Pensieve header: The PBW multiplication tensor for $gl_{n,\epsilon}$ using the nilpotent-only $\lambda$-tangent formalism. Some material from pensive://Projects/glneps and from pensieve://Projects/UEA.

```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

## Conventions

$x_{\alpha,\beta}$ is always with $\alpha \leq \beta$ and represents a basis element of the upper triangular matrices. $y_{\alpha,\beta}$ is always with $\beta \leq \alpha$ and represents a basis element of the lower triangular matrices. The adjoint of $x_{\alpha,\beta}$ is $y_{\beta,\alpha}$.

Variable declarations:

```
VarPattern = ((y | x | η | ξ) | (y | x | η | ξ)__)[_];
```

Greek - Latin duality:

```
{y*, x*, η*, ξ*} = {η, ξ, y, x};
(u_αβ__)* := (u*)αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

```
{x1,2, y2,3, η1,2[7]}*
```

```
{ξ1,2, η2,3, y1,2[7]}
```

Weights:

```
Wt[xα_,β_] := {1, α - β}; Wt[yα_,β_] := {0, α - β};
Wt[ξα_,β_] := {0, β - α}; Wt[ηα_,β_] := {1, β - α}; ;
Wt[u_[_]] := Wt[u];
```

## The Engine

Canonical Forms:

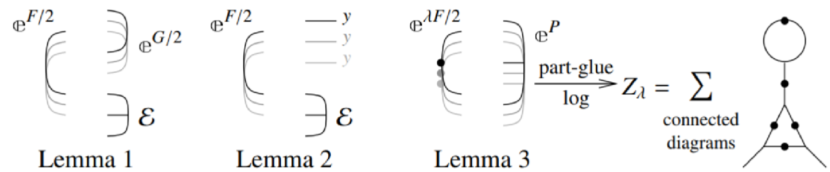
```
LogReduce[ℰ_] := ℰ /. c_ * Log[a_] :→ Log@Factor[a^c] //. Log[a_] + Log[b_] :→ Log@Factor[a b];
CF[ℰ_] := Expand@Collect[ℰ, Cases[ℰ, VarPattern, ∞], CCF] /. CCF → Factor;
CF[ℰ_𝔼] := CF /@ MapAt[LogReduce, ℰ, 1];
CF[ℰ : 𝔼__[___]] := CF /@ ℰ;
```

𝔼 operations:

```
ℰ_𝔼[$] := Length[ℰ] - 1; 𝔼_[ℰs___][$] := 𝔼[ℰs][$];
ℰ_𝔼[k_Integer] := ℰ⟦k + 1⟧; 𝔼_[ℰs___][k_Integer] := {ℰs}⟦k + 1⟧;
𝔼 /: ℰ1_𝔼 ≡ ℰ2_𝔼 := Inner[CF@#1 == CF@#2 &, ℰ1, ℰ2, And];
𝔼d1_→r1_[ℰ1s___] ≡ 𝔼d2_→r2_[ℰ2s___] ^:= (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[ℰ1s] ≡ 𝔼[ℰ2s]);
𝔼 /: ℰ1_𝔼 * ℰ2_𝔼 := 𝔼 @@ Table[CF[ℰ1[kk] + ℰ2[kk]], {kk, 0, Min[ℰ1[$], ℰ2[$]]}];
𝔼d1_→r1_[ℰ1s___] 𝔼d2_→r2_[ℰ2s___] ^:= 𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[ℰ1s] × 𝔼[ℰ2s]);
```

```
𝔼d1_→r1_[ℰ1s___] // 𝔼d2_→r2_[ℰ2s___] := Module[{is = r1 ⋂ d2, lvs, llvs},
  lvs = Cases[{ℰ1s}, (x | y)__[i_] /; MemberQ[is, i], ∞] ⋃
    Cases[{ℰ2s}, (ξ | η)__[i_] /; MemberQ[is, i], ∞]*;
  llvs = Map[$, lvs, {2}];
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ (Zip_{llvs⋃llvs*}[{(llvs)*.(llvs), Times[
      𝔼[ℰ1s] /. Thread[lvs → llvs],
      𝔼[ℰ2s] /. Thread[lvs* → llvs*]
     ]}])
 ]
```

Zipping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.




Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight $n$ variables in perturbations with the weight 0 variables in the coefficients of $Q$.

dvs stands for “Diagonal Variables”. In $\mathrm{gl}_n$ they are the $x_{ii}$’s and the $y_{ii}$’s. They have weights {1,0} and {0,0}.

```
Zip_vs_[{𝓕_, 𝓔_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
   {𝓕, 𝓔} // Zip1_vs (*//Zip2_Complement[vs,dvs]*) // Zip2_dvs // Zip3_Complement[vs,dvs] // Zip3_dvs
  ] // Last
```

Getting rid of the quadratic.

**Lemma 1.** With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}G_{ij}z_iz_j}\right\rangle_B=\det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B$$

```
Zip1_{} = Identity;
Zip1_vs_ @{𝓕_, 𝔼[Q_, P___]} := Module[{𝓘, F, G, u, v},
  𝓘 = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v} 𝓕, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂_{u,v} Q, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[𝓘 - G.F]).(vs) / 2],
   𝔼[CF[Q - PowerExpand@Log[Det[𝓘 - G.F]] / 2 - vs.G.vs / 2], P]}
 ]
```

Getting rid of linear terms.

**Lemma 2.** $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

```
Zip2_{} = Identity;
Zip2_vs_ @{𝓕_, 𝔼[Q_, P___]} := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂_{u,v} 𝓕], 0], {u, vs}, {v, vs}];
  Y = Table[∂_v Q, {v, vs}] /. Table[v → 0, {v, vs}];
  CF /@ {𝓕, 𝔼[Q - Y.vs + Y.F.Y / 2, P] /. Thread[vs → vs + F.Y]}
 ]
```

Dealing with Feynman diagrams.

**Lemma 3.** With an extra variable $\lambda$, $Z_\lambda:=\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

Note that the power $m$ of $\lambda$ is at most $k-1+\frac{2k+2}{2}=2k$. We write $Z_\lambda=\sum Z[m]\,\lambda^m$.

```
Zip3vs_@{𝓕_, ℰ_𝔼} := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] = ℰ⟦kk + 1⟧, {kk, 0, ℰ@$}];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0}, ∂u,v 𝓕, 0];
  Z[m_, kk_, u_] := Z[m, kk, u] = ∂u Z[m, kk];
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] = ∂v Z[m, kk, u];
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤ ℰ@$, ++kk,
    Z[m + 1, kk] = CF@Sum[
       If[F[u, v] === 0, 0,
        F[u, v]/(2 (m + 1)) (Z[m, kk, u, v] + Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m},
            {jj, 0, kk}])],
       {u, vs}, {v, vs}];
    If[Z[m + 1, kk] =!= 0, $m = m + 1]
   ]];
  CF /@ ({
      𝓕 - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
      𝔼 @@ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0, ℰ@$}]
     } /. Table[v → 0, {v, vs}])
 ]
Λ2𝔼d_→r_[Λ_] := Λ2𝔼d→r[$k, Λ];
Λ2𝔼d_→r_[$k_, Λ_] := Module[{k}, 𝔼d→r @@ Table[SeriesCoefficient[Λ, {ϵ, 0, k}], {k, 0, $k}]];
```

# Prolog

```
(*BeginPackage["UEA`"];*)
Print[
  "UEA` does computations in general universal enveloping algebras and PBW algebras.
   It is in the public domain, available at
   http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror Bar-Natan is
   committed to support it within reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
  Sort@{"**", ϵ, $k, CF, SSQ, NilQ, B, m, SetAlgebra, U, UB, UProducts, USimp, UU,
   $Basis, $PBWRule, δ, adPower, adExp},
  "."];
(*Begin["`Private`"];*)
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.

UEA` implements / extends {**, adExp, adPower, B, CF, m, NilQ, SetAlgebra, SSQ, U, UB, UProducts, USimp, UU, δ, ϵ, $Basis, $k, $PBWRule}.

# Utilities

```
δi_,j_ := If[i == j, 1, 0];
```

```
SSQ[c_. * x_] /; MemberQ[$Basis, x] := ¬ NilQ[x]; (* Semi-Simple Q *)
NilQ[c_ * x_] /; MemberQ[$Basis, x] := NilQ[x];
SSQ[x_Plus] := And @@ (SSQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
$k = 1;
```

```
CF[ℰ_] := Expand[ℰ /. e^L_ :→ e^Expand[L]] /. {ϵ^k_. /; k > $k → 0};
```

## Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := CF[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := CF[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := CF[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :→ U_i[xs],
    x_ /; MemberQ[$Basis, x] :→ U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k}) c // USimp
  ];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :→ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⊠ y - y ⊠ x];
```

## Epilog

```
(*End[]; EndPackage[];*)
```

## Predefined Algebras

### sl(2)

```
Print["UEA`SetAlgebra knows \"sl2\"."];
```

```
UEA`SetAlgebra knows "sl2".
```

```
SetAlgebra["sl2"] := (
   Print["In sl2: ⟨e,h,f⟩/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
   B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
   $Algebra = "sl2";
   $Basis = {e, h, f};
   $PBWRule = {e → 1, h → 2, f → 3};
   NilQ[e] = NilQ[f] = True; NilQ[h] = False;
  );
```

### $gl_{n,\epsilon}$

```
Print["UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}."];
```

```
UEA`SetAlgebra knows the ϵ-nilpotent algebra gl_{n,ϵ}.
```

```
SetAlgebra[gl_{n_,ϵ}] := (
   $Algebra = gl_{n,ϵ};
   $Basis = Flatten@{
      Table[y_{α,β}, {β, 2, n}, {α, 1, β - 1}],
      Table[x_{α,β}, {β, 1, n}, {α, 1, β}]
     };
   (*$Basis=Reverse@SortBy[$Basis,If[#[[1]]===x,{0,#[[3]]-#[[2]]}, {1,#[[2]]-#[[3]]}]&];*)
   NilQ[y__] = True; NilQ[x_{α_,β_}] := (α =!= β); (* Nilpotent Q *)
   $PBWRule = Thread[$Basis → Range@Length@$Basis];
   B[x_{i_,j_}, x_{k_,l_}] := δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j};
   B[y_{i_,j_}, y_{k_,l_}] := CF[-ϵ δ_{j,k} y_{i,l} + ϵ δ_{l,i} y_{k,j}];
   B[x_{i_,j_}, y_{l_,k_}] := CF[δ_{j,k} x_{i,l} - δ_{l,i} x_{k,j} /. x_{α_,β_} :→ If[α ≤ β, ϵ x_{α,β}, y_{β,α}]];
   DefRep[0] = Normal@SparseArray[{}, {n, n}];
   DefRep[ℰ_] := ℰ /. {
      x_{α_,β_} :→ Normal@SparseArray[{α, β} → 1, {n, n}],
      y_{α_,β_} :→ Normal@SparseArray[{β, α} → ϵ, {n, n}]
     }
  );
```

```
adPower[x_, k_, Env_][c_ * y_] /; MemberQ[$Basis, y] := CF[c adPower[x, k, Env][y]];
adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_][y_] := y;
adPower[x_, k_, Env_][0] = 0;
adPower[x_, k_, Env_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k, Env][y] = B[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, {$Algebra, $k}][y];
```

```
adExp[x_][y_] /; SSQ[x] := Expand@Module[{A, b, c},
   A = Table[Coefficient[B[b, x], c], {c, $Basis}, {b, $Basis}];
   CF[$Basis.Normal[Series[MatrixExp[A], {ϵ, 0, $k}]].Table[Coefficient[y, b], {b, $Basis}]]
  ]
```

```
adExp[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂_λ x;
λTangent[xs_List] := λTangent @@ xs
TriangularDSolve[eqns_, funs_, iv_] := Module[{sol = {}, e, fun, der},
  MapThread[
   {e, fun} ↦ sol = Join[sol,
      Echo@
       MapAt[CF,
        First@DSolve[CF[e /. Derivative[1][f_][v_] ⧴ der[f[v], v] /. sol /. der[f_, v_] ⧴ ∂_v f],
          fun, iv], {1, 2}]
      ],
   {eqns, funs}
  ];
  sol
 ]
```

## $gl_{4,\epsilon}$ Experiments

```
SetAlgebra[gl_{4,ϵ}];
$k = 1
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
MatrixForm@Table[{b1, b2} → (DefRep[b1].DefRep[b2] - DefRep[b2].DefRep[b1] == DefRep[B[b1, b2]]),
  {b1, $Basis}, {b2, $Basis}]
DeleteCases[
 Flatten[Table[{a, b, c} → B[a, B[b, c]] + B[b, B[c, a]] + B[c, B[a, b]], {a, $Basis},
   {b, $Basis}, {c, $Basis}]], _ → 0]
```

1

$\{y_{1,2} \to 1,\ y_{1,3} \to 2,\ y_{2,3} \to 3,\ y_{1,4} \to 4,\ y_{2,4} \to 5,\ y_{3,4} \to 6,\ x_{1,1} \to 7,\ x_{1,2} \to 8,\ x_{2,2} \to 9,\ x_{1,3} \to 10,\ x_{2,3} \to 11,\ x_{3,3} \to 12,\ x_{1,4} \to 13,\ x_{2,4} \to 14,\ x_{3,4} \to 15,\ x_{4,4} \to 16\}$

| | | | | |
|---|---|---|---|---|
| $\{y_{1,2}, y_{1,2}\} \to 0$ | $\{y_{1,3}, y_{1,2}\} \to 0$ | $\{y_{2,3}, y_{1,2}\} \to \in y_{1,3}$ | $\{y_{1,4}, y_{1,2}\} \to 0$ | $\{y_{2,4}, y_{1,2}\} \to \in y_{1,4}$ |
| $\{y_{1,2}, y_{1,3}\} \to 0$ | $\{y_{1,3}, y_{1,3}\} \to 0$ | $\{y_{2,3}, y_{1,3}\} \to 0$ | $\{y_{1,4}, y_{1,3}\} \to 0$ | $\{y_{2,4}, y_{1,3}\} \to 0$ |
| $\{y_{1,2}, y_{2,3}\} \to -\in y_{1,3}$ | $\{y_{1,3}, y_{2,3}\} \to 0$ | $\{y_{2,3}, y_{2,3}\} \to 0$ | $\{y_{1,4}, y_{2,3}\} \to 0$ | $\{y_{2,4}, y_{2,3}\} \to 0$ |
| $\{y_{1,2}, y_{1,4}\} \to 0$ | $\{y_{1,3}, y_{1,4}\} \to 0$ | $\{y_{2,3}, y_{1,4}\} \to 0$ | $\{y_{1,4}, y_{1,4}\} \to 0$ | $\{y_{2,4}, y_{1,4}\} \to 0$ |
| $\{y_1$ | | | | |

Rows of the first block (read row by row):

| | | | | |
|---|---|---|---|---|
| $\{y_{1,2}, y_{1,2}\} \to 0$ | $\{y_{1,2}, y_{1,3}\} \to 0$ | $\{y_{1,2}, y_{2,3}\} \to -\in y_{1,3}$ | $\{y_{1,2}, y_{1,4}\} \to 0$ | $\{y_1$ |
| $\{y_{1,3}, y_{1,2}\} \to 0$ | $\{y_{1,3}, y_{1,3}\} \to 0$ | $\{y_{1,3}, y_{2,3}\} \to 0$ | $\{y_{1,3}, y_{1,4}\} \to 0$ | |
| $\{y_{2,3}, y_{1,2}\} \to \in y_{1,3}$ | $\{y_{2,3}, y_{1,3}\} \to 0$ | $\{y_{2,3}, y_{2,3}\} \to 0$ | $\{y_{2,3}, y_{1,4}\} \to 0$ | |
| $\{y_{1,4}, y_{1,2}\} \to 0$ | $\{y_{1,4}, y_{1,3}\} \to 0$ | $\{y_{1,4}, y_{2,3}\} \to 0$ | $\{y_{1,4}, y_{1,4}\} \to 0$ | |
| $\{y_{2,4}, y_{1,2}\} \to \in y_{1,4}$ | $\{y_{2,4}, y_{1,3}\} \to 0$ | $\{y_{2,4}, y_{2,3}\} \to 0$ | $\{y_{2,4}, y_{1,4}\} \to 0$ | |
| $\{y_{3,4}, y_{1,2}\} \to 0$ | $\{y_{3,4}, y_{1,3}\} \to \in y_{1,4}$ | $\{y_{3,4}, y_{2,3}\} \to \in y_{2,4}$ | $\{y_{3,4}, y_{1,4}\} \to 0$ | |
| $\{x_{1,1}, y_{1,2}\} \to -y_{1,2}$ | $\{x_{1,1}, y_{1,3}\} \to -y_{1,3}$ | $\{x_{1,1}, y_{2,3}\} \to 0$ | $\{x_{1,1}, y_{1,4}\} \to -y_{1,4}$ | |
| $\{x_{1,2}, y_{1,2}\} \to \in x_{1,1} - \in x_{2,2}$ | $\{x_{1,2}, y_{1,3}\} \to -y_{2,3}$ | $\{x_{1,2}, y_{2,3}\} \to 0$ | $\{x_{1,2}, y_{1,4}\} \to -y_{2,4}$ | |
| $\{x_{2,2}, y_{1,2}\} \to y_{1,2}$ | $\{x_{2,2}, y_{1,3}\} \to 0$ | $\{x_{2,2}, y_{2,3}\} \to -y_{2,3}$ | $\{x_{2,2}, y_{1,4}\} \to 0$ | $\{x$ |
| $\{x_{1,3}, y_{1,2}\} \to -\in x_{2,3}$ | $\{x_{1,3}, y_{1,3}\} \to \in x_{1,1} - \in x_{3,3}$ | $\{x_{1,3}, y_{2,3}\} \to \in x_{1,2}$ | $\{x_{1,3}, y_{1,4}\} \to -y_{3,4}$ | |
| $\{x_{2,3}, y_{1,2}\} \to 0$ | $\{x_{2,3}, y_{1,3}\} \to y_{1,2}$ | $\{x_{2,3}, y_{2,3}\} \to \in x_{2,2} - \in x_{3,3}$ | $\{x_{2,3}, y_{1,4}\} \to 0$ | $\{x$ |
| $\{x_{3,3}, y_{1,2}\} \to 0$ | $\{x_{3,3}, y_{1,3}\} \to y_{1,3}$ | $\{x_{3,3}, y_{2,3}\} \to y_{2,3}$ | $\{x_{3,3}, y_{1,4}\} \to 0$ | |
| $\{x_{1,4}, y_{1,2}\} \to -\in x_{2,4}$ | $\{x_{1,4}, y_{1,3}\} \to -\in x_{3,4}$ | $\{x_{1,4}, y_{2,3}\} \to 0$ | $\{x_{1,4}, y_{1,4}\} \to \in x_{1,1} - \in x_{4,4}$ | $\{x$ |
| $\{x_{2,4}, y_{1,2}\} \to 0$ | $\{x_{2,4}, y_{1,3}\} \to 0$ | $\{x_{2,4}, y_{2,3}\} \to -\in x_{3,4}$ | $\{x_{2,4}, y_{1,4}\} \to y_{1,2}$ | $\{x_{2,4},$ |
| $\{x_{3,4}, y_{1,2}\} \to 0$ | $\{x_{3,4}, y_{1,3}\} \to 0$ | $\{x_{3,4}, y_{2,3}\} \to 0$ | $\{x_{3,4}, y_{1,4}\} \to y_{1,3}$ | $\{$ |
| $\{x_{4,4}, y_{1,2}\} \to 0$ | $\{x_{4,4}, y_{1,3}\} \to 0$ | $\{x_{4,4}, y_{2,3}\} \to 0$ | $\{x_{4,4}, y_{1,4}\} \to y_{1,4}$ | $\{$ |

| | | | | | |
|---|---|---|---|---|---|
| $\{y_{1,2}, y_{1,2}\} \to$ True | $\{y_{1,2}, y_{1,3}\} \to$ True | $\{y_{1,2}, y_{2,3}\} \to$ True | $\{y_{1,2}, y_{1,4}\} \to$ True | $\{y_{1,2}, y_{2,4}\} \to$ True | $\{y_{1,2}, y_{3,4}\} \to$ True |
| $\{y_{1,3}, y_{1,2}\} \to$ True | $\{y_{1,3}, y_{1,3}\} \to$ True | $\{y_{1,3}, y_{2,3}\} \to$ True | $\{y_{1,3}, y_{1,4}\} \to$ True | $\{y_{1,3}, y_{2,4}\} \to$ True | $\{y_{1,3}, y_{3,4}\} \to$ True |
| $\{y_{2,3}, y_{1,2}\} \to$ True | $\{y_{2,3}, y_{1,3}\} \to$ True | $\{y_{2,3}, y_{2,3}\} \to$ True | $\{y_{2,3}, y_{1,4}\} \to$ True | $\{y_{2,3}, y_{2,4}\} \to$ True | $\{y_{2,3}, y_{3,4}\} \to$ True |
| $\{y_{1,4}, y_{1,2}\} \to$ True | $\{y_{1,4}, y_{1,3}\} \to$ True | $\{y_{1,4}, y_{2,3}\} \to$ True | $\{y_{1,4}, y_{1,4}\} \to$ True | $\{y_{1,4}, y_{2,4}\} \to$ True | $\{y_{1,4}, y_{3,4}\} \to$ True |
| $\{y_{2,4}, y_{1,2}\} \to$ True | $\{y_{2,4}, y_{1,3}\} \to$ True | $\{y_{2,4}, y_{2,3}\} \to$ True | $\{y_{2,4}, y_{1,4}\} \to$ True | $\{y_{2,4}, y_{2,4}\} \to$ True | $\{y_{2,4}, y_{3,4}\} \to$ True |
| $\{y_{3,4}, y_{1,2}\} \to$ True | $\{y_{3,4}, y_{1,3}\} \to$ True | $\{y_{3,4}, y_{2,3}\} \to$ True | $\{y_{3,4}, y_{1,4}\} \to$ True | $\{y_{3,4}, y_{2,4}\} \to$ True | $\{y_{3,4}, y_{3,4}\} \to$ True |
| $\{x_{1,1}, y_{1,2}\} \to$ True | $\{x_{1,1}, y_{1,3}\} \to$ True | $\{x_{1,1}, y_{2,3}\} \to$ True | $\{x_{1,1}, y_{1,4}\} \to$ True | $\{x_{1,1}, y_{2,4}\} \to$ True | $\{x_{1,1}, y_{3,4}\} \to$ True |
| $\{x_{1,2}, y_{1,2}\} \to$ True | $\{x_{1,2}, y_{1,3}\} \to$ True | $\{x_{1,2}, y_{2,3}\} \to$ True | $\{x_{1,2}, y_{1,4}\} \to$ True | $\{x_{1,2}, y_{2,4}\} \to$ True | $\{x_{1,2}, y_{3,4}\} \to$ True |
| $\{x_{2,2}, y_{1,2}\} \to$ True | $\{x_{2,2}, y_{1,3}\} \to$ True | $\{x_{2,2}, y_{2,3}\} \to$ True | $\{x_{2,2}, y_{1,4}\} \to$ True | $\{x_{2,2}, y_{2,4}\} \to$ True | $\{x_{2,2}, y_{3,4}\} \to$ True |
| $\{x_{1,3}, y_{1,2}\} \to$ True | $\{x_{1,3}, y_{1,3}\} \to$ True | $\{x_{1,3}, y_{2,3}\} \to$ True | $\{x_{1,3}, y_{1,4}\} \to$ True | $\{x_{1,3}, y_{2,4}\} \to$ True | $\{x_{1,3}, y_{3,4}\} \to$ True |
| $\{x_{2,3}, y_{1,2}\} \to$ True | $\{x_{2,3}, y_{1,3}\} \to$ True | $\{x_{2,3}, y_{2,3}\} \to$ True | $\{x_{2,3}, y_{1,4}\} \to$ True | $\{x_{2,3}, y_{2,4}\} \to$ True | $\{x_{2,3}, y_{3,4}\} \to$ True |
| $\{x_{3,3}, y_{1,2}\} \to$ True | $\{x_{3,3}, y_{1,3}\} \to$ True | $\{x_{3,3}, y_{2,3}\} \to$ True | $\{x_{3,3}, y_{1,4}\} \to$ True | $\{x_{3,3}, y_{2,4}\} \to$ True | $\{x_{3,3}, y_{3,4}\} \to$ True |
| $\{x_{1,4}, y_{1,2}\} \to$ True | $\{x_{1,4}, y_{1,3}\} \to$ True | $\{x_{1,4}, y_{2,3}\} \to$ True | $\{x_{1,4}, y_{1,4}\} \to$ True | $\{x_{1,4}, y_{2,4}\} \to$ True | $\{x_{1,4}, y_{3,4}\} \to$ True |
| $\{x_{2,4}, y_{1,2}\} \to$ True | $\{x_{2,4}, y_{1,3}\} \to$ True | $\{x_{2,4}, y_{2,3}\} \to$ True | $\{x_{2,4}, y_{1,4}\} \to$ True | $\{x_{2,4}, y_{2,4}\} \to$ True | $\{x_{2,4}, y_{3,4}\} \to$ True |
| $\{x_{3,4}, y_{1,2}\} \to$ True | $\{x_{3,4}, y_{1,3}\} \to$ True | $\{x_{3,4}, y_{2,3}\} \to$ True | $\{x_{3,4}, y_{1,4}\} \to$ True | $\{x_{3,4}, y_{2,4}\} \to$ True | $\{x_{3,4}, y_{3,4}\} \to$ True |
| $\{x_{4,4}, y_{1,2}\} \to$ True | $\{x_{4,4}, y_{1,3}\} \to$ True | $\{x_{4,4}, y_{2,3}\} \to$ True | $\{x_{4,4}, y_{1,4}\} \to$ True | $\{x_{4,4}, y_{2,4}\} \to$ True | $\{x_{4,4}, y_{3,4}\} \to$ True |

$\{\}$

```
SBasis = SortBy[$Basis, If[#[[1]] === x, {0, #[[3]] - #[[2]]}, {1, #[[2]] - #[[3]]}] &];
sol[-1] = Table[If[SSQ[bb], ((bb /. x → ξ1) + (bb /. x → ξ2)), 0], {bb, $Basis}];
Do[
  Block[{$k = k},
   s0 = Thread[$Basis → sol[k - 1]];
   lhs = λTangent[
     Join[Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ1, y → η1}), {bb, $Basis}],
      Table[If[NilQ[bb], λ, 1] bb (bb /. {x → ξ2, y → η2}), {bb, $Basis}]]
    ];
   rhs = λTangent[
     $Basis ($Basis /. {x_αβ__ :→ (x_αβ /. s0) + ϵ^k f_αβ[λ], y_αβ__ :→ (y_αβ /. s0) + ϵ^k g_αβ[λ]})
    ];
   eqns = Transpose@{
      (Coefficient[CF[lhs - rhs], #] == 0) & /@ SBasis,
      SBasis /. {x_αβ__ :→ f_αβ[0] == 0, y_αβ__ :→ g_αβ[0] == 0}
     };
   unknowns = SBasis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]};
   s = TriangularDSolve[eqns, unknowns, λ];
   sol[k] = CF[sol[k - 1] + ϵ^k ($Basis /. {x_αβ__ → f_αβ[λ], y_αβ__ → g_αβ[λ]} /. s)]
  ],
  {k, 0, $k}
 ];
cm[i_, j_ → k_] :=
  Λ2𝔼_{i,j}→{k}[sol[$k].$Basis /. {x_αβ__ :→ x_αβ[k], y_αβ__ :→ y_αβ[k], ξ1_αβ__ :→ ξ_αβ[i], η1_αβ__ :→ η_αβ[i],
     ξ2_αβ__ :→ ξ_αβ[j], η2_αβ__ :→ η_αβ[j], λ → 1}];
Short[cm[i, j → k], 10]
```

$$\{f_{1,1}[\lambda] \to 0\}$$

$$\{f_{2,2}[\lambda] \to 0\}$$

$$\{f_{3,3}[\lambda] \to 0\}$$

$$\{f_{4,4}[\lambda] \to 0\}$$

$$\left\{f_{1,2}[\lambda] \to e^{-\xi 2_{1,1}}\,\lambda\,\xi 1_{1,2} + e^{-\xi 1_{2,2}}\,\lambda\,\xi 2_{1,2}\right\}$$

$$\left\{f_{2,3}[\lambda] \to e^{-\xi 2_{2,2}}\,\lambda\,\xi 1_{2,3} + e^{-\xi 1_{3,3}}\,\lambda\,\xi 2_{2,3}\right\}$$

$$\left\{f_{3,4}[\lambda] \to e^{-\xi 2_{3,3}}\,\lambda\,\xi 1_{3,4} + e^{-\xi 1_{4,4}}\,\lambda\,\xi 2_{3,4}\right\}$$

$$\left\{f_{1,3}[\lambda] \to e^{-\xi 2_{1,1}}\,\lambda\,\xi 1_{1,3} - \lambda^2\,\xi 1_{2,3}\,\xi 2_{1,2} + e^{-\xi 1_{3,3}}\,\lambda\,\xi 2_{1,3}\right\}$$

$$\left\{f_{2,4}[\lambda] \to e^{-\xi 2_{2,2}}\,\lambda\,\xi 1_{2,4} - \lambda^2\,\xi 1_{3,4}\,\xi 2_{2,3} + e^{-\xi 1_{4,4}}\,\lambda\,\xi 2_{2,4}\right\}$$

$$\left\{f_{1,4}[\lambda] \to e^{-\xi 2_{1,1}}\,\lambda\,\xi 1_{1,4} - \lambda^2\,\xi 1_{2,4}\,\xi 2_{1,2} - \lambda^2\,\xi 1_{3,4}\,\xi 2_{1,3} + e^{-\xi 1_{4,4}}\,\lambda\,\xi 2_{1,4}\right\}$$

$$\left\{g_{1,4}[\lambda] \to \lambda\,\eta 1_{1,4} + e^{-\xi 1_{1,1}+\xi 1_{4,4}}\,\lambda\,\eta 2_{1,4}\right\}$$

$$\left\{g_{1,3}[\lambda] \to \lambda\,\eta 1_{1,3} + e^{-\xi 1_{1,1}+\xi 1_{3,3}}\,\lambda\,\eta 2_{1,3} + e^{-\xi 1_{1,1}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^2\,\eta 2_{1,4}\,\xi 1_{3,4}\right\}$$

$$\left\{g_{2,4}[\lambda] \to \lambda\,\eta 1_{2,4} + e^{-\xi 1_{2,2}+\xi 1_{4,4}}\,\lambda\,\eta 2_{2,4} - e^{\xi 1_{4,4}}\,\lambda^2\,\eta 2_{1,4}\,\xi 1_{1,2}\right\}$$

$$\left\{g_{1,2}[\lambda] \to \lambda\,\eta 1_{1,2} + e^{-\xi 1_{1,1}+\xi 1_{2,2}}\,\lambda\,\eta 2_{1,2} + e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\,\lambda^2\,\eta 2_{1,3}\,\xi 1_{2,3} + e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{4,4}}\,\lambda^2\,\eta 2_{1,4}\,\xi 1_{2,4} + e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,4}\,\xi 1_{2,3}\,\xi 1_{3,4}\right\}$$

$$\left\{g_{2,3}[\lambda] \to \lambda\,\eta 1_{2,3} + e^{-\xi 1_{2,2}+\xi 1_{3,3}}\,\lambda\,\eta 2_{2,3} - e^{\xi 1_{3,3}}\,\lambda^2\,\eta 2_{1,3}\,\xi 1_{1,2} + e^{-\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^2\,\eta 2_{2,4}\,\xi 1_{3,4} - e^{\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{3,4}\right\}$$

$$\left\{g_{3,4}[\lambda] \to \lambda\,\eta 1_{3,4} + e^{-\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda\,\eta 2_{3,4} - e^{\xi 1_{4,4}}\,\lambda^2\,\eta 2_{1,4}\,\xi 1_{1,3} - e^{\xi 1_{4,4}}\,\lambda^2\,\eta 2_{2,4}\,\xi 1_{2,3}\right\}$$

$$\{f_{1,1}[\lambda] \to e^{\xi1_{2,2}} \lambda^2 \eta2_{1,2} \xi1_{1,2} + e^{\xi1_{3,3}} \lambda^2 \eta2_{1,3} \xi1_{1,3} + e^{\xi1_{4,4}} \lambda^2 \eta2_{1,4} \xi1_{1,4} + e^{\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta2_{1,3} \xi1_{1,2} \xi1_{2,3} + e^{\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta2_{1,4} \xi1_{1,2} \xi1_{2,4} + e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{1,4} \xi1_{1,3} \xi1_{3,4} + e^{\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,2} \xi1_{2,3} \xi1_{3,4}\}$$

$$\{f_{2,2}[\lambda] \to -e^{\xi1_{2,2}} \lambda^2 \eta2_{1,2} \xi1_{1,2} + e^{\xi1_{3,3}} \lambda^2 \eta2_{2,3} \xi1_{2,3} - e^{\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta2_{1,3} \xi1_{1,2} \xi1_{2,3} + e^{\xi1_{4,4}} \lambda^2 \eta2_{2,4} \xi1_{2,4} - e^{\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta2_{1,4} \xi1_{1,2} \xi1_{2,4} + e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{2,4} \xi1_{2,3} \xi1_{3,4} - e^{\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,2} \xi1_{2,3} \xi1_{3,4}\}$$

$$\{f_{3,3}[\lambda] \to -e^{\xi1_{3,3}} \lambda^2 \eta2_{1,3} \xi1_{1,3} - e^{\xi1_{3,3}} \lambda^2 \eta2_{2,3} \xi1_{2,3} + e^{\xi1_{4,4}} \lambda^2 \eta2_{3,4} \xi1_{3,4} - e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{1,4} \xi1_{1,3} \xi1_{3,4} - e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{2,4} \xi1_{2,3} \xi1_{3,4}\}$$

$$\{f_{4,4}[\lambda] \to -e^{\xi1_{4,4}} \lambda^2 \eta2_{1,4} \xi1_{1,4} - e^{\xi1_{4,4}} \lambda^2 \eta2_{2,4} \xi1_{2,4} - e^{\xi1_{4,4}} \lambda^2 \eta2_{3,4} \xi1_{3,4}\}$$

$$\{f_{1,2}[\lambda] \to e^{-\xi1_{2,2}+\xi1_{3,3}-\xi2_{1,1}} \lambda^2 \eta2_{2,3} \xi1_{1,3} - e^{\xi1_{3,3}-\xi2_{1,1}} \lambda^3 \eta2_{1,3} \xi1_{1,2} \xi1_{1,3} + e^{-\xi1_{2,2}+\xi1_{4,4}-\xi2_{1,1}} \lambda^2 \eta2_{2,4} \xi1_{1,4} - e^{\xi1_{4,4}-\xi2_{1,1}} \lambda^3 \eta2_{1,4} \xi1_{1,2} \xi1_{1,4} + e^{-\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}-\xi2_{1,1}} \lambda^3 \eta2_{2,4} \xi1_{1,3} \xi1_{3,4} - e^{\xi1_{3,3}+\xi1_{4,4}-\xi2_{1,1}} \lambda^4 \eta2_{1,4} \xi1_{1,2} \xi1_{1,3} \xi1_{3,4} + \lambda^3 \eta2_{1,2} \xi1_{1,2} \xi2_{1,2} - e^{-\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta2_{2,3} \xi1_{2,3} \xi2_{1,2} + e^{\xi1_{3,3}} \lambda^4 \eta2_{1,3} \xi1_{1,2} \xi1_{2,3} \xi2_{1,2} - e^{-\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta2_{2,4} \xi1_{2,4} \xi2_{1,2} + e^{\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,2} \xi1_{2,4} \xi2_{1,2} - e^{-\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{2,4} \xi1_{2,3} \xi1_{3,4} \xi2_{1,2} + e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^5 \eta2_{1,4} \xi1_{1,2} \xi1_{2,3} \xi1_{3,4} \xi2_{1,2}\}$$

$$\{f_{2,3}[\lambda] \to -e^{-\xi2_{2,2}} \lambda^2 \eta2_{1,2} \xi1_{1,3} + e^{-\xi1_{3,3}+\xi1_{4,4}-\xi2_{2,2}} \lambda^2 \eta2_{3,4} \xi1_{2,4} - e^{\xi1_{4,4}-\xi2_{2,2}} \lambda^3 \eta2_{1,4} \xi1_{1,3} \xi1_{2,4} - e^{\xi1_{4,4}-\xi2_{2,2}} \lambda^3 \eta2_{2,4} \xi1_{2,3} \xi1_{2,4} + \lambda^3 \eta2_{1,3} \xi1_{1,3} \xi2_{2,3} + \lambda^3 \eta2_{2,3} \xi1_{2,3} \xi2_{2,3} - e^{-\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{3,4} \xi1_{3,4} \xi2_{2,3} + e^{\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,3} \xi1_{3,4} \xi2_{2,3} + e^{\xi1_{4,4}} \lambda^4 \eta2_{2,4} \xi1_{2,3} \xi1_{3,4} \xi2_{2,3}\}$$

$$\{f_{3,4}[\lambda] \to -e^{-\xi2_{3,3}} \lambda^2 \eta2_{1,3} \xi1_{1,4} - e^{-\xi2_{3,3}} \lambda^2 \eta2_{2,3} \xi1_{2,4} + \lambda^3 \eta2_{1,4} \xi1_{1,4} \xi2_{3,4} + \lambda^3 \eta2_{2,4} \xi1_{2,4} \xi2_{3,4} + \lambda^3 \eta2_{3,4} \xi1_{3,4} \xi2_{3,4}\}$$

$$\{f_{1,3}[\lambda] \to e^{-\xi1_{3,3}+\xi1_{4,4}-\xi2_{1,1}} \lambda^2 \eta2_{3,4} \xi1_{1,4} - e^{\xi1_{4,4}-\xi2_{1,1}} \lambda^3 \eta2_{1,4} \xi1_{1,3} \xi1_{1,4} - e^{\xi1_{4,4}-\xi2_{1,1}} \lambda^3 \eta2_{2,4} \xi1_{1,4} \xi1_{2,3} + \lambda^3 \eta2_{1,2} \xi1_{1,3} \xi2_{1,2} - e^{-\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{3,4} \xi1_{2,4} \xi2_{1,2} + e^{\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,3} \xi1_{2,4} \xi2_{1,2} + e^{\xi1_{4,4}} \lambda^4 \eta2_{2,4} \xi1_{2,3} \xi1_{2,4} \xi2_{1,2} + \lambda^3 \eta2_{1,3} \xi1_{1,3} \xi2_{1,3} + \lambda^3 \eta2_{2,3} \xi1_{2,3} \xi2_{1,3} - e^{-\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{3,4} \xi1_{3,4} \xi2_{1,3} + e^{\xi1_{4,4}} \lambda^4 \eta2_{1,4} \xi1_{1,3} \xi1_{3,4} \xi2_{1,3} + e^{\xi1_{4,4}} \lambda^4 \eta2_{2,4} \xi1_{2,3} \xi1_{3,4} \xi2_{1,3}\}$$

$$\{f_{2,4}[\lambda] \to -e^{-\xi2_{2,2}} \lambda^2 \eta2_{1,2} \xi1_{1,4} + \lambda^3 \eta2_{1,3} \xi1_{1,4} \xi2_{2,3} + \lambda^3 \eta2_{2,3} \xi1_{2,4} \xi2_{2,3} + \lambda^3 \eta2_{1,4} \xi1_{1,4} \xi2_{2,4} + \lambda^3 \eta2_{2,4} \xi1_{2,4} \xi2_{2,4} + \lambda^3 \eta2_{3,4} \xi1_{3,4} \xi2_{2,4}\}$$

$$\{f_{1,4}[\lambda] \to \lambda^3 \eta2_{1,2} \xi1_{1,4} \xi2_{1,2} + \lambda^3 \eta2_{1,3} \xi1_{1,4} \xi2_{1,3} + \lambda^3 \eta2_{2,3} \xi1_{2,4} \xi2_{1,3} + \lambda^3 \eta2_{1,4} \xi1_{1,4} \xi2_{1,4} + \lambda^3 \eta2_{2,4} \xi1_{2,4} \xi2_{1,4} + \lambda^3 \eta2_{3,4} \xi1_{3,4} \xi2_{1,4}\}$$

$$\{g_{1,4}[\lambda] \to e^{-\xi1_{1,1}+\xi1_{2,2}} \lambda^2 \eta1_{2,4} \eta2_{1,2} + e^{-\xi1_{1,1}+\xi1_{3,3}} \lambda^2 \eta1_{3,4} \eta2_{1,3} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta2_{1,2} \eta2_{1,4} \xi1_{1,2} - e^{-\xi1_{1,1}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{1,3} \eta2_{1,4} \xi1_{1,3} - e^{-\xi1_{1,1}+2\xi1_{4,4}} \lambda^3 \eta2_{1,4}^2 \xi1_{1,4} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta1_{2,4} \eta2_{1,3} \xi1_{2,3} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,3} \eta2_{1,4} \xi1_{1,2} \xi1_{2,3} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta1_{2,4} \eta2_{1,4} \xi1_{2,4} - e^{-\xi1_{1,1}+\xi1_{2,2}+2\xi1_{4,4}} \lambda^4 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{2,4} + e^{-\xi1_{1,1}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta1_{3,4} \eta2_{1,4} \xi1_{3,4} - e^{-\xi1_{1,1}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^4 \eta2_{1,4}^2 \xi1_{1,3} \xi1_{3,4} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta1_{2,4} \eta2_{1,4} \xi1_{2,3} \xi1_{3,4} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^5 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{2,3} \xi1_{3,4}\}$$

$$\{g_{1,3}[\lambda] \to e^{-\xi1_{1,1}+\xi1_{2,2}} \lambda^2 \eta1_{2,3} \eta2_{1,2} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta2_{1,2} \eta2_{1,3} \xi1_{1,2} - e^{-\xi1_{1,1}+2\xi1_{3,3}} \lambda^3 \eta2_{1,3}^2 \xi1_{1,3} - e^{-\xi1_{1,1}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{1,3} \eta2_{1,4} \xi1_{1,4} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}} \lambda^3 \eta1_{2,3} \eta2_{1,3} \xi1_{2,3} - e^{-\xi1_{1,1}+\xi1_{2,2}+2\xi1_{3,3}} \lambda^4 \eta2_{1,3}^2 \xi1_{1,2} \xi1_{2,3} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{4,4}} \lambda^3 \eta1_{2,3} \eta2_{1,4} \xi1_{2,4} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,3} \eta2_{1,4} \xi1_{1,2} \xi1_{2,4} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,2} \eta2_{1,4} \xi1_{1,2} \xi1_{3,4} - 2\, e^{-\xi1_{1,1}+2\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,3} \eta2_{1,4} \xi1_{1,3} \xi1_{3,4} - e^{-\xi1_{1,1}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^4 \eta2_{1,4}^2 \xi1_{1,4} \xi1_{3,4} + e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta1_{2,3} \eta2_{1,4} \xi1_{2,3} \xi1_{3,4} - 2\, e^{-\xi1_{1,1}+\xi1_{2,2}+2\xi1_{3,3}+\xi1_{4,4}} \lambda^5 \eta2_{1,3} \eta2_{1,4} \xi1_{1,2} \xi1_{2,3} \xi1_{3,4} - e^{-\xi1_{1,1}+\xi1_{2,2}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^5 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{2,4} \xi1_{3,4} - e^{-\xi1_{1,1}+2\xi1_{3,3}+2\xi1_{4,4}} \lambda^5 \eta2_{1,4}^2 \xi1_{1,3} \xi1_{3,4}^2 - e^{-\xi1_{1,1}+\xi1_{2,2}+2\xi1_{3,3}+2\xi1_{4,4}} \lambda^6 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{2,3} \xi1_{3,4}^2\}$$

$$\{g_{2,4}[\lambda] \to e^{-\xi1_{2,2}+\xi1_{3,3}} \lambda^2 \eta1_{3,4} \eta2_{2,3} - e^{\xi1_{3,3}} \lambda^3 \eta1_{3,4} \eta2_{1,3} \xi1_{1,2} + e^{\xi1_{4,4}} \lambda^3 \eta2_{1,2} \eta2_{2,4} \xi1_{1,2} - e^{-\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{1,4} \eta2_{2,3} \xi1_{1,3} + e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,3} \eta2_{1,4} \xi1_{1,2} \xi1_{1,3} - e^{-\xi1_{2,2}+2\xi1_{4,4}} \lambda^3 \eta2_{1,4} \eta2_{2,4} \xi1_{1,4} + e^{2\xi1_{4,4}} \lambda^4 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{1,4} - e^{-\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta2_{2,3} \eta2_{2,4} \xi1_{2,3} + e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta2_{1,3} \eta2_{2,4} \xi1_{1,2} \xi1_{2,3} - e^{-\xi1_{2,2}+2\xi1_{4,4}} \lambda^3 \eta2_{2,4}^2 \xi1_{2,4} + e^{2\xi1_{4,4}} \lambda^4 \eta2_{1,4} \eta2_{2,4} \xi1_{1,2} \xi1_{2,4} + e^{-\xi1_{2,2}+\xi1_{3,3}+\xi1_{4,4}} \lambda^3 \eta1_{3,4} \eta2_{2,4} \xi1_{3,4} - e^{\xi1_{3,3}+\xi1_{4,4}} \lambda^4 \eta1_{3,4} \eta2_{1,4} \xi1_{1,2} \xi1_{3,4} - e^{-\xi1_{2,2}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^4 \eta2_{1,4} \eta2_{2,4} \xi1_{1,3} \xi1_{3,4} + e^{\xi1_{3,3}+2\xi1_{4,4}} \lambda^5 \eta2_{1,4}^2 \xi1_{1,2} \xi1_{1,3} \xi1_{3,4} - e^{-\xi1_{2,2}+\xi1_{3,3}+2\xi1_{4,4}} \lambda^4 \eta2_{2,4}^2 \xi1_{2,3} \xi1_{3,4} + e^{\xi1_{3,3}+2\xi1_{4,4}} \lambda^5 \eta2_{1,4} \eta2_{2,4} \xi1_{1,2} \xi1_{2,3} \xi1_{3,4}\}$$

$$\{g_{1,2}[\lambda] \to -e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}}\,\lambda^3\,\eta 2_{1,2}^2\,\xi 1_{1,2} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}}\,\lambda^3\,\eta 2_{1,2}\,\eta 2_{1,3}\,\xi 1_{1,3} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,2}\,\eta 2_{1,4}\,\xi 1_{1,4} -$$
$$2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}}\,\lambda^4\,\eta 2_{1,2}\,\eta 2_{1,3}\,\xi 1_{1,2}\,\xi 1_{2,3} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\lambda^4\,\eta 2_{1,3}^2\,\xi 1_{1,3}\,\xi 1_{2,3} -$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,4}\,\xi 1_{2,3} - e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\lambda^5\,\eta 2_{1,3}^2\,\xi 1_{1,2}\,\xi 1_{2,3}^2 -$$
$$2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,2}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{2,4} - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,3}\,\xi 1_{2,4} -$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}^2\,\xi 1_{1,4}\,\xi 1_{2,4} - 2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{2,4} -$$
$$e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}^2\,\xi 1_{1,2}\,\xi 1_{2,4}^2 - e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,2}\,\eta 2_{1,4}\,\xi 1_{1,3}\,\xi 1_{3,4} -$$
$$2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,2}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{3,4} - 2\,e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,3}\,\xi 1_{2,3}\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}^2\,\xi 1_{1,4}\,\xi 1_{2,3}\,\xi 1_{3,4} - 2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^6\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{2,3}^2\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}^2\,\xi 1_{1,3}\,\xi 1_{2,4}\,\xi 1_{3,4} - 2\,e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^6\,\eta 2_{1,4}^2\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{2,4}\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{1,1}+\xi 1_{2,2}+2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^6\,\eta 2_{1,4}^2\,\xi 1_{1,3}\,\xi 1_{2,3}\,\xi 1_{3,4}^2 - e^{-\xi 1_{1,1}+2\,\xi 1_{2,2}+2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^7\,\eta 2_{1,4}^2\,\xi 1_{1,2}\,\xi 1_{2,3}^2\,\xi 1_{3,4}^2\}$$

$$\{g_{2,3}[\lambda] \to e^{\xi 1_{3,3}}\,\lambda^3\,\eta 2_{1,2}\,\eta 2_{2,3}\,\xi 1_{1,2} - e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\lambda^3\,\eta 2_{1,3}\,\eta 2_{2,3}\,\xi 1_{1,3} +$$
$$e^{2\,\xi 1_{3,3}}\,\lambda^4\,\eta 2_{1,3}^2\,\xi 1_{1,2}\,\xi 1_{1,3} - e^{-\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,3}\,\eta 2_{2,4}\,\xi 1_{1,4} + e^{\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{1,4} -$$
$$e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}}\,\lambda^3\,\eta 2_{2,3}^2\,\xi 1_{2,3} + e^{2\,\xi 1_{3,3}}\,\lambda^4\,\eta 2_{1,3}\,\eta 2_{2,3}\,\xi 1_{1,2}\,\xi 1_{2,3} - e^{-\xi 1_{2,2}+\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^3\,\eta 2_{2,3}\,\eta 2_{2,4}\,\xi 1_{2,4} +$$
$$e^{\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{2,3}\,\xi 1_{1,2}\,\xi 1_{2,4} + e^{\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,2}\,\eta 2_{2,4}\,\xi 1_{1,2}\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{2,3}\,\xi 1_{1,3}\,\xi 1_{3,4} - e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,3}\,\eta 2_{2,4}\,\xi 1_{1,3}\,\xi 1_{3,4} +$$
$$2\,e^{2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,3}\,\eta 2_{1,4}\,\xi 1_{1,2}\,\xi 1_{1,3}\,\xi 1_{3,4} - e^{-\xi 1_{2,2}+\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,4}\,\xi 1_{3,4} +$$
$$e^{\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}^2\,\xi 1_{1,2}\,\xi 1_{1,4}\,\xi 1_{3,4} - 2\,e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^4\,\eta 2_{2,3}\,\eta 2_{2,4}\,\xi 1_{2,3}\,\xi 1_{3,4} +$$
$$e^{2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}\,\eta 2_{2,3}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{3,4} + e^{2\,\xi 1_{3,3}+\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,3}\,\eta 2_{2,4}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{2,2}+\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{2,4}^2\,\xi 1_{2,4}\,\xi 1_{3,4} + e^{\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,2}\,\xi 1_{2,4}\,\xi 1_{3,4} -$$
$$e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,3}\,\xi 1_{3,4}^2 + e^{2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^6\,\eta 2_{1,4}^2\,\xi 1_{1,2}\,\xi 1_{1,3}\,\xi 1_{3,4}^2 -$$
$$e^{-\xi 1_{2,2}+2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^5\,\eta 2_{2,4}^2\,\xi 1_{2,3}\,\xi 1_{3,4}^2 + e^{2\,\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^6\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,2}\,\xi 1_{2,3}\,\xi 1_{3,4}^2\}$$

$$\{g_{3,4}[\lambda] \to e^{\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,2}\,\eta 2_{2,4}\,\xi 1_{1,3} + e^{\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,3}\,\eta 2_{3,4}\,\xi 1_{1,3} - e^{-\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^3\,\eta 2_{1,4}\,\eta 2_{3,4}\,\xi 1_{1,4} +$$
$$e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}^2\,\xi 1_{1,3}\,\xi 1_{1,4} + e^{\xi 1_{4,4}}\,\lambda^3\,\eta 2_{2,3}\,\eta 2_{3,4}\,\xi 1_{2,3} + e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,4}\,\xi 1_{2,3} -$$
$$e^{-\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^3\,\eta 2_{2,4}\,\eta 2_{3,4}\,\xi 1_{2,4} + e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{2,4}\,\xi 1_{1,3}\,\xi 1_{2,4} + e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{2,4}^2\,\xi 1_{2,3}\,\xi 1_{2,4} -$$
$$e^{-\xi 1_{3,3}+2\,\xi 1_{4,4}}\,\lambda^3\,\eta 2_{3,4}^2\,\xi 1_{3,4} + e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{1,4}\,\eta 2_{3,4}\,\xi 1_{1,3}\,\xi 1_{3,4} + e^{2\,\xi 1_{4,4}}\,\lambda^4\,\eta 2_{2,4}\,\eta 2_{3,4}\,\xi 1_{2,3}\,\xi 1_{3,4}\}$$

$$\mathbb{E}_{\{i,j\}\to\{k\}}\Big[y_{1,4}[k]\,\left(\eta_{1,4}[i] + e^{-\xi_{1,1}[i]+\xi_{4,4}[i]}\,\eta_{1,4}[j]\right) +$$
$$x_{1,1}[k]\,(\xi_{1,1}[i] + \xi_{1,1}[j]) + y_{2,4}[k]\,\left(\eta_{2,4}[i] + e^{-\xi_{2,2}[i]+\xi_{4,4}[i]}\,\eta_{2,4}[j] - e^{\xi_{4,4}[i]}\,\eta_{1,4}[j]\,\xi_{1,2}[i]\right) +$$
$$x_{1,2}[k]\,\left(e^{-\xi_{1,1}[j]}\,\xi_{1,2}[i] + e^{-\xi_{2,2}[i]}\,\xi_{1,2}[j]\right) + x_{2,2}[k]\,(\xi_{2,2}[i] + \xi_{2,2}[j]) +$$
$$y_{3,4}[k]\,\left(\eta_{3,4}[i] + e^{-\xi_{3,3}[i]+\xi_{4,4}[i]}\,\eta_{3,4}[j] - e^{\xi_{4,4}[i]}\,\eta_{1,4}[j]\,\xi_{1,3}[i] - e^{\xi_{4,4}[i]}\,\eta_{2,4}[j]\,\xi_{2,3}[i]\right) +$$
$$\ll 4 \gg + y_{2,3}[k]\,\left(\eta_{2,3}[i] + e^{-\xi_{2,2}[i]+\xi_{3,3}[i]}\,\eta_{2,3}[j] - e^{\xi_{3,3}[i]}\,\eta_{1,3}[j]\,\xi_{1,2}[i] +\right.$$
$$\left.e^{-\xi_{2,2}[i]+\xi_{3,3}[i]+\xi_{4,4}[i]}\,\eta_{2,4}[j]\,\xi_{3,4}[i] - e^{\xi_{3,3}[i]+\xi_{4,4}[i]}\,\eta_{1,4}[j]\,\xi_{1,2}[i]\,\xi_{3,4}[i]\right) +$$
$$x_{1,4}[k]\,\left(e^{-\xi_{1,1}[j]}\,\xi_{1,4}[i] + e^{-\xi_{4,4}[i]}\,\xi_{1,4}[j] - \xi_{1,2}[j]\,\xi_{2,4}[i] - \xi_{1,3}[j]\,\xi_{3,4}[i]\right) +$$
$$e^{-\xi_{1,1}[i]}\,y_{1,2}[k]\,\left(e^{\xi_{1,1}[i]}\,\eta_{1,2}[i] + e^{\xi_{2,2}[i]}\,\eta_{1,2}[j] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]}\,\eta_{1,3}[j]\,\xi_{2,3}[i] +\right.$$
$$\left.e^{\xi_{2,2}[i]+\xi_{4,4}[i]}\,\eta_{1,4}[j]\,\xi_{2,4}[i] + e^{\xi_{2,2}[i]+\xi_{3,3}[i]+\xi_{4,4}[i]}\,\eta_{1,4}[j]\,\xi_{2,3}[i]\,\xi_{3,4}[i]\right) +$$
$$x_{2,4}[k]\,\left(e^{-\xi_{2,2}[j]}\,\xi_{2,4}[i] + e^{-\xi_{4,4}[i]}\,\xi_{2,4}[j] - \xi_{2,3}[j]\,\xi_{3,4}[i]\right) + x_{3,4}[k]\,\left(e^{-\xi_{3,3}[j]}\,\xi_{3,4}[i] + e^{-\xi_{4,4}[i]}\,\xi_{3,4}[j]\right) +$$
$$x_{4,4}[k]\,(\xi_{4,4}[i] + \xi_{4,4}[j]),\ \ll 1 \gg\Big]$$

**(cm[1, 2 → 2] // cm[2, 3 → 1]) ≡ (cm[2, 3 → 2] // cm[1, 2 → 1])**

True