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In[ ]:= SetAlgebra[gl2,ε];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@
Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
{b1, $Basis}, {b2, $Basis}]

Out[ ]:=
{y1,2 → 1, x1,1 → 2, x1,2 → 3, x2,2 → 4}

Out[ ]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \rightarrow 0 & \{y_{1,2}, x_{1,1}\} \rightarrow y_{1,2} & \{y_{1,2}, x_{1,2}\} \rightarrow -\epsilon x_{1,1} + \epsilon x_{2,2} & \{y_{1,2}, x_{2,2}\} \rightarrow -y_{1,2} \\ \{x_{1,1}, y_{1,2}\} \rightarrow -y_{1,2} & \{x_{1,1}, x_{1,1}\} \rightarrow 0 & \{x_{1,1}, x_{1,2}\} \rightarrow x_{1,2} & \{x_{1,1}, x_{2,2}\} \rightarrow 0 \\ \{x_{1,2}, y_{1,2}\} \rightarrow \epsilon x_{1,1} - \epsilon x_{2,2} & \{x_{1,2}, x_{1,1}\} \rightarrow -x_{1,2} & \{x_{1,2}, x_{1,2}\} \rightarrow 0 & \{x_{1,2}, x_{2,2}\} \rightarrow x_{1,2} \\ \{x_{2,2}, y_{1,2}\} \rightarrow y_{1,2} & \{x_{2,2}, x_{1,1}\} \rightarrow 0 & \{x_{2,2}, x_{1,2}\} \rightarrow -x_{1,2} & \{x_{2,2}, x_{2,2}\} \rightarrow 0 \end{pmatrix}$$


Out[ ]//MatrixForm=

$$\begin{pmatrix} \{y_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{y_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{y_{1,2}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{1,1}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,1}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,1}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{1,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{1,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{1,2}, x_{2,2}\} \rightarrow \text{True} \\ \{x_{2,2}, y_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,1}\} \rightarrow \text{True} & \{x_{2,2}, x_{1,2}\} \rightarrow \text{True} & \{x_{2,2}, x_{2,2}\} \rightarrow \text{True} \end{pmatrix}$$


In[ ]:= SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
sol[-1] = 0 $Basis;
Do[
Block[{$k = k},
s0 = Echo@Thread[$Basis → sol[k - 1]];
lhs = λTangent@Echo[
Join[λ $Basis ($Basis /. {x → ξ1, y → η1}), λ $Basis ($Basis /. {x → ξ2, y → η2})]
];
rhs = λTangent@Echo[
$Basis ($Basis /. {xαβ → (xαβ /. s0) + εk fαβ[λ], yαβ → (yαβ /. s0) + εk gαβ[λ]})
];
eqns = Transpose@{
(Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
SBasis /. {xαβ → fαβ[0] == 0, yαβ → gαβ[0] == 0}
};
unknowns = SBasis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]};
s = TriangularDSolve[eqns, unknowns, λ];
sol[k] = CF[sol[k - 1] + εk ($Basis /. {xαβ → fαβ[λ], yαβ → gαβ[λ]} /. s)]
],
{k, 0, $k}
];
cm[i_, j_ → k_] := Δ2E{i,j}→{k}[Echo[sol[$k].SBasis] /. {xαβ → xαβ[k], yαβ → yαβ[k],
ξ1αβ → ξαβ[i], η1αβ → ηαβ[i], ξ2αβ → ξαβ[j], η2αβ → ηαβ[j], λ → 1}];
cm[i, j → k]

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- »  $\{y_{1,2} \rightarrow 0, x_{1,1} \rightarrow 0, x_{1,2} \rightarrow 0, x_{2,2} \rightarrow 0\}$
- »  $\{\lambda y_{1,2} \eta_{1,2}, \lambda x_{1,1} \xi_{1,1}, \lambda x_{1,2} \xi_{1,2}, \lambda x_{2,2} \xi_{2,2}, \lambda y_{1,2} \eta_{2,2}, \lambda x_{1,1} \xi_{2,1}, \lambda x_{1,2} \xi_{2,2}, \lambda x_{2,2} \xi_{2,2}\}$
- »  $\{y_{1,2} g_{1,2}[\lambda], x_{1,1} f_{1,1}[\lambda], x_{1,2} f_{1,2}[\lambda], x_{2,2} f_{2,2}[\lambda]\}$
- »  $\{f_{1,1}[\lambda] \rightarrow \lambda \xi_{1,1} + \lambda \xi_{2,1}\}$
- »  $\{f_{2,2}[\lambda] \rightarrow \lambda \xi_{1,2} + \lambda \xi_{2,2}\}$
- »  $\{f_{1,2}[\lambda] \rightarrow e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{2,2}} \lambda \xi_{2,2}\}$
- »  $\{g_{1,2}[\lambda] \rightarrow \lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2}\}$
- »  $\{y_{1,2} \rightarrow \lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2}, x_{1,1} \rightarrow \lambda \xi_{1,1} + \lambda \xi_{2,1}, x_{1,2} \rightarrow e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{2,2}} \lambda \xi_{2,2}, x_{2,2} \rightarrow \lambda \xi_{1,2} + \lambda \xi_{2,2}\}$
- »  $\{\lambda y_{1,2} \eta_{1,2}, \lambda x_{1,1} \xi_{1,1}, \lambda x_{1,2} \xi_{1,2}, \lambda x_{2,2} \xi_{2,2}, \lambda y_{1,2} \eta_{2,2}, \lambda x_{1,1} \xi_{2,1}, \lambda x_{1,2} \xi_{2,2}, \lambda x_{2,2} \xi_{2,2}\}$
- »  $\{y_{1,2} (\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} + e g_{1,2}[\lambda]), x_{1,1} (\lambda \xi_{1,1} + \lambda \xi_{2,1} + e f_{1,1}[\lambda]), x_{1,2} (e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{2,2}} \lambda \xi_{2,2} + e f_{1,2}[\lambda]), x_{2,2} (\lambda \xi_{1,2} + \lambda \xi_{2,2} + e f_{2,2}[\lambda])\}$
- »  $\{f_{1,1}[\lambda] \rightarrow e^{\lambda \xi_{2,2}} \lambda^2 \eta_{2,2} \xi_{1,2}\}$
- »  $\{f_{2,2}[\lambda] \rightarrow -e^{\lambda \xi_{2,2}} \lambda^2 \eta_{2,2} \xi_{1,2}\}$
- »  $\{f_{1,2}[\lambda] \rightarrow \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}\}$
- »  $\{g_{1,2}[\lambda] \rightarrow -e^{-\lambda \xi_{1,1} + 2 \lambda \xi_{1,2}} \lambda^3 \eta_{2,2}^2 \xi_{1,2}\}$
- »  $x_{1,1} (\lambda \eta_{1,2} + e^{-\lambda \xi_{1,1} + \lambda \xi_{1,2}} \lambda \eta_{2,2} - e^{-\lambda \xi_{1,1} + 2 \lambda \xi_{1,2}} \lambda^3 \eta_{2,2}^2 \xi_{1,2}) + x_{2,2} (\lambda \xi_{1,1} + e^{\lambda \xi_{2,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{2,1}) + x_{1,2} (e^{-\lambda \xi_{2,1}} \lambda \xi_{1,2} + e^{-\lambda \xi_{2,2}} \lambda \xi_{2,2} + e \lambda^3 \eta_{2,2} \xi_{1,2} \xi_{2,2}) + y_{1,2} (-e^{\lambda \xi_{2,2}} \lambda^2 \eta_{2,2} \xi_{1,2} + \lambda \xi_{1,2} + \lambda \xi_{2,2})$

Table: Iterator {k\$13352, 0, Global`\$k} does not have appropriate bounds.

Out[\*]=

$$E_{\{i,j\} \rightarrow \{k\}} \left[ \begin{array}{l} x_{2,2}[k] (\xi_{1,1}[i] + \xi_{1,1}[j] + e^{\xi_{2,2}[i]} \in \eta_{1,2}[j] \xi_{1,2}[i]) + \\ x_{1,1}[k] (\eta_{1,2}[i] - e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} \eta_{1,2}[j] (-1 + e^{\xi_{2,2}[i]} \in \eta_{1,2}[j] \xi_{1,2}[i])) + \\ x_{1,2}[k] (e^{-\xi_{2,2}[i]} \xi_{1,2}[j] + \xi_{1,2}[i] (e^{-\xi_{1,1}[j]} + e \in \eta_{1,2}[j] \xi_{1,2}[j])) + \\ y_{1,2}[k] (-e^{\xi_{2,2}[i]} \in \eta_{1,2}[j] \xi_{1,2}[i] + \xi_{2,2}[i] + \xi_{2,2}[j]) \\ \emptyset \end{array} \right] \quad \begin{array}{l} k \\ \\ \\ \\ \text{Tr} \end{array}$$

In[\*]:=  $CF[x_{1,1}[k] (\eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} \eta_{1,2}[j]) + x_{2,2}[k] (\xi_{1,1}[i] + \xi_{1,1}[j]) + x_{1,2}[k] (e^{-\xi_{1,1}[j]} \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} \xi_{1,2}[j]) + y_{1,2}[k] (\xi_{2,2}[i] + \xi_{2,2}[j]), e^{-\xi_{1,1}[i]} \eta_{1,2}[j] \xi_{1,2}[i] (e^{\xi_{1,1}[i] + \xi_{2,2}[i]} x_{2,2}[k] - e^{\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] - e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] + e^{\xi_{1,1}[i]} x_{1,2}[k] \xi_{1,2}[j])]$

In[\*]:=  $Expand[\{x_{1,1}[k] (\eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} \eta_{1,2}[j]) + x_{2,2}[k] (\xi_{1,1}[i] + \xi_{1,1}[j]) + x_{1,2}[k] (e^{-\xi_{1,1}[j]} \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} \xi_{1,2}[j]) + y_{1,2}[k] (\xi_{2,2}[i] + \xi_{2,2}[j]), e^{-\xi_{1,1}[i]} \eta_{1,2}[j] \xi_{1,2}[i] (e^{\xi_{1,1}[i] + \xi_{2,2}[i]} x_{2,2}[k] - e^{\xi_{1,1}[i] + \xi_{2,2}[i]} y_{1,2}[k] - e^{2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] + e^{\xi_{1,1}[i]} x_{1,2}[k] \xi_{1,2}[j])\}]$

Out[\*]=

$$\{x_{1,1}[k] \eta_{1,2}[i] + e^{-\xi_{1,1}[i] + \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j] + x_{2,2}[k] \xi_{1,1}[i] + x_{2,2}[k] \xi_{1,1}[j] + e^{-\xi_{1,1}[j]} x_{1,2}[k] \xi_{1,2}[i] + e^{-\xi_{2,2}[i]} x_{1,2}[k] \xi_{1,2}[j] + y_{1,2}[k] \xi_{2,2}[i] + y_{1,2}[k] \xi_{2,2}[j], e^{\xi_{2,2}[i]} x_{2,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{\xi_{2,2}[i]} y_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] - e^{-\xi_{1,1}[i] + 2 \xi_{2,2}[i]} x_{1,1}[k] \eta_{1,2}[j]^2 \xi_{1,2}[i] + x_{1,2}[k] \eta_{1,2}[j] \xi_{1,2}[i] \xi_{1,2}[j]\}$$

In[\*]:= (cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])

Out[\*]=

$$\begin{aligned}
 & x_{2,2}[1] \eta_{1,2}[1] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} x_{2,2}[1] \eta_{1,2}[2] + \\
 & e^{-\eta_{1,2}[1] - e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} \eta_{1,2}[2] + \xi_{1,1}[1] + \xi_{1,1}[2]} x_{1,1}[1] \eta_{1,2}[3] + y_{1,2}[1] \xi_{1,1}[1] + y_{1,2}[1] \xi_{1,1}[2] + \\
 & x_{2,2}[1] \xi_{1,1}[3] + e^{-\xi_{1,1}[2] - \xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] + e^{-\xi_{1,1}[3] - \xi_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] + \\
 & e^{-\xi_{1,1}[1] - \xi_{1,1}[2]} x_{1,2}[1] \xi_{1,2}[3] + x_{1,1}[1] \xi_{2,2}[1] + x_{1,1}[1] \xi_{2,2}[2] + y_{1,2}[1] \xi_{2,2}[3] = \\
 & x_{1,1}[1] \eta_{1,2}[1] + x_{2,2}[1] \eta_{1,2}[2] + e^{-\xi_{1,1}[2] + \xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3] + x_{2,2}[1] \xi_{1,1}[1] + \\
 & y_{1,2}[1] \xi_{1,1}[2] + y_{1,2}[1] \xi_{1,1}[3] + e^{-\eta_{1,2}[2] - e^{-\xi_{1,1}[2] + \xi_{2,2}[2]} \eta_{1,2}[3]} x_{1,2}[1] \xi_{1,2}[1] + \\
 & e^{-\xi_{1,1}[3] - \xi_{2,2}[1]} x_{1,2}[1] \xi_{1,2}[2] + e^{-\xi_{2,2}[1] - \xi_{2,2}[2]} x_{1,2}[1] \xi_{1,2}[3] + y_{1,2}[1] \xi_{2,2}[1] + \\
 & e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} x_{1,1}[1] \xi_{2,2}[2] + e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} x_{1,1}[1] \xi_{2,2}[3] \&\& \\
 & - e^{\xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[2] \xi_{1,2}[1] + e^{\xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] - e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} x_{2,2}[1] \\
 & \eta_{1,2}[2]^2 \xi_{1,2}[1] + e^{\xi_{1,1}[1]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] - e^{\xi_{1,1}[1]} y_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] + \\
 & e^{-\eta_{1,2}[1] - e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} \eta_{1,2}[2] + \xi_{1,1}[1] + \xi_{1,1}[2] + \xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[2] \eta_{1,2}[3] \xi_{1,2}[1] + \\
 & e^{-\eta_{1,2}[1] - e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} \eta_{1,2}[2] + \xi_{1,1}[1] + \xi_{1,1}[2] + 2 \xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[2]^2 \eta_{1,2}[3] \xi_{1,2}[1] - \\
 & e^{-\eta_{1,2}[1] - e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} \eta_{1,2}[2] + 2 \xi_{1,1}[1] + \xi_{1,1}[2]} x_{1,1}[1] \eta_{1,2}[3]^2 \xi_{1,2}[1] + \\
 & e^{\xi_{1,1}[1] + \xi_{1,1}[2] - \xi_{2,2}[1]} x_{2,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] - e^{\xi_{1,1}[1] + \xi_{1,1}[2] - \xi_{2,2}[1]} y_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] - \\
 & e^{-\eta_{1,2}[1] - e^{-\xi_{1,1}[1] + \xi_{2,2}[1]} \eta_{1,2}[2] + 2 \xi_{1,1}[1] + 2 \xi_{1,1}[2] - \xi_{2,2}[1]} x_{1,1}[1] \eta_{1,2}[3]^2 \xi_{1,2}[2] + \\
 & e^{-\xi_{1,1}[3]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[2] - e^{-\xi_{1,1}[1] - \xi_{1,1}[2] + \xi_{2,2}[1]} x_{1,2}[1] \eta_{1,2}[2] \xi_{1,2}[1] \xi_{1,2}[3] + \\
 & e^{-\xi_{1,1}[2]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[1] \xi_{1,2}[3] + e^{-\xi_{2,2}[1]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] \xi_{1,2}[3] = \\
 & - e^{-\xi_{1,1}[1] + \xi_{2,2}[1] + \xi_{2,2}[2]} x_{1,1}[1] \eta_{1,2}[3] \xi_{1,2}[2] + e^{\xi_{2,2}[2]} y_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] - \\
 & e^{-\xi_{1,1}[2] + 2 \xi_{2,2}[2]} x_{2,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[2] + \\
 & e^{-\eta_{1,2}[2] - e^{-\xi_{1,1}[2] + \xi_{2,2}[2]} \eta_{1,2}[3] - \xi_{1,1}[2] + 2 \xi_{2,2}[2]} x_{1,2}[1] \eta_{1,2}[3]^2 \xi_{1,2}[1] \xi_{1,2}[2] + \\
 & e^{-\xi_{2,2}[1]} x_{1,2}[1] \eta_{1,2}[3] \xi_{1,2}[2] \xi_{1,2}[3] + e^{\xi_{2,2}[1]} x_{2,2}[1] \xi_{1,2}[1] \xi_{2,2}[2] - \\
 & e^{\xi_{2,2}[1]} y_{1,2}[1] \xi_{1,2}[1] \xi_{2,2}[2] + e^{-\xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] \xi_{1,2}[2] \xi_{2,2}[2] + \\
 & e^{-\xi_{2,2}[2]} x_{1,2}[1] \xi_{1,2}[1] \xi_{1,2}[3] \xi_{2,2}[2] - e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} x_{1,1}[1] \xi_{1,2}[1] \xi_{2,2}[2]^2 + \\
 & e^{\xi_{2,2}[1]} x_{2,2}[1] \xi_{1,2}[1] \xi_{2,2}[3] - e^{\xi_{2,2}[1]} y_{1,2}[1] \xi_{1,2}[1] \xi_{2,2}[3] + \\
 & e^{-\xi_{1,1}[3]} x_{1,2}[1] \xi_{1,2}[1] \xi_{1,2}[2] \xi_{2,2}[3] + e^{-\xi_{2,2}[2]} x_{1,2}[1] \xi_{1,2}[1] \xi_{1,2}[3] \xi_{2,2}[3] - \\
 & 2 e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} x_{1,1}[1] \xi_{1,2}[1] \xi_{2,2}[2] \xi_{2,2}[3] - e^{-\xi_{1,1}[1] + 2 \xi_{2,2}[1]} x_{1,1}[1] \xi_{1,2}[1] \xi_{2,2}[3]^2
 \end{aligned}$$