the multiplication tensor of QU.

The implementation of QU is modified from pensieve://Projects/SL2Portfolio/SL2PortfolioProgram.nb.

Initialization / Utilities

```
In[*]:=
$k = 1; $U = QU; $E := {$k};
$trim := {e^{k-} /; k > $k -> 0};
SetAttributes[{SS, SST}, HoldAll];
q_{h-} := e^{\gamma \epsilon^h};
(* Upper to lower and lower to Upper: *)
U2l = {B_{i-}^{p-} -> e^{-p h \gamma} b_i, B_{i-}^{p-} -> e^{-p h \gamma} b, T_{i-}^{p-} -> e^{p h} t_i, T_{i-}^{p-} -> e^{p h} t, \mathcal{A}_{i-}^{p-} -> e^{p \gamma \alpha_i}, \mathcal{A}_{i-}^{p-} -> e^{p \gamma \alpha}};
l2U = {e_{-}^{c-} b_{i-} + d_{-} -> B_{i-}^{-c/(h \gamma)} e^d, e_{-}^{c-} b + d_{-} -> B^{-c/(h \gamma)} e^d,
e_{-}^{c-} t_{i-} + d_{-} -> T_{i-}^{c/h} e^d, e_{-}^{c-} t + d_{-} -> T^{c/h} e^d,
e_{-}^{c-} \alpha_{i-} + d_{-} -> \mathcal{A}_{i-}^{c/\gamma} e^d, e_{-}^{c-} \alpha + d_{-} -> \mathcal{A}^{c/\gamma} e^d,
e_{-}^{\epsilon} -> e^{Expand@e}};
SS[e_, op_] := Collect[Normal@Series[e, {e, 0, $k}], e, op];
SS[e_] := SS[e, Together];
SST[e_, op___] := SS[e /. U2l, op];
Simp[e_, op_] := Collect[e, _CU | _QU, op];
Simp[e_] := Simp[e, SS[#, Expand] &];
SimpT[e_] := Collect[e, _CU | _QU, SST[#, Expand] &];
K\delta /: K\delta_{i,j} := If[i == j, 1, 0];
c_Integer_{k_Integer} := c + O[\epsilon]^{k+1};
```

```
In[*]:=
CF[e_] := ExpandDenominator@
ExpandNumerator@Together[Expand[e] /. e^{x-} e^{y-} -> e^{x+y} /. e^{x-} -> e^{CF[x]}];
```

```
In[*]:=
Unprotect[SeriesData];
SeriesData /: CF[sd_SeriesData] := MapAt[CF, sd, 3];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
SeriesData /: Simplify[sd_SeriesData] := MapAt[Simplify, sd, 3];
SeriesData /: Together[sd_SeriesData] := MapAt[Together, sd, 3];
SeriesData /: Collect[sd_SeriesData, specs___] := MapAt[Collect[#, specs] &, sd, 3];
Protect[SeriesData];
```

Self-Pair (SP):

```
In[*]:=
SP_{\{}}[P_] := P; SP_{\{\epsilon \rightarrow x, ps\_}}[P_] := Expand[P // SP_{ps}] /. f_{-} \cdot \epsilon^d -> \partial_{\{x,d\}} f
```

DeclareAlgebra

```
In[*]:= Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z_] := (x ⨗ y) ⨗ z;
0 ⨗ _ = _ ⨗ 0 = 0;
(x_Plus) ⨗ y_ := (# ⨗ y) & /@ x; x_ ⨗ (y_Plus) := (x ⨗ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⨗ y - y ⨗ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

```
In[*]:= DeclareAlgebra[U_Symbol, opts__Rule] := Module[
{gp, sr, g, cp, M, pow, k = 0,
gs = Generators /. {opts},
cs = Centrals /. {opts} /. Centrals -> {}
},
U[Generators] = gs;
(#_ = U@#) & /@ gs;
gp = Alternatives @@ gs; gp = gp | gp_ (* gens *)
sr = Flatten@Table[{g -> ++k, gi_ -> {i, k}}, {g, gs}]; (* sorting -> *)
cp = Alternatives @@ cs; (* cents *)
SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
CE[ε_] := Collect[ε, _U, Expand] /. e^x_ -> e^Expand[x] /. $trim;
Ui_[ε_] := ε /. {t : cp -> ti_, u_U -> (#i &) /@ u};
Ui_[NCM[]] = pow[ε_, 0] = U@{ } = 1_U = U[];
B[U@(x_)i_, U@(y_)i_] := Ui_@B[U@x, U@y];
B[U@(x_)i_, U@(y_)j_] /; i != j := 0;
B[U@y_, U@x_] := CE[-B[U@x, U@y]];
x_ ⨗ (c_. 1_U) := CE[c x]; (c_. 1_U) ⨗ x_ := CE[c x];
(a_. U[xx_, x_]) ⨗ (b_. U[y_, yy_]) := If[OrderedQ[{x, y} /. sr],
CE@M[a b /. $trim, U[xx, x, y, yy]],
U@xx ⨗ CE@M[a b /. $trim, U@y ⨗ U@x + B[U@x, U@y, $E]] ⨗ U@yy];
U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⨗ U@{r}];
U@{c_. * l : gp, r___} := CE[c U[l] ⨗ U@{r}];
U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
U@{l_, r___} := U@{Expand[l], r};
U[ε_NonCommutativeMultiply] := U /@ ε;
o_U[specs___, poly_] := Module[{sp, null, vs, us},
sp = Replace[{specs}, l_List -> l_null, {1}];
vs = Join@@ (First /@ sp);
us = Join@@ (sp /. l_s_ -> (l /. x_i_ -> x_s));
CE[Total[
CoefficientRules[poly, vs] /. (p_ -> c_) -> c U@{us^p}
]] /. x_null -> x];
```

```

 $\mathbb{O}_U[\text{specs\_}, \mathbb{E}[L_, Q_, P_]] := \mathbb{O}_U[\text{specs}, \text{SS@Normal}[P e^{L+Q}]];$ 
 $\text{pow}[\mathcal{E}_, n_] := \text{pow}[\mathcal{E}, n - 1] \otimes \mathcal{E};$ 
 $\mathbb{S}_U[\mathcal{E}_, \text{ss\_Rule}] := \text{CE@Total}[\text{CoefficientRules}[\mathcal{E}, \text{First} / @ \{\text{ss}\}] /. \text{ }]$ 
 $(p_ \rightarrow c_ ) \Rightarrow c \text{ NCM} @ \text{MapThread}[\text{pow}, \{\text{Last} / @ \{\text{ss}\}, p\}]]];$ 
 $\sigma_{rs\_} [c_ . * u\_U] := (c / . (t : cp)_{j_} \Rightarrow t_{j_} /. \{rs\}) U[\text{List} @ (u / . v_{-j_} \Rightarrow v_{j_} /. \{rs\})];$ 
 $m_{j \rightarrow k\_} [c_ . * u\_U] := \text{CE}[(c / . (t : cp)_{j_} \rightarrow t_{k_}) \text{DeleteCases}[u, \_j | k] \otimes U @ \text{Cases}[u, w_{-j} \Rightarrow w_k] \otimes U @ \text{Cases}[u, \_k];$ 
 $U / : c_ . * u\_U * v\_U := \text{CE}[c u \otimes v];$ 
 $S_{i\_} [c_ . * u\_U] := \text{CE}[(c / . S_i[U, \text{Centrals}]) \text{DeleteCases}[u, \_i] \otimes U_i[\text{NCM} @ \text{Reverse} @ \text{Cases}[u, x_{-i} \Rightarrow S @ U @ x]]];$ 
 $\Delta_{i \rightarrow j\_ , k\_} [c_ . * u\_U] := \text{CE}[(c / . \Delta_{i \rightarrow j, k}[U, \text{Centrals}]) \text{DeleteCases}[u, \_i] \otimes (\text{NCM} @ \text{Cases}[u, x_{-i} \Rightarrow \sigma_{1 \rightarrow j, 2 \rightarrow k} @ \Delta @ U @ x] /. \text{NCM}[] \rightarrow U[])]];$ 

```

Meta-Operations

```

In[*]:=  $\sigma_{rs\_} [\mathcal{E}_{Plus}] := \sigma_{rs} / @ \mathcal{E};$ 
 $m_{j \rightarrow j\_} = \text{Identity}; m_{j \rightarrow k\_} [0] = 0;$ 
 $m_{j \rightarrow k\_} [\mathcal{E}_{Plus}] := \text{Simp}[m_{j \rightarrow k} / @ \mathcal{E};$ 
 $m_{is\_ , i\_ , j \rightarrow k\_} [\mathcal{E}_] := m_{j \rightarrow k} @ m_{is, i \rightarrow j} @ \mathcal{E};$ 
 $S_{i\_} [\mathcal{E}_{Plus}] := \text{Simp}[S_i / @ \mathcal{E};$ 
 $\Delta_{is\_} [\mathcal{E}_{Plus}] := \text{Simp}[\Delta_{is} / @ \mathcal{E};$ 

```

Implementing $\text{CU} = \mathcal{U}(\text{sl}_2^{\vee \mathbb{E}})$

Verify σ and Δ ! Also Generalize Δ to $\Delta_{i,j_1,j_2,\dots}$.

```

In[*]:= DeclareAlgebra[CU, Generators  $\rightarrow \{y, a, x\}$ , Centrals  $\rightarrow \{t\}$ ];
 $B[a_{CU}, y_{CU}] = -y_{CU}; B[x_{CU}, a_{CU}] = -x_{CU};$ 
 $B[x_{CU}, y_{CU}] = 2 e_{a_{CU}} - t_{1_{CU}};$ 
 $(S @ y_{CU} = -y_{CU}; S @ a_{CU} = -a_{CU}; S @ x_{CU} = -x_{CU};)$ 
 $S_{i\_} [CU, \text{Centrals}] = \{t_i \rightarrow -t_i\};$ 
 $\Delta @ y_{CU} = CU @ y_1 + CU @ y_2; \Delta @ a_{CU} = CU @ a_1 + CU @ a_2; \Delta @ x_{CU} = CU @ x_1 + CU @ x_2;$ 
 $\Delta_{i \rightarrow j\_ , k\_} [CU, \text{Centrals}] = \{t_i \rightarrow t_j + t_k\};$ 

```

Implementing $QU = \mathcal{U}_q(\mathfrak{sl}_2^{\vee \epsilon})$

```
In[*]:=  $\hbar = \gamma = 1$ ;
DeclareAlgebra[QU, Generators  $\rightarrow \{y, a, x\}$ , Centrals  $\rightarrow \{t, T\}$ ];
B[aQU, yQU] = - $\gamma$  yQU; B[xQU, aQU] = - $\gamma$  QU@x;
B[xQU, yQU] := SS[q $\hbar$  - 1] QU@{y, x} + OQU[{a}, SS[(1 - T e-2 $\epsilon$  a  $\hbar$ )] /  $\hbar$ ]];
(S@yQU := OQU[{a, y}, SS[-T-1 e $\hbar \epsilon$  a y]]; S@aQU = -aQU; S@xQU := OQU[{a, x}, SS[-e $\hbar \epsilon$  a x]]);
Si_[QU, Centrals] = {ti  $\rightarrow$  -ti, Ti  $\rightarrow$  Ti-1};
 $\Delta$ @yQU := OQU[{y1, a1}, {y2}, SS[y1 + T1 e- $\hbar \epsilon$  a1 y2]];
 $\Delta$ @aQU = QU@a1 + QU@a2;  $\Delta$ @xQU := OQU[{a1, x1}, {x2}, SS[x1 + e- $\hbar \epsilon$  a1 x2]];
 $\Delta$ i $\rightarrow$ j-,k-[QU, Centrals] = {ti  $\rightarrow$  tj + tk, Ti  $\rightarrow$  Tj Tk};
```

```
In[*]:= B[xQU, yQU]
```

```
Out[*]=
```

(1 - T) QU[] + 2 T $\in$ QU[a] + ϵ QU[y, x]

tSW

Logoi from Pensieve://Talks/Toulouse-1705/DogmaDemo.nb and from Pensieve://Talks/Sydney-1708/ExtraDetails@@.nb.

Goal. In either U , compute $F = e^{-\eta y} e^{\xi x} e^{\eta y} e^{-\xi x}$. First compute $G = e^{\xi x} y e^{-\xi x}$, a finite sum. Now F satisfies the ODE $\partial_\eta F = \partial_\eta (e^{-\eta y} e^{\eta G}) = -yF + FG$ with initial conditions $F(\eta = 0) = 1$. So we set it up and solve:

```
In[*]:= SWxy[U_, kk_] :=
  SWxy[U, kk] = Block[{ $\$U = U$ ,  $\$k = kk$ ,  $\$p = kk$ }, Module[{G, F, fs, f, bs, e, b, es},
    G = Simp[Table[ $\xi^k / k!$ , {k, 0,  $\$k + 1$ }] . NestList[Simp[B[xU, #]] &, yU,  $\$k + 1$ ]];
    fs = Flatten@Table[f1,i,j,k[ $\eta$ ], {1, 0,  $\$k$ }, {i, 0, 1}, {j, 0, 1}, {k, 0, 1}];
    F = fs . (bs = fs /. fL-,i-,j-,k-[ $\eta$ ]  $\Rightarrow$   $\epsilon^L U @ \{y^i, a^j, x^k\}$ );
    es = Flatten[
      Table[Coefficient[e, b] == 0, {e, {F - 1U /.  $\eta \rightarrow 0$ , F  $\boxtimes$  G - yU  $\boxtimes$  F -  $\partial_\eta F$ }}, {b, bs}]];
    F = F /. DSolve[es, fs,  $\eta$ ][[1]];
     $\mathbb{E}$ [0,
       $\xi x + \eta y + (U /. \{CU \rightarrow -t \eta \xi, QU \rightarrow \eta \xi (1 - T) / \hbar\})$ ,
      F + 0 $\$k$  /. {e $\rightarrow$  1, U  $\rightarrow$  Times}
    ] /. (v :  $\eta | \xi | t | T | y | a | x$ )  $\rightarrow$  v1
  ]];
tSWxy,i-,j- $\rightarrow$ k- := SWxy[$U, $k] /. { $\xi_1 \rightarrow \xi_i, \eta_1 \rightarrow \eta_j, (v : t | T | y | a | x)_1 \rightarrow v_k$ };
tSWxa,i-,j- $\rightarrow$ k- :=  $\mathbb{E}[\alpha_j a_k, e^{-\gamma \alpha_j} \xi_i x_k, 1]$ ;
tSWay,i-,j- $\rightarrow$ k- :=  $\mathbb{E}[\alpha_i a_k, e^{-\gamma \alpha_i} \eta_j y_k, 1]$ ;
```

Exponentials as needed.

Task. Define $\text{Exp}_{U_i,k}[\xi, P]$ which computes $e^{\xi \mathcal{O}(P)}$ to ϵ^k in the algebra U_i , where ξ is a scalar, X is x_i or y_i , and P is an ϵ -dependent near-docile element, giving the answer in $\mathbb{E}$ -form. Should satisfy

$$U @ \text{Exp}_{U,k}[\xi, P] == \mathbb{S}_U[e^{\xi x}, x \rightarrow \mathcal{O}(P)].$$

Methodology. If $P_0 := P_{\epsilon=0}$ and $\mathcal{O}^{\xi \mathcal{O}(P)} = \mathcal{O}(e^{\xi P_0} F(\xi))$, then $F(\xi=0) = 1$ and we have:

$$\mathcal{O}(e^{\xi P_0} (P_0 F(\xi) + \partial_{\xi} F)) = \mathcal{O}(\partial_{\xi} e^{\xi P_0} F(\xi)) = \partial_{\xi} \mathcal{O}(e^{\xi P_0} F(\xi)) = \partial_{\xi} \mathcal{O}^{\xi \mathcal{O}(P)} = \mathcal{O}^{\xi \mathcal{O}(P)} \mathcal{O}(P) = \mathcal{O}(e^{\xi P_0} F(\xi)) \mathcal{O}(P).$$

This is an ODE for F . Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for φ .

```
In[*]:= (* Bug: The first line is valid only if  $\mathcal{O}(e^{P_0}) = e^{\mathcal{O}(P_0)}$ . *)
(* Bug:  $\xi$  must be a symbol. *)
Exp_{U,i}_,0[\xi_, P_] := Module[{LQ = Normal@P /.  $\epsilon \rightarrow 0$ },
  E[\xi LQ /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  LQ /. (t | a)_i  $\rightarrow 0, 1$ ]];
Exp_{U,i}_,k[\xi_, P_] := Block[{$U = U, $k = k},
  Module[{P0,  $\varphi$ ,  $\varphi S$ , F, j, rhs, at0, at\xi},
    P0 = Normal@P /.  $\epsilon \rightarrow 0$ ;
     $\varphi S =$ 
      Flatten@Table[\varphi_{j1,j2,j3}[\xi], {j2, 0, k}, {j1, 0, 2k+1-j2}, {j3, 0, 2k+1-j2-j1}];
    F = Normal@Last@Exp_{U,i}_,k-1[\xi, P] +  $\epsilon^k \varphi S$ . (\varphi S /. \varphi_{js}[\xi] := Times@@{y_i, a_i, x_i}^{js});
    rhs = Normal@
      Last@m_{i,j} \rightarrow i [E[\xi P0 /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  P0 /. (t | a)_i  $\rightarrow 0$ , F +  $\theta_k$ ] m_{i,j} \rightarrow j E[0, 0, P +  $\theta_k$ ]];
    at0 = (# == 0) & /@ Flatten@CoefficientList[F - 1 /.  $\xi \rightarrow 0$ , {y_i, a_i, x_i}];
    at\xi = (# == 0) & /@ Flatten@CoefficientList[(\partial_{\xi} F) + P0 F - rhs, {y_i, a_i, x_i}];
    E[\xi P0 /. (x | y)_i  $\rightarrow 0$ ,  $\xi$  P0 /. (t | a)_i  $\rightarrow 0$ , F +  $\theta_k$ ] /.
    DSolve[And@@(at0 \cup at\xi),  $\varphi S$ , \xi][[1]]]
```

In[*]:= \$k = 1

Out[*]=

1

In[*]:= \$Basis = {a_{QU}, x_{QU}, y_{QU}};

Module[{a, b}, Table[{a, b} $\rightarrow$ CE[B[a, b]], {a, \$Basis}, {b, \$Basis}]] // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} \{QU[a], QU[a]\} \rightarrow 0 & \{QU[a], QU[x]\} \rightarrow \gamma QU[x] \\ \{QU[x], QU[a]\} \rightarrow -\gamma QU[x] & \{QU[x], QU[x]\} \rightarrow 0 \\ \{QU[y], QU[a]\} \rightarrow \gamma QU[y] & \{QU[y], QU[x]\} \rightarrow \left(-\frac{1}{\hbar} + \frac{T}{\hbar}\right) QU[] - 2 T \in QU[a] - \gamma \in \hbar QU[y, x] \end{pmatrix} \quad \{QU[x], QU[x]\} \rightarrow 0$$

```

In[*]:= $Basis = {aQU, xQU, yQU};
Module[{a, b},
  DeleteCases[
    Flatten[Table[{a, b, c} → CE[B[a, b] + B[b, a]], {a, $Basis}, {b, $Basis}]], _ → 0]
]
Module[{a, b, c},
  DeleteCases[Flatten[Table[{a, b, c} → CE[B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]]],
    {a, $Basis}, {b, $Basis}, {c, $Basis}]], _ → 0]
]
$ABasis = {aQU, xQU, yQU, QU[y, y], QU[y, a], QU[y, x], QU[a, a], QU[a, x], QU[x, x]};
Module[{a, b, c},
  DeleteCases[Flatten[Table[{a, b, c} → CE[a ⊗ CE[b ⊗ c] - CE[a ⊗ b] ⊗ c],
    {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]], _ → 0]
]

Out[*]=
{}

Out[*]=
{}

Out[*]=
{}

In[*]:= Module[{a, b, c},
  Table[{a, b, c} → CE[a ⊗ CE[b ⊗ c]], {a, $ABasis}, {b, $ABasis}, {c, $ABasis}]
];

In[*]:= B[xQU, yQU] // CE
Out[*]=

$$\left(\frac{1}{\hbar} - \frac{T}{\hbar}\right) QU[] + 2 T \in QU[a] + \gamma \in \hbar QU[y, x]$$


In[*]:= B[xQU, B[xQU, yQU]] // CE
Out[*]=

$$(\gamma \in -3 T \gamma \in) QU[x]$$


In[*]:= B[xQU, B[xQU, B[xQU, yQU]]] // CE
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]] // CE
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]]] // Expand
Out[*]=
0

In[*]:= B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, B[xQU, yQU]]]]]] // Expand
Out[*]=
0

```

```
In[*]:= adPower[x_, k_, Env_] [c_ * y_QU] := CE[c adPower[x, k, Env] [y]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y;
adPower[_ , 0, Env_] [y_] := y;
adPower[x_, k_, Env_] [0] = 0;
adPower[x_, k_, Env_] [y_QU] :=
  adPower[x, k, Env] [y] = CE@B[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];
```

```
In[*]:= adPower[x_QU, 1] [y_QU]
```

```
Out[*]=
```

$$\left(-\frac{1}{\hbar} + \frac{T}{\hbar}\right) \text{QU}[] - 2 T \in \text{QU}[a] - \gamma \in \hbar \text{QU}[y, x]$$

```
In[*]:= adPower[x_QU, 2] [y_QU]
```

```
Out[*]=
```

$$(\gamma \in -3 T \gamma \in) \text{QU}[x]$$

```
In[*]:= adPower[x_QU, 3] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= adPower[x_QU, 4] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= adPower[x_QU, 5] [y_QU]
```

```
Out[*]=
```

$$0$$

```
In[*]:= NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
PotQ[c_ * x_QU] := ~ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And@@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And@@ (NilQ /@ (List @@ x));
```

```
In[*]:= Ad[0] = Identity;
Ad[c_ . a_QU] [z_] := z /. w_QU => e^{c \gamma (Count[w,y]-Count[w,x])} w;
Ad[c_ . QU[]] [z_] := z;
```

```
In[*]:= Ad[x_] [y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k] [y] != 0,
    s += adPower[x, k] [y] / k!;
    ++k
  ];
  s
]
```

In[*]:= **Ad**[**x_{QU}**][**y_{QU}**]

Out[*]=

$$-QU[] + T QU[] - 2 T \in QU[a] + \frac{1}{2} \in QU[x] - \frac{3}{2} T \in QU[x] + QU[y] - \in QU[y, x]$$

In[*]:= **\$k**

Out[*]=

1

In[*]:= **Ad**[**c a_{QU}**]/@{**QU**[], **x_{QU}**, **y_{QU}**, **QU**[**x**, **y**]}

Out[*]=

$$\{QU[], e^{-c\gamma} QU[x], e^{c\gamma} QU[y], QU[x, y]\}$$

In[*]:= **Ad**[(**α1** + **α2** + **ε fa**[**λ**]) **a_{QU}**]/@{**QU**[], **x_{QU}**, **y_{QU}**, **QU**[**x**, **y**]}

Out[*]=

$$\{QU[], e^{-\gamma(\alpha1+\alpha2+\epsilon fa[\lambda])} QU[x], e^{\gamma(\alpha1+\alpha2+\epsilon fa[\lambda])} QU[y], QU[x, y]\}$$

In[*]:= **Normal@Series**[**e**^{-**γ** (**α1**+**α2**+**ε fa**[**λ**])}, {**ε**, **θ**, **\$k**}]

Out[*]=

$$e^{-(\alpha1+\alpha2)\gamma} - e^{-(\alpha1+\alpha2)\gamma} \gamma \in fa[\lambda]$$

In[*]:= **CE**[

Ad[(**α1** + **α2** + **ε fa**[**λ**]) **a_{QU}**]/@{**QU**[], **x_{QU}**, **y_{QU}**, **QU**[**x**, **y**]}/. **e**^{**ε**}→**Normal@Series**[**e**^{**ε**}, {**ε**, **θ**, **\$k**}]

Out[*]=

$$\{QU[], (e^{-\alpha1\gamma-\alpha2\gamma} - e^{-\alpha1\gamma-\alpha2\gamma} \gamma \in fa[\lambda]) QU[x], (e^{\alpha1\gamma+\alpha2\gamma} + e^{\alpha1\gamma+\alpha2\gamma} \gamma \in fa[\lambda]) QU[y], QU[x, y]\}$$

In[*]:= **Ad**[**a_{QU}**][**Ad**[**x_{QU}**][**y_{QU}**]

Out[*]=

$$-\frac{QU[]}{\hbar} + \frac{T QU[]}{\hbar} - 2 T \in QU[a] + \frac{1}{2} e^{-\gamma} \gamma \in QU[x] - \frac{3}{2} e^{-\gamma} T \gamma \in QU[x] + e^{\gamma} QU[y] - \gamma \in \hbar QU[y, x]$$

In[*]:= **λTangent**[] = 0;

λTangent[**xs___**, **x_**] := **CE**[**Ad**[**x**][**λTangent**[**xs**]] + **∂_λ****x**];

λTangent[**xs_List**] := **λTangent**@@**xs**

In[*]:= **λTangent**[**λ y_{QU}**, **λ a_{QU}**, **λ x_{QU}**]

Out[*]=

$$\left(-\frac{e^{\gamma\lambda}\lambda}{\hbar} + \frac{e^{\gamma\lambda}T\lambda}{\hbar}\right) QU[] + (1 - 2 e^{\gamma\lambda} T \in \lambda) QU[a] +$$

$$\left(1 + \gamma\lambda + \frac{1}{2} e^{\gamma\lambda} \gamma \in \lambda^2 - \frac{3}{2} e^{\gamma\lambda} T \gamma \in \lambda^2\right) QU[x] + e^{\gamma\lambda} QU[y] - e^{\gamma\lambda} \gamma \in \lambda \hbar QU[y, x]$$

In[*]:= **Seepage_u**[{ }][**c_**] = **c**;

Seepage_u[**ξs_List**][**P_**] := **Module**[{**gs** = **U**[**Generators**], **lx**},

lx = **gs**[[**Length**@**ξs**]];

Collect[**P**, **lx**, (**Ad**[**Last**[**ξs**] **lx_u**][**0_u**[**gs**, **Seepage_u**[**Most**@**ξs**]@#]] /. **U**→**Times**) &]

In[*]:= **Seepage**_{QU}[{ η , α , ξ }] [y]

Out[*]=

$$e^{\alpha} y - e^{\alpha} \xi + e^{\alpha} T \xi - 2 a e^{\alpha} T \in \xi - e^{\alpha} x y \in \xi + \frac{1}{2} e^{\alpha} x \in \xi^2 - \frac{3}{2} e^{\alpha} T x \in \xi^2$$

In[*]:= **O**_{QU}[{ y , a , x }, $x y$]

Out[*]=

QU [y , x]

In[*]:= **Ad**[QU[x]] [y_{QU}]

Out[*]=

\$Aborted

In[*]:= **Expand**[**Seepage**_{QU}[{ η }] [7] /. $\epsilon \rightarrow 0$]

Out[*]=

7

In[*]:= **Expand**[**Seepage**_{QU}[{ η , α , ξ }] [7 x] /. $\epsilon \rightarrow 0$]

Out[*]=

7 x

In[*]:= **Expand**[**Seepage**_{QU}[{ η , α , ξ }] [7 a] /. $\epsilon \rightarrow 0$]

Expand[**Seepage**_{QU}[{ η , α , ξ }] [7 y] /. $\epsilon \rightarrow 0$]

Expand[**Seepage**_{QU}[{ η , α , ξ }] [7 $a + y$] /. $\epsilon \rightarrow 0$]

Out[*]=

7 $a + 7 x \xi$

Out[*]=

\$Aborted

Out[*]=

\$Aborted

In[*]:= **\$k = 1**

Out[*]=

1

In[*]:= **lhs = λ Tangent** [**λ η_1 y_{QU} , α_1 a_{QU} , λ ξ_1 x_{QU} , λ η_2 y_{QU} , α_2 a_{QU} , λ ξ_2 x_{QU}**]

Out[*]=

$$\begin{aligned} & \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 + \frac{1}{2} e^{\alpha_1} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - 2 e^{\alpha_1} T \in \eta_1 \eta_2 \lambda^3 \xi_1^2 + \right. \\ & \quad \frac{3}{2} e^{\alpha_1} T^2 \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - e^{\alpha_1 + \alpha_2} \eta_1 \lambda \xi_2 + e^{\alpha_1 + \alpha_2} T \eta_1 \lambda \xi_2 - e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} T \eta_2 \lambda \xi_2 + \\ & \quad e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - 4 e^{\alpha_1 + \alpha_2} T \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 + 3 e^{\alpha_1 + \alpha_2} T^2 \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - \\ & \quad \left. \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 + 2 e^{\alpha_2} T \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \frac{3}{2} e^{\alpha_2} T^2 \in \eta_2^2 \lambda^3 \xi_1 \xi_2 \right) QU[] + \\ & \left(-2 e^{\alpha_1} T \in \eta_1 \lambda \xi_1 + 2 T \in \eta_2 \lambda \xi_1 - 2 e^{\alpha_1 + \alpha_2} T \in \eta_1 \lambda \xi_2 - 2 e^{\alpha_2} T \in \eta_2 \lambda \xi_2 \right) QU[a] + \\ & \left(e^{-\alpha_2} \xi_1 + \frac{1}{2} e^{\alpha_1 - \alpha_2} \in \eta_1 \lambda^2 \xi_1^2 - \frac{3}{2} e^{\alpha_1 - \alpha_2} T \in \eta_1 \lambda^2 \xi_1^2 + \xi_2 + e^{\alpha_1} \in \eta_1 \lambda^2 \xi_1 \xi_2 - \right. \\ & \quad 3 e^{\alpha_1} T \in \eta_1 \lambda^2 \xi_1 \xi_2 - \in \eta_2 \lambda^2 \xi_1 \xi_2 + 3 T \in \eta_2 \lambda^2 \xi_1 \xi_2 + \frac{1}{2} e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda^2 \xi_2^2 - \\ & \quad \left. \frac{3}{2} e^{\alpha_1 + \alpha_2} T \in \eta_1 \lambda^2 \xi_2^2 + \frac{1}{2} e^{\alpha_2} \in \eta_2 \lambda^2 \xi_2^2 - \frac{3}{2} e^{\alpha_2} T \in \eta_2 \lambda^2 \xi_2^2 \right) QU[x] + \\ & \left(e^{\alpha_1 + \alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1 + \alpha_2} \in \eta_1 \eta_2 \lambda^2 \xi_1 + 3 e^{\alpha_1 + \alpha_2} T \in \eta_1 \eta_2 \lambda^2 \xi_1 + \right. \\ & \quad \left. \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^2 \xi_1 - \frac{3}{2} e^{\alpha_2} T \in \eta_2^2 \lambda^2 \xi_1 \right) QU[y] + \\ & \left(-e^{\alpha_1} \in \eta_1 \lambda \xi_1 + \in \eta_2 \lambda \xi_1 - e^{\alpha_1 + \alpha_2} \in \eta_1 \lambda \xi_2 - e^{\alpha_2} \in \eta_2 \lambda \xi_2 \right) QU[y, x] \end{aligned}$$

In[*]:= **seep = Seepage_{QU}** [{ **$fy_0[\lambda] + \epsilon fy_1[\lambda]$, $\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]$, $fx_0[\lambda] + \epsilon fx_1[\lambda]$** }] [**$\partial_\lambda$ ((**$ft_0[\lambda] + \epsilon ft_1[\lambda]$**) **$t$** + (**$fy_0[\lambda] + \epsilon fy_1[\lambda]$**) **$y$** + **$fa_1[\lambda]$** **$a$** + (**$fx_0[\lambda] + \epsilon fx_1[\lambda]$**) **$x$**)]**

Out[*]=

$$\begin{aligned} & a \in fa_1'[\lambda] + x \in fx_0[\lambda] fa_1'[\lambda] + t ft_0'[\lambda] + t \in ft_1'[\lambda] + x (fx_0'[\lambda] + \epsilon fx_1'[\lambda]) + \\ & e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} y fy_0'[\lambda] - e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} fx_0[\lambda] fy_0'[\lambda] + e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} T fx_0[\lambda] fy_0'[\lambda] - \\ & 2 a e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} T \in fx_0[\lambda] fy_0'[\lambda] - e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} x y \in fx_0[\lambda] fy_0'[\lambda] + \\ & \frac{1}{2} e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} x \in fx_0[\lambda]^2 fy_0'[\lambda] - \frac{3}{2} e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} T x \in fx_0[\lambda]^2 fy_0'[\lambda] - \\ & e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} \in fx_1[\lambda] fy_0'[\lambda] + e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} T \in fx_1[\lambda] fy_0'[\lambda] + \\ & e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} y \in fy_1'[\lambda] - e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} \in fx_0[\lambda] fy_1'[\lambda] + e^{\alpha_1 + \alpha_2 + \epsilon fa_1[\lambda]} T \in fx_0[\lambda] fy_1'[\lambda] \end{aligned}$$

In[*]:= rhs = OQu[{y, a, x}, seep] /. e^ε -> Normal@Series[e^ε, {ε, 0, \$k}]

Out[*]=

$$\begin{aligned}
& -\epsilon \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{QU}[y, x] \text{fy0}'[\lambda] + \\
& \text{QU}[a] \left(\epsilon \text{fa1}'[\lambda] - 2 \text{T} \in \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) + \\
& \text{QU}[x] \left(\epsilon \text{fx0}[\lambda] \text{fa1}'[\lambda] + \text{fx0}'[\lambda] + \epsilon \text{fx1}'[\lambda] + \frac{1}{2} \in \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] - \right. \\
& \quad \left. \frac{3}{2} \text{T} \in \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] \right) + \\
& \text{QU}[y] \left(\left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fy0}'[\lambda] + \epsilon \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fy1}'[\lambda] \right) + \\
& \text{QU}[] \left(\text{t ft0}'[\lambda] + \text{t} \in \text{ft1}'[\lambda] - \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{fy0}'[\lambda] + \right. \\
& \quad \text{T} \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{fy0}'[\lambda] - \epsilon \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx1}[\lambda] \text{fy0}'[\lambda] + \\
& \quad \text{T} \in \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx1}[\lambda] \text{fy0}'[\lambda] - \\
& \quad \left. \epsilon \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{fy1}'[\lambda] + \text{T} \in \left(e^{\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \right) \text{fx0}[\lambda] \text{fy1}'[\lambda] \right)
\end{aligned}$$

In[*]:= diff = CE[lhs - rhs]

Out[*]=

$$\begin{aligned}
& \text{QU}[y, x] \left(-e^{\alpha_1} \in \eta_1 \lambda \xi_1 + \epsilon \eta_2 \lambda \xi_1 - e^{\alpha_1+\alpha_2} \in \eta_1 \lambda \xi_2 - e^{\alpha_2} \in \eta_2 \lambda \xi_2 + e^{\alpha_1+\alpha_2} \in \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) + \\
& \text{QU}[a] \left(-2 e^{\alpha_1} \text{T} \in \eta_1 \lambda \xi_1 + 2 \text{T} \in \eta_2 \lambda \xi_1 - 2 e^{\alpha_1+\alpha_2} \text{T} \in \eta_1 \lambda \xi_2 - \right. \\
& \quad \left. 2 e^{\alpha_2} \text{T} \in \eta_2 \lambda \xi_2 - \epsilon \text{fa1}'[\lambda] + 2 e^{\alpha_1+\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) + \\
& \text{QU}[x] \left(e^{-\alpha_2} \xi_1 + \frac{1}{2} e^{\alpha_1-\alpha_2} \in \eta_1 \lambda^2 \xi_1^2 - \frac{3}{2} e^{\alpha_1-\alpha_2} \text{T} \in \eta_1 \lambda^2 \xi_1^2 + \xi_2 + e^{\alpha_1} \in \eta_1 \lambda^2 \xi_1 \xi_2 - \right. \\
& \quad \left. 3 e^{\alpha_1} \text{T} \in \eta_1 \lambda^2 \xi_1 \xi_2 - \epsilon \eta_2 \lambda^2 \xi_1 \xi_2 + 3 \text{T} \in \eta_2 \lambda^2 \xi_1 \xi_2 + \frac{1}{2} e^{\alpha_1+\alpha_2} \in \eta_1 \lambda^2 \xi_2^2 - \right. \\
& \quad \left. \frac{3}{2} e^{\alpha_1+\alpha_2} \text{T} \in \eta_1 \lambda^2 \xi_2^2 + \frac{1}{2} e^{\alpha_2} \in \eta_2 \lambda^2 \xi_2^2 - \frac{3}{2} e^{\alpha_2} \text{T} \in \eta_2 \lambda^2 \xi_2^2 - \epsilon \text{fx0}[\lambda] \text{fa1}'[\lambda] - \right. \\
& \quad \left. \text{fx0}'[\lambda] - \epsilon \text{fx1}'[\lambda] - \frac{1}{2} e^{\alpha_1+\alpha_2} \in \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] + \frac{3}{2} e^{\alpha_1+\alpha_2} \text{T} \in \text{fx0}[\lambda]^2 \text{fy0}'[\lambda] \right) + \\
& \text{QU}[y] \left(e^{\alpha_1+\alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1+\alpha_2} \in \eta_1 \eta_2 \lambda^2 \xi_1 + 3 e^{\alpha_1+\alpha_2} \text{T} \in \eta_1 \eta_2 \lambda^2 \xi_1 + \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^2 \xi_1 - \right. \\
& \quad \left. \frac{3}{2} e^{\alpha_2} \text{T} \in \eta_2^2 \lambda^2 \xi_1 - e^{\alpha_1+\alpha_2} \text{fy0}'[\lambda] - e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1+\alpha_2} \in \text{fy1}'[\lambda] \right) + \\
& \text{QU}[] \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} \text{T} \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - \text{T} \eta_2 \lambda \xi_1 + \frac{1}{2} e^{\alpha_1} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - \right. \\
& \quad \left. 2 e^{\alpha_1} \text{T} \in \eta_1 \eta_2 \lambda^3 \xi_1^2 + \frac{3}{2} e^{\alpha_1} \text{T}^2 \in \eta_1 \eta_2 \lambda^3 \xi_1^2 - e^{\alpha_1+\alpha_2} \eta_1 \lambda \xi_2 + e^{\alpha_1+\alpha_2} \text{T} \eta_1 \lambda \xi_2 - \right. \\
& \quad \left. e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} \text{T} \eta_2 \lambda \xi_2 + e^{\alpha_1+\alpha_2} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - 4 e^{\alpha_1+\alpha_2} \text{T} \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 + \right. \\
& \quad \left. 3 e^{\alpha_1+\alpha_2} \text{T}^2 \in \eta_1 \eta_2 \lambda^3 \xi_1 \xi_2 - \frac{1}{2} e^{\alpha_2} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 + 2 e^{\alpha_2} \text{T} \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \frac{3}{2} e^{\alpha_2} \text{T}^2 \in \eta_2^2 \lambda^3 \xi_1 \xi_2 - \right. \\
& \quad \left. \text{t ft0}'[\lambda] - \text{t} \in \text{ft1}'[\lambda] + e^{\alpha_1+\alpha_2} \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1+\alpha_2} \text{T} \text{fx0}[\lambda] \text{fy0}'[\lambda] + \right. \\
& \quad \left. e^{\alpha_1+\alpha_2} \in \text{fa1}[\lambda] \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1+\alpha_2} \text{T} \in \text{fa1}[\lambda] \text{fx0}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1+\alpha_2} \in \text{fx1}[\lambda] \text{fy0}'[\lambda] - \right. \\
& \quad \left. e^{\alpha_1+\alpha_2} \text{T} \in \text{fx1}[\lambda] \text{fy0}'[\lambda] + e^{\alpha_1+\alpha_2} \in \text{fx0}[\lambda] \text{fy1}'[\lambda] - e^{\alpha_1+\alpha_2} \text{T} \in \text{fx0}[\lambda] \text{fy1}'[\lambda] \right)
\end{aligned}$$

In[*]:= **diff /. $\epsilon \rightarrow 0$**

Out[*]=

$$\begin{aligned} & \text{QU}[x] \left(e^{-\alpha^2} \xi_1 + \xi_2 - \text{fx0}'[\lambda] \right) + \text{QU}[y] \left(e^{\alpha_1 + \alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1 + \alpha_2} \text{fy0}'[\lambda] \right) + \\ & \text{QU}[] \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 - e^{\alpha_1 + \alpha_2} \eta_1 \lambda \xi_2 + e^{\alpha_1 + \alpha_2} T \eta_1 \lambda \xi_2 - \right. \\ & \left. e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} T \eta_2 \lambda \xi_2 - t \text{ft0}'[\lambda] + e^{\alpha_1 + \alpha_2} \text{fx0}[\lambda] \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} T \text{fx0}[\lambda] \text{fy0}'[\lambda] \right) \end{aligned}$$

In[*]:= **{sol1} = DSolve[Coefficient[diff /. $\epsilon \rightarrow 0$, x_{QU}] == 0 ∧ fx0[0] == 0, fx0, λ]**

Out[*]=

$$\left\{ \left\{ \text{fx0} \rightarrow \text{Function} \left[\{\lambda\}, e^{-\alpha^2} \lambda \left(\xi_1 + e^{\alpha^2} \xi_2 \right) \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. $\epsilon \rightarrow 0$]**

Out[*]=

$$\begin{aligned} & \text{QU}[y] \left(e^{\alpha_1 + \alpha_2} \eta_1 + e^{\alpha_2} \eta_2 - e^{\alpha_1 + \alpha_2} \text{fy0}'[\lambda] \right) + \\ & \text{QU}[] \left(-e^{\alpha_1} \eta_1 \lambda \xi_1 + e^{\alpha_1} T \eta_1 \lambda \xi_1 + \eta_2 \lambda \xi_1 - T \eta_2 \lambda \xi_1 - e^{\alpha_1 + \alpha_2} \eta_1 \lambda \xi_2 + \right. \\ & \left. e^{\alpha_1 + \alpha_2} T \eta_1 \lambda \xi_2 - e^{\alpha_2} \eta_2 \lambda \xi_2 + e^{\alpha_2} T \eta_2 \lambda \xi_2 - t \text{ft0}'[\lambda] + e^{\alpha_1} \lambda \xi_1 \text{fy0}'[\lambda] - \right. \\ & \left. e^{\alpha_1} T \lambda \xi_1 \text{fy0}'[\lambda] + e^{\alpha_1 + \alpha_2} \lambda \xi_2 \text{fy0}'[\lambda] - e^{\alpha_1 + \alpha_2} T \lambda \xi_2 \text{fy0}'[\lambda] \right) \end{aligned}$$

In[*]:= **{sol2} = DSolve[Coefficient[CE[diff /. sol1 /. $\epsilon \rightarrow 0$], y_{QU}] == 0 ∧ fy0[0] == 0, fy0, λ]**

Out[*]=

$$\left\{ \left\{ \text{fy0} \rightarrow \text{Function} \left[\{\lambda\}, e^{-\alpha_1} \left(e^{\alpha_1} \eta_1 + \eta_2 \right) \lambda \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. sol2 /. $\epsilon \rightarrow 0$]**

Out[*]=

$$\text{QU}[] \left(2 \eta_2 \lambda \xi_1 - 2 T \eta_2 \lambda \xi_1 - t \text{ft0}'[\lambda] \right)$$

In[*]:= **{sol3} = DSolve[Coefficient[CE[diff /. sol1 /. sol2 /. $\epsilon \rightarrow 0$], QU[]] == 0 ∧ ft0[0] == 0, ft0, λ]**

Out[*]=

$$\left\{ \left\{ \text{ft0} \rightarrow \text{Function} \left[\{\lambda\}, -\frac{(-1 + T) \eta_2 \lambda^2 \xi_1}{t} \right] \right\} \right\}$$

In[*]:= **CE[diff /. sol1 /. sol2 /. sol3]**

Out[*]=

$$\begin{aligned}
& 2 \in \eta^2 \lambda \xi^1 \text{QU}[y, x] + \text{QU}[a] \left(4 T \in \eta^2 \lambda \xi^1 - \epsilon \text{fa1}'[\lambda] \right) + \\
& \text{QU}[x] \left(-\frac{1}{2} e^{-\alpha^2} \in \eta^2 \lambda^2 \xi^1{}^2 + \frac{3}{2} e^{-\alpha^2} T \in \eta^2 \lambda^2 \xi^1{}^2 - 2 \in \eta^2 \lambda^2 \xi^1 \xi^2 + \right. \\
& \quad \left. 6 T \in \eta^2 \lambda^2 \xi^1 \xi^2 - e^{-\alpha^2} \in \lambda \xi^1 \text{fa1}'[\lambda] - \epsilon \lambda \xi^2 \text{fa1}'[\lambda] - \epsilon \text{fx1}'[\lambda] \right) + \\
& \text{QU}[y] \left(-e^{\alpha^1 + \alpha^2} \in \eta^1 \eta^2 \lambda^2 \xi^1 + 3 e^{\alpha^1 + \alpha^2} T \in \eta^1 \eta^2 \lambda^2 \xi^1 + \frac{1}{2} e^{\alpha^2} \in \eta^2 \lambda^2 \xi^1 - \right. \\
& \quad \left. \frac{3}{2} e^{\alpha^2} T \in \eta^2 \lambda^2 \xi^1 - e^{\alpha^1 + \alpha^2} \in \eta^1 \text{fa1}[\lambda] - e^{\alpha^2} \in \eta^2 \text{fa1}[\lambda] - e^{\alpha^1 + \alpha^2} \in \text{fy1}'[\lambda] \right) + \\
& \text{QU}[] \left(\frac{1}{2} e^{\alpha^1} \in \eta^1 \eta^2 \lambda^3 \xi^1{}^2 - 2 e^{\alpha^1} T \in \eta^1 \eta^2 \lambda^3 \xi^1{}^2 + \frac{3}{2} e^{\alpha^1} T^2 \in \eta^1 \eta^2 \lambda^3 \xi^1{}^2 + \right. \\
& \quad e^{\alpha^1 + \alpha^2} \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 - 4 e^{\alpha^1 + \alpha^2} T \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 + 3 e^{\alpha^1 + \alpha^2} T^2 \in \eta^1 \eta^2 \lambda^3 \xi^1 \xi^2 - \\
& \quad \frac{1}{2} e^{\alpha^2} \in \eta^2 \lambda^3 \xi^1 \xi^2 + 2 e^{\alpha^2} T \in \eta^2 \lambda^3 \xi^1 \xi^2 - \frac{3}{2} e^{\alpha^2} T^2 \in \eta^2 \lambda^3 \xi^1 \xi^2 + e^{\alpha^1} \in \eta^1 \lambda \xi^1 \text{fa1}[\lambda] - \\
& \quad e^{\alpha^1} T \in \eta^1 \lambda \xi^1 \text{fa1}[\lambda] + \epsilon \eta^2 \lambda \xi^1 \text{fa1}[\lambda] - T \in \eta^2 \lambda \xi^1 \text{fa1}[\lambda] + e^{\alpha^1 + \alpha^2} \in \eta^1 \lambda \xi^2 \text{fa1}[\lambda] - \\
& \quad e^{\alpha^1 + \alpha^2} T \in \eta^1 \lambda \xi^2 \text{fa1}[\lambda] + e^{\alpha^2} \in \eta^2 \lambda \xi^2 \text{fa1}[\lambda] - e^{\alpha^2} T \in \eta^2 \lambda \xi^2 \text{fa1}[\lambda] + \\
& \quad e^{\alpha^1 + \alpha^2} \in \eta^1 \text{fx1}[\lambda] - e^{\alpha^1 + \alpha^2} T \in \eta^1 \text{fx1}[\lambda] + e^{\alpha^2} \in \eta^2 \text{fx1}[\lambda] - e^{\alpha^2} T \in \eta^2 \text{fx1}[\lambda] - t \in \text{ft1}'[\lambda] + \\
& \quad \left. e^{\alpha^1} \in \lambda \xi^1 \text{fy1}'[\lambda] - e^{\alpha^1} T \in \lambda \xi^1 \text{fy1}'[\lambda] + e^{\alpha^1 + \alpha^2} \in \lambda \xi^2 \text{fy1}'[\lambda] - e^{\alpha^1 + \alpha^2} T \in \lambda \xi^2 \text{fy1}'[\lambda] \right)
\end{aligned}$$