Pensieve header: An isolated attempt to compute the multiplication tensor of ${\mathcal U}_q(sl^+_{3,\epslion})$. Continues SjabboQUSL3Seepage.nb.

```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

Initialization / Utilities

```
$k = 1; $U = QU; $E := {$k};
$trim := {ϵ^k_. /; k > $k → 0};
SetAttributes[{SS, SST}, HoldAll];
q_ℏ_ := e^(ϒ ϵ ℏ);
(* Upper to lower and lower to Upper: *)
U2l = {B_i_^p_. → e^(-p ℏ ϒ b_i), B^p_. → e^(-p ℏ ϒ b), T_i_^p_. → e^(p ℏ t_i), T^p_. → e^(p ℏ t), 𝒜_i_^p_. → e^(p ϒ α_i), 𝒜^p_. → e^(p ϒ α)};
l2U = {e^(c_. b_i_ + d_.) ⧴ B_i^(-c/(ℏ ϒ)) e^d, e^(c_. b + d_.) ⧴ B^(-c/(ℏ ϒ)) e^d,
   e^(c_. t_i_ + d_.) ⧴ T_i^(c/ℏ) e^d, e^(c_. t + d_.) ⧴ T^(c/ℏ) e^d,
   e^(c_. α_i_ + d_.) ⧴ 𝒜_i^(c/ϒ) e^d, e^(c_. α + d_.) ⧴ 𝒜^(c/ϒ) e^d,
   e^ℰ_ ⧴ e^(Expand@ℰ)};
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
SST[ℰ_, op___] := SS[ℰ /. U2l, op];
Simp[ℰ_, op_] := Collect[ℰ, _CU | _QU, op];
Simp[ℰ_] := Simp[ℰ, SS[#, Expand] &];
SimpT[ℰ_] := Collect[ℰ, _CU | _QU, SST[#, Expand] &];
Kδ /: Kδ_i_,j_ := If[i === j, 1, 0];
c_Integer_k_Integer := c + O[ϵ]^(k+1);
```

DeclareAlgebr

```
Unprotect[NonCommutativeMultiply]; Attributes[NonCommutativeMultiply] = {};
(NCM = NonCommutativeMultiply)[x_] := x;
NCM[x_, y_, z__] := (x ⊠ y) ⊠ z;
0 ⊠ _ = _ ⊠ 0 = 0;
(x_Plus) ⊠ y_ := (# ⊠ y) & /@ x; x_ ⊠ (y_Plus) := (x ⊠ #) & /@ y;
B[x_, x_] = 0; B[x_, y_] := x ⊠ y - y ⊠ x;
B[x_, y_, e_] := B[x, y, e] = B[x, y];
```

```
DeclareAlgebra[U_Symbol, opts__Rule] := Module[
  {gp, sr, g, cp, M, pow, k = 0,
   gs = Generators /. {opts},
   cs = Centrals /. {opts} /. Centrals → {}
  },
  U[Generators] = gs;
  (#_U = U@#) & /@ gs;
  gp = Alternatives @@ gs; gp = gp | gp_; (* gens *)
  sr = Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, gs}]; (* sorting → *)
  cp = Alternatives @@ cs; (* cents *)
  SetAttributes[M, HoldRest]; M[0, _] = 0; M[a_, x_] := a x;
  CE[ℰ_] := Collect[ℰ, _U, Expand] /. ⅇ^x_ :> ⅇ^Expand[x] /. $trim;
  U_i_[ℰ_] := ℰ /. {t : cp :> t_i, u_U :> (#_i &) /@ u};
  U_i_[NCM[]] = pow[ℰ_, 0] = U@{} = 1_U = U[];
  B[U@(x_)_i_, U@(y_)_i_] := U_i@B[U@x, U@y];
  B[U@(x_)_i_, U@(y_)_j_] /; i =!= j := 0;
  B[U@y_, U@x_] := CE[-B[U@x, U@y]];
  x_ ⊠ (c_. 1_U) := CE[c x]; (c_. 1_U) ⊠ x_ := CE[c x];
  (a_. U[xx___, x_]) ⊠ (b_. U[y_, yy___]) := If[OrderedQ[{x, y} /. sr],
    CE@M[a b /. $trim, U[xx, x, y, yy]],
    U@xx ⊠ CE@M[a b /. $trim, U@y ⊠ U@x + B[U@x, U@y, $E]] ⊠ U@yy ];
  U@{c_. * (l : gp)^n_, r___} /; FreeQ[c, gp] := CE[c U@Table[l, {n}] ⊠ U@{r}];
  U@{c_. * l : gp, r___} := CE[c U[l] ⊠ U@{r}];
  U@{c_, r___} /; FreeQ[c, gp] := CE[c U@{r}];
  U@{l_Plus, r___} := CE[U@{#, r} & /@ l];
  U@{l_, r___} := U@{Expand[l], r};
  U[ℰ_NonCommutativeMultiply] := U /@ ℰ;
  𝕆_U[specs___, poly_] := Module[{sp, null, vs, us},
    sp = Replace[{specs}, l_List :> l_null, {1}];
    vs = Join @@ (First /@ sp);
    us = Join @@ (sp /. l_s_ :> (l /. x_i_ :> x_s));
    CE[Total[
        CoefficientRules[poly, vs] /. (p_ → c_) :> c U@(us^p)
      ]] /. x_null :> x];
  𝕆_U[specs___, 𝔼[L_, Q_, P_]] := 𝕆_U[specs, SS@Normal[P ⅇ^(L+Q)]];
  pow[ℰ_, n_] := pow[ℰ, n - 1] ⊠ ℰ;
  𝕊_U[ℰ_, ss___Rule] := CE@Total[
     CoefficientRules[ℰ, First /@ {ss}] /. (p_ → c_) :> c NCM @@ MapThread[pow, {Last /@ {ss}, p}]];
  σ_rs___[c_. * u_U] := (c /. (t : cp)_j_ :> t_(j /. {rs})) U[List @@ (u /. v_j_ :> v_(j /. {rs}))];
  m_j_→k_[c_. * u_U] :=
   CE[((c /. (t : cp)_j → t_k) DeleteCases[u, _j|k]) ⊠ U @@ Cases[u, w_j :> w_k] ⊠ U @@ Cases[u, _k]];
  U /: c_. * u_U * v_U := CE[c u ⊠ v];
  S_i_[c_. * u_U] :=
   CE[((c /. S_i[U, Centrals]) DeleteCases[u, _i]) ⊠ U_i[NCM @@ Reverse@Cases[u, x_i :> S@U@x]]];
  Δ_i_→j_,k_[c_. * u_U] :=
   CE[((c /. Δ_i→j,k[U, Centrals]) DeleteCases[u, _i]) ⊠
     (NCM @@ Cases[u, x_i :> σ_1→j,2→k@Δ@U@x] /. NCM[] → U[])]; ]
```

```
Invert_U_[ℰ_] := Module[{A, B, β}, (* ℰ=A U[]+∈ B *)
  A = Coefficient[ℰ, 1_U] /. ϵ → 0;
  B = CE[ϵ^-1 (ℰ - A 1_U)];
  𝕊_U[SS[(A + ϵ β)^-1], β → B]
 ]
```

Implementing QU = $\mathcal{U}_q(\mathrm{sl}^+_{3,\gamma\epsilon})$

Greek - Latin duality:

```
{X*, Y*, Z*, b*, a*, z*, y*, x*, Ξ*, ϒ*, Ω*, β*, α*, ξ*, η*, ξ*} =
  {Ξ, ϒ, Ω, β, α, ξ, η, ξ, X, Y, Z, b, a, z, y, x};
(u_αβ__)* := (u*)_αβ;
((u : ((y | x | η | ξ) | (y | x | η | ξ)__))[i_])* := u*[i];
(vs_List)* := (v ↦ v*) /@ vs;
```

```
ℏ = γ = 1;
DeclareAlgebra[QU, Generators → {X, Y, Z, b, a, z, y, x}, Centrals → {s, t}];
B[a_QU, x_QU] = -2 QU@x;
B[b_QU, x_QU] = QU@x;
B[b_QU, y_QU] = -2 QU@y;
B[a_QU, y_QU] = QU@y;
B[a_QU, z_QU] = -QU@z;
B[b_QU, z_QU] = -QU@z;
B[y_QU, x_QU] = -z_QU + ϵ ℏ QU[y, x];
B[z_QU, x_QU] = -ϵ ℏ QU[z, x];
B[b_QU, a_QU] = 0;
B[z_QU, y_QU] = ϵ ℏ QU[z, y];
B[a_QU, X_QU] = 2 X_QU;
B[b_QU, X_QU] = -X_QU;
B[a_QU, Y_QU] = -Y_QU;
B[b_QU, Y_QU] = 2 Y_QU;
B[a_QU, Z_QU] = Z_QU;
B[b_QU, Z_QU] = Z_QU;
B[x_QU, X_QU] = -2 ϵ ℏ QU@{X, x} + (1 - s) QU[]/ℏ + 2 s ϵ QU[a];
B[y_QU, Y_QU] = -2 ϵ ℏ QU@{Y, y} + (1 - t) QU[]/ℏ + 2 t ϵ QU[b];
B[z_QU, Z_QU] = -2 ϵ ℏ QU@{Z, z} + (1 - s t) QU[]/ℏ + 2 s t ϵ QU[a] + 2 s t ϵ QU[b];
B[y_QU, Z_QU] = X_QU - ϵ ℏ QU@{Z, y};
B[x_QU, Z_QU] =  -ϵ ℏ QU@{Z, x} - (s + s ϵ ℏ) QU[Y] + 2 s ϵ ℏ QU[Y, a];
B[z_QU, X_QU] = 2 ϵ y_QU - ϵ ℏ QU@{X, z};
B[z_QU, Y_QU] = -2 t ϵ QU[x] - ϵ ℏ QU[Y, z];
B[y_QU, X_QU] = ϵ ℏ QU[X, y];
B[x_QU, Y_QU] = ϵ ℏ QU[Y, x];
B[X_QU, Y_QU] = 2 ϵ Z_QU - ϵ ℏ QU[X, Y];
B[X_QU, Z_QU] = ϵ ℏ QU[X, Z];
B[Z_QU, Y_QU] = ϵ ℏ QU[Y, Z];
```

The Multiplication Tensor

```
adPower[x_, k_, Env_][c_ * y_QU] := CE[c adPower[x, k, Env][y]];
adPower[x_, k_, Env_][y_Plus] := adPower[x, k, Env] /@ y;
adPower[_, 0, Env_][y_] := y;
adPower[x_, k_, Env_][0] = 0;
adPower[x_, k_, Env_][y_QU] := adPower[x, k, Env][y] = CE@B[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

```
NilQ[z_QU] = NilQ[Z_QU] = NilQ[X_QU] = NilQ[Y_QU] = NilQ[x_QU] = NilQ[y_QU] = True; NilQ[a_QU] = False;
NilQ[b_QU] = False;
PotQ[c_. * x_QU] := ¬ NilQ[x]; (* Potent Q *)
NilQ[c_ * x_QU] := NilQ[x];
PotQ[x_Plus] := And @@ (PotQ /@ (List @@ x));
NilQ[x_Plus] := And @@ (NilQ /@ (List @@ x));
```

```
Ad[0] = Identity;
Ad[c_. a_QU][z_] := z /. w_QU :→ ⅇ^(c ϒ (Sum[B[g_QU, a_QU]/g_QU Count[w, g], {g, QU@Generators}])) w;
Ad[c_. b_QU][z_] := z /. w_QU :→ ⅇ^(c ϒ (Sum[B[g_QU, b_QU]/g_QU Count[w, g], {g, QU@Generators}])) w;
Ad[c_. QU[]][z_] := z;
```

```
Ad[x_][y_] /; NilQ[x] := Expand@Module[{k = 0, s = 0},
   While[adPower[x, k][y] =!= 0,
    s += adPower[x, k][y] / k!;
    ++k
   ];
   s
  ]
```

```
TT_λ_[] = 0; (* Translated Tangent *)
TT_λ_[xs___, x_] := CE[Ad[x][TT_λ[xs]] + ∂_λ x];
TT_λ_[xs_List] := TT_λ @@ xs
```

```
Seepage_U_[{}, c_] = c;
Seepage_U_[ξs_List, P_] := Module[{gs = U[Generators], lx},
  lx = gs⟦Length@ξs⟧;
  Collect[P, lx, (Ad[Last[ξs] lx_U][𝕆_U[gs, Seepage_U[Most@ξs, #]]] /. U → Times) &]
 ]
```

$L_\lambda = \mathbb{CE}(\lambda \xi_{i,1} x)\,\mathbb{CE}(\lambda \xi_{i,2} x)$

Seepage is defined by: $\underset{\text{off}}{\mathbb{CE}(f_i x_i)}\,\mathbb{O}(\text{Seepage}(\{f_i\}, A)) = \mathbb{O}(e^{f_i x_i} A)$

$TT(L_\lambda) = L^{-1} \partial_\lambda L_\lambda$

$S(B) = \text{Seepage}(\{f_i\}, B).$

$R_\lambda = \mathbb{CE}(f+g)$, $f = f_i x_i$ $g = \sum_{|w| \geq 1} g_w w$ $e^f = F$ $e^g = G$ $R_\lambda = \mathbb{O}(FG)$ $\partial_\lambda F = F \partial_\lambda f$

$TT(R_\lambda) = R_\lambda^{-1} \partial_\lambda R_\lambda) = \mathbb{O}(FG)^{-1} \mathbb{O}(\partial_\lambda FG) = S(G)^{-1} \mathbb{O}(F)^{-1} \mathbb{O}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)^{-1} S(\partial_\lambda f G + \partial_\lambda G)$

```
TTL = CE[TT_λ @@ (Flatten@Table[
      (If[PotQ[#_QU], 1, λ] #*[k] #_QU) & /@ QU@Generators,
      {k, {i, j}}
     ])]
```

```
QU[z, y] (e^(3 β[j]) ∈ λ ξ[i] × η[i] - e^(-α[j]+2 β[j]) ∈ λ ξ[j] × η[i] + e^(α[j]+β[j]) ∈ λ ξ[i] × η[j] + ∈ λ ξ[j] × η[j]) +
 QU[z, x] (… 1 …) + … 24 … + QU[Y] (… 1 …) +
 QU[] (-e^(-2 α[i]+β[i]) λ ξ[i] × Ξ[i] + e^(-2 α[i]+β[i]) s λ ξ[i] × Ξ[i] - e^(-2 α[i]-2 α[j]+β[i]+β[j]) λ ξ[j] × Ξ[i] + … 716 … +
   2 e^(-3 β[j]) s^2 ∈ λ^5 η[j]^2 ξ[i] × ξ[j] Ω[j]^2 + 2 e^(-3 β[j]) s t ∈ λ^5 η[j]^2 ξ[i] × ξ[j] Ω[j]^2 - 2 e^(-3 β[j]) s^2 t ∈ λ^5 η[j]^2 ξ[i] × ξ[j] Ω[j]^2)
```

Size in memory: 1 MB | + Show more | Show all | Iconize | Store full expression in notebook

```
mons[k_] := Union@Cases[Coefficient[TTL, ϵ, k], _QU, ∞];
mons /@ Range[0, $k]
```

```
{{QU[], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z]},
 {QU[], QU[a], QU[b], QU[x], QU[X], QU[y], QU[Y], QU[z], QU[Z], QU[X, x], QU[X, X],
  QU[X, y], QU[X, Y], QU[X, z], QU[X, Z], QU[y, x], QU[Y, a], QU[Y, x], QU[Y, y], QU[Y, Y],
  QU[Y, z], QU[Y, Z], QU[z, x], QU[z, y], QU[z, z], QU[Z, x], QU[Z, y], QU[Z, z]}}
```

```
ff = Table[
   Sum[ϵ^k f_{k,u}[λ], {k, 0, $k}],
   {u, QU@Generators}
  ] /. f_{0,u_}[λ] /; PotQ[u_QU] :→ u*[i] + u*[j]
```

```
{f_{0,x}[λ] + ∈ f_{1,x}[λ], f_{0,Y}[λ] + ∈ f_{1,Y}[λ], f_{0,Z}[λ] + ∈ f_{1,Z}[λ], β[i] + β[j] + ∈ f_{1,b}[λ],
 α[i] + α[j] + ∈ f_{1,a}[λ], f_{0,z}[λ] + ∈ f_{1,z}[λ], f_{0,y}[λ] + ∈ f_{1,y}[λ], f_{0,x}[λ] + ∈ f_{1,x}[λ]}
```

```
Short[fs = Flatten[Table[
    f_{k,Times@@m},
    {k, 0, $k}, {m, mons[k]}
   ]], 10]
```

```
{f_{0,1}, f_{0,x}, f_{0,X}, f_{0,y}, f_{0,Y}, f_{0,z}, f_{0,Z}, f_{1,1}, f_{1,a}, f_{1,b}, f_{1,x}, f_{1,X}, f_{1,y}, f_{1,Y}, f_{1,z}, f_{1,Z}, f_{1,x X}, f_{1,X^2}, f_{1,X y},
 f_{1,X Y}, f_{1,X z}, f_{1,X Z}, f_{1,x y}, f_{1,a Y}, f_{1,x Y}, f_{1,y Y}, f_{1,Y^2}, f_{1,Y z}, f_{1,Y Z}, f_{1,x z}, f_{1,y z}, f_{1,z^2}, f_{1,x Z}, f_{1,y Z}, f_{1,z Z}}
```

```
gg = Sum[
  ϵ^k f_{k,Times@@u}[λ] Times @@ u,
  {k, 0, $k}, {u, Complement[mons[k], QU /@ QU@Generators]}
 ]
```

```
f_{0,1}[λ] + ∈ f_{1,1}[λ] + x X ∈ f_{1,x X}[λ] + X^2 ∈ f_{1,X^2}[λ] + x y ∈ f_{1,x y}[λ] + X y ∈ f_{1,X y}[λ] + a Y ∈ f_{1,a Y}[λ] + x Y ∈ f_{1,x Y}[λ] +
 X Y ∈ f_{1,X Y}[λ] + y Y ∈ f_{1,y Y}[λ] + Y^2 ∈ f_{1,Y^2}[λ] + x z ∈ f_{1,x z}[λ] + X z ∈ f_{1,X z}[λ] + y z ∈ f_{1,y z}[λ] +
 Y z ∈ f_{1,Y z}[λ] + z^2 ∈ f_{1,z^2}[λ] + x Z ∈ f_{1,x Z}[λ] + X Z ∈ f_{1,X Z}[λ] + y Z ∈ f_{1,y Z}[λ] + Y Z ∈ f_{1,Y Z}[λ] + z Z ∈ f_{1,z Z}[λ]
```

```
G = SS[e^gg]
```

```
e^(f_{0,1}[λ]) + e^(f_{0,1}[λ]) ∈ (f_{1,1}[λ] + x X f_{1,x X}[λ] + X^2 f_{1,X^2}[λ] + x y f_{1,x y}[λ] + X y f_{1,X y}[λ] + a Y f_{1,a Y}[λ] +
   x Y f_{1,x Y}[λ] + X Y f_{1,X Y}[λ] + y Y f_{1,y Y}[λ] + Y^2 f_{1,Y^2}[λ] + x z f_{1,x z}[λ] + X z f_{1,X z}[λ] + y z f_{1,y z}[λ] +
   Y z f_{1,Y z}[λ] + z^2 f_{1,z^2}[λ] + x Z f_{1,x Z}[λ] + X Z f_{1,X Z}[λ] + y Z f_{1,y Z}[λ] + Y Z f_{1,Y Z}[λ] + z Z f_{1,z Z}[λ])
```

```
𝒮[B_] := CE[SS@𝕆_QU[QU@Generators, Seepage_QU[ff, SS@CE@B]]];
Short[𝒮G = 𝒮[G], 10]
```

```
e^(f_{0,1}[λ]) ∈ QU[y, x] f_{1,x y}[λ] + <<25>> +
 QU[z] (e^(-2 α[i]-2 α[j]+β[i]+β[j]+f_{0,1}[λ]) ∈ f_{0,x}[λ]^2 f_{1,X y}[λ] - e^(-2 α[i]-2 α[j]+β[i]+β[j]+f_{0,1}[λ]) s ∈ f_{0,x}[λ]^2 f_{1,X y}[λ] +
   e^(α[i]+α[j]-2 β[i]-2 β[j]+f_{0,1}[λ]) ∈ f_{0,x}[λ] f_{0,y}[λ]^2 f_{1,a Y}[λ] -
   e^(α[i]+α[j]-2 β[i]-2 β[j]+f_{0,1}[λ]) t ∈ f_{0,x}[λ] f_{0,y}[λ]^2 f_{1,a Y}[λ] + <<18>> +
   e^(-α[i]-α[j]-β[i]-β[j]+f_{0,1}[λ]) ∈ f_{0,x}[λ] f_{0,y}[λ] f_{1,z z}[λ] - e^(-α[i]-α[j]-β[i]-β[j]+f_{0,1}[λ]) s ∈ f_{0,x}[λ] f_{0,y}[λ] f_{1,z z}[λ] -
   e^(-α[i]-α[j]-β[i]-β[j]+f_{0,1}[λ]) ∈ f_{0,z}[λ] f_{1,z z}[λ] + e^(-α[i]-α[j]-β[i]-β[j]+f_{0,1}[λ]) s t ∈ f_{0,z}[λ] f_{1,z z}[λ])
```

```
TTR = CE[Invert_QU[𝒮G] ⊠ 𝒮[(∂_λ (ff.QU@Generators) G + ∂_λ G)]]
```

$$\mathrm{QU}[b]\left(-2\,e^{\alpha[i]+\alpha[j]-2\beta[i]-2\beta[j]}\,t\,\epsilon\, f_{0,y}[\lambda]\, f_{0,Y}'[\lambda]-2\,e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]}\,s\,t\,\epsilon\, f_{0,z}[\lambda]\, f_{0,Z}'[\lambda]+\epsilon\, f_{1,b}'[\lambda]\right)+$$
$$\mathrm{QU}[y,x]\left(\epsilon\, f_{0,x}[\lambda]\, f_{0\ \ll 1 \gg\ \ll 1 \gg}'[\lambda]+\epsilon \ll 1 \gg'[\lambda]\right)+\ll 24 \gg+$$
$$\mathrm{QU}[X,z]\left(\ll 1 \gg\right)+\mathrm{QU}[z]\left(-e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]}\,\epsilon\, f_{0,x}[\lambda]^2\, f_{0,y}[\lambda]\, f_{0,X}'[\lambda]+\right.$$
$$\left.e^{-2\alpha[i]-2\alpha[j]+\beta[i]+\beta[j]}\,s\,\epsilon\, f_{0,x}[\lambda]^2\, f_{0,y}[\lambda]\, f_{0,X}'[\lambda]+\ll 95 \gg+e^{-\alpha[i]-\alpha[j]-\beta[i]-\beta[j]}\,s\,t\,\epsilon\, f_{0,z}[\lambda]\, f_{1,zZ}'[\lambda]\right)$$

Full expression not available (original memory size: 0.6 MB)

```
Short[diff = CE[TTL - TTR], 10];
Short[eqns = Flatten[Table[
     Coefficient[Coefficient[diff, m], ϵ, k] == 0,
     {k, 0, $k}, {m, mons[k]}
    ]], 10];
Short[inits = (#[0] == 0) & /@ fs, 10];
DSolve[eqns ⋃ inits, fs, λ]
```

$$\{\{f_{0,x} \to \text{Function}[\{\lambda\}, e^{-\beta[j]} \lambda (e^{2\alpha[j]} \xi[i] + e^{\beta[j]} \xi[j])],$$

$$f_{0,y} \to \text{Function}[\{\lambda\}, e^{-\alpha[j]} \lambda (e^{2\beta[j]} \eta[i] + e^{\alpha[j]} \eta[j])],$$

$$f_{0,X} \to \text{Function}[\{\lambda\}, e^{-\beta[i]} \lambda (e^{\beta[i]} \Xi[i] + e^{2\alpha[i]} \Xi[j] + e^{2\alpha[i]} \lambda \eta[i] \times \Omega[j])],$$

$$f_{0,Y} \to \text{Function}[\{\lambda\}, e^{-\alpha[i]} \lambda (e^{\alpha[i]} \Upsilon[i] + e^{2\beta[i]} \Upsilon[j] - e^{2\beta[i]} s \lambda \xi[i] \times \Omega[j])],$$

$$f_{0,z} \to \text{Function}[\{\lambda\}, e^{-\beta[j]} \lambda (e^{\alpha[j]+2\beta[j]} \zeta[i] + e^{\beta[j]} \zeta[j] + e^{2\alpha[j]} \lambda \eta[j] \times \xi[i])],$$

$$f_{0,1} \to \text{Function}[\{\lambda\}, \lambda^2 \xi[i] \times \Xi[j] - s \lambda^2 \xi[i] \times \Xi[j] + \lambda^2 \eta[i] \times \Upsilon[j] - t \lambda^2 \eta[i] \times \Upsilon[j] +$$
$$\lambda^2 \zeta[i] \times \Omega[j] - s t \lambda^2 \zeta[i] \times \Omega[j] - s \lambda^3 \eta[i] \times \xi[i] \times \Omega[j] + s t \lambda^3 \eta[i] \times \xi[i] \times \Omega[j]],$$

$$f_{0,Z} \to \text{Function}[\{\lambda\}, \lambda (\Omega[i] + e^{\alpha[i]+\beta[i]} \Omega[j])],$$

$$f_{1,a} \to \text{Function}[\{\lambda\}, -2 s \lambda^2 (-\xi[i] \times \Xi[j] - t \zeta[i] \times \Omega[j] - \lambda \eta[i] \times \xi[i] \times \Omega[j] + t \lambda \eta[i] \times \xi[i] \times \Omega[j])],$$

$$f_{1,b} \to \text{Function}[\{\lambda\}, -2 t \lambda^2 (-\eta[i] \times \Upsilon[j] - s \zeta[i] \times \Omega[j] + s \lambda \eta[i] \times \xi[i] \times \Omega[j])],$$

$$f_{1,x y} \to \text{Function}[\{\lambda\}, -e^{2\alpha[j]-\beta[j]} \lambda^2 \eta[j] \times \xi[i]],$$

$$f_{1,x X} \to \text{Function}[\{\lambda\}, -e^{2\alpha[i]+2\alpha[j]-\beta[i]-\beta[j]} \lambda^2 \xi[i] (2 \Xi[j] + \lambda \eta[i] \times \Omega[j])],$$

$$f_{1,X y} \to \text{Function}[\{\lambda\}, e^{2\alpha[i]-\alpha[j]-\beta[i]+2\beta[j]} \lambda^2 \eta[i] \times \Xi[j]],$$

$$f_{1,X^2} \to \text{Function}[\{\lambda\}, e^{4\alpha[i]-2\beta[i]} \lambda^3 \eta[i] \times \Xi[j] \times \Omega[j]],$$

$$f_{1,a Y} \to \text{Function}[\{\lambda\}, 2 e^{-\alpha[i]+2\beta[i]} s \lambda^2 \xi[i] \times \Omega[j]],$$

$$f_{1,x Y} \to \text{Function}[\{\lambda\}, e^{-\alpha[i]+2\alpha[j]+2\beta[i]-\beta[j]} \lambda^2 \xi[i] (\Upsilon[j] + 2 s \lambda \xi[i] \times \Omega[j])],$$

$$f_{1,y Y} \to \text{Function}[\{\lambda\}, -2 e^{-\alpha[i]-\alpha[j]+2\beta[i]+2\beta[j]} \lambda^2 \eta[i] \times \Upsilon[j]],$$

$$f_{1,X Y} \to \text{Function}[\{\lambda\}, e^{\alpha[i]-\beta[i]} \lambda^2 (e^{\alpha[i]} \Xi[j] \times \Upsilon[i] + 2 e^{2\beta[i]} s \lambda \xi[i] \times \Xi[j] \times \Omega[j] +$$
$$e^{\alpha[i]} \lambda \eta[i] \times \Upsilon[i] \times \Omega[j] - e^{2\beta[i]} \lambda \eta[i] \times \Upsilon[j] \times \Omega[j] + 2 e^{2\beta[i]} s \lambda^2 \eta[i] \times \xi[i] \Omega[j]^2)],$$

$$f_{1,Y^2} \to \text{Function}[\{\lambda\}, -e^{-2\alpha[i]+4\beta[i]} s \lambda^3 \xi[i] \times \Upsilon[j] \times \Omega[j]],$$

$$f_{1,x z} \to \text{Function}[\{\lambda\}, e^{2\alpha[j]-\beta[j]} \lambda^2 \zeta[j] \times \xi[i]], f_{1,x} \to \text{Function}[\{\lambda\}, -e^{2\alpha[j]-\beta[j]} \lambda^2$$
$$(\lambda \xi[i]^2 \Xi[j] - 3 s \lambda \xi[i]^2 \Xi[j] + 2 t \zeta[i] \times \Upsilon[j] - \lambda \eta[i] \times \xi[i] \times \Upsilon[j] + t \lambda \eta[i] \times \xi[i] \times \Upsilon[j] + \lambda \zeta[i] \times$$
$$\xi[i] \times \Omega[j] - 3 s t \lambda \zeta[i] \times \xi[i] \times \Omega[j] - 2 s \lambda^2 \eta[i] \xi[i]^2 \Omega[j] + 2 s t \lambda^2 \eta[i] \xi[i]^2 \Omega[j])], f_{1,y z} \to$$
$$\text{Function}[\{\lambda\}, -e^{-\alpha[j]-\beta[j]} \lambda^2 (e^{3\beta[j]} \zeta[j] \times \eta[i] + e^{2\alpha[j]+2\beta[j]} \lambda \eta[i] \times \eta[j] \times \xi[i] + e^{3\alpha[j]} \lambda \eta[j]^2 \xi[i])],$$

$$f_{1,y} \to \text{Function}[\{\lambda\}, -e^{-\alpha[j]+2\beta[j]} \lambda^2 (-2 \zeta[i] \times \Xi[j] + 2 s \lambda \eta[i] \times \xi[i] \times \Xi[j] + \lambda \eta[i]^2 \Upsilon[j] - 3 t \lambda \eta[i]^2 \Upsilon[$$
$$j] - \lambda \zeta[i] \times \eta[i] \times \Omega[j] - s t \lambda \zeta[i] \times \eta[i] \times \Omega[j] + s \lambda^2 \eta[i]^2 \xi[i] \times \Omega[j] + s t \lambda^2 \eta[i]^2 \xi[i] \times \Omega[j])],$$

$$f_{1,X z} \to \text{Function}[\{\lambda\}, -e^{2\alpha[i]+\alpha[j]-\beta[i]-\beta[j]} \lambda^2 (e^{2\beta[j]} \zeta[i] \times \Xi[j] + 2 e^{\alpha[j]} \lambda \eta[j] \times \xi[i] \times \Xi[j] +$$
$$e^{2\beta[j]} \lambda \zeta[i] \times \eta[i] \times \Omega[j] + e^{\alpha[j]} \lambda^2 \eta[i] \times \eta[j] \times \xi[i] \times \Omega[j])],$$

$$f_{1,X} \to \text{Function}[\{\lambda\}, -e^{2\alpha[i]-\beta[i]} \lambda^3 (\xi[i] \Xi[j]^2 - 3 s \xi[i] \Xi[j]^2 - \eta[i] \times \Xi[j] \times \Upsilon[j] +$$
$$t \eta[i] \times \Xi[j] \times \Upsilon[j] - \zeta[i] \times \Xi[j] \times \Omega[j] - s t \zeta[i] \times \Xi[j] \times \Omega[j] - 3 s \lambda \eta[i] \times \xi[i] \times \Xi[j] \times \Omega[j] +$$
$$s t \lambda \eta[i] \times \xi[i] \times \Xi[j] \times \Omega[j] + \lambda \eta[i]^2 \Upsilon[j] \times \Omega[j] - t \lambda \eta[i]^2 \Upsilon[j] \times \Omega[j] -$$
$$2 s t \lambda \zeta[i] \times \eta[i] \Omega[j]^2 - 2 s \lambda^2 \eta[i]^2 \xi[i] \Omega[j]^2 + 2 s t \lambda^2 \eta[i]^2 \xi[i] \Omega[j]^2)],$$

$$f_{1,Y z} \to \text{Function}[\{\lambda\}, e^{-\alpha[i]+\alpha[j]+2\beta[i]-\beta[j]} \lambda^2 (-e^{2\beta[j]} \zeta[i] \times \Upsilon[j] + e^{\alpha[j]} \lambda \eta[j] \times \xi[i] \times \Upsilon[j] +$$
$$3 e^{2\beta[j]} s \lambda \zeta[i] \times \xi[i] \times \Omega[j] + 2 e^{\alpha[j]} s \lambda^2 \eta[j] \xi[i]^2 \Omega[j])],$$

$$f_{1,Y} \to \text{Function}[\{\lambda\}, e^{-\alpha[i]+2\beta[i]} \lambda^2 (-2 s \lambda \xi[i] \times \Xi[j] \times \Upsilon[j] - \lambda \eta[i] \Upsilon[j]^2 + 3 t \lambda \eta[i] \Upsilon[j]^2 -$$
$$s \xi[i] \times \Omega[j] + s \lambda^2 \xi[i]^2 \Xi[j] \times \Omega[j] - s^2 \lambda^2 \xi[i]^2 \Xi[j] \times \Omega[j] - \lambda \zeta[i] \times \Upsilon[j] \times \Omega[j] +$$
$$3 s t \lambda \zeta[i] \times \Upsilon[j] \times \Omega[j] - 4 s t \lambda^2 \eta[i] \times \xi[i] \times \Upsilon[j] \times \Omega[j] + 2 s \lambda^2 \zeta[i] \times \xi[i] \Omega[j]^2 -$$
$$4 s^2 t \lambda^2 \zeta[i] \times \xi[i] \Omega[j]^2 - s^2 \lambda^3 \eta[i] \xi[i]^2 \Omega[j]^2 + 3 s^2 t \lambda^3 \eta[i] \xi[i]^2 \Omega[j]^2)],$$

$$f_{1,z^2} \to \text{Function}[\{\lambda\}, e^{2\alpha[j]-\beta[j]} \lambda^3 \zeta[j] \times \eta[j] \times \xi[i]], f_{1,z} \to \text{Function}[\{\lambda\},$$
$$-e^{\alpha[j]-\beta[j]} (-2 e^{2\beta[j]} s \lambda^3 \zeta[i] \times \xi[i] \times \Xi[j] + e^{\alpha[j]} \lambda^4 \eta[j] \xi[i]^2 \Xi[j] - 3 e^{\alpha[j]} s \lambda^4 \eta[j] \xi[i]^2 \Xi[j] +$$
$$2 e^{\alpha[j]} t \lambda^3 \zeta[i] \times \eta[j] \times \Upsilon[j] - e^{\alpha[j]} \lambda^4 \eta[i] \times \eta[j] \times \xi[i] \times \Upsilon[j] + e^{\alpha[j]} t \lambda^4 \eta[i] \times \eta[j] \times \xi[i] \times \Upsilon[j] +$$
$$e^{2\beta[j]} \lambda^3 \zeta[i]^2 \Omega[j] - 3 e^{2\beta[j]} s t \lambda^3 \zeta[i]^2 \Omega[j] - 2 e^{2\beta[j]} s \lambda^4 \zeta[i] \times \eta[i] \times \xi[i] \times \Omega[j] +$$
$$2 e^{2\beta[j]} s t \lambda^4 \zeta[i] \times \eta[i] \times \xi[i] \times \Omega[j] + e^{\alpha[j]} \lambda^4 \zeta[i] \times \eta[j] \times \xi[i] \times \Omega[j] - 3 e^{\alpha[j]} s t \lambda^4 \zeta[i] \times$$
$$\eta[j] \times \xi[i] \times \Omega[j] - 2 e^{\alpha[j]} s \lambda^5 \eta[i] \times \eta[j] \xi[i]^2 \Omega[j] + 2 e^{\alpha[j]} s t \lambda^5 \eta[i] \times \eta[j] \xi[i]^2 \Omega[j])],$$

$$f_{1,1} \to \text{Function}[\{\lambda\}, \frac{1}{2} (-\lambda^4 \xi[i]^2 \Xi[j]^2 + 4 s \lambda^4 \xi[i]^2 \Xi[j]^2 - 3 s^2 \lambda^4 \xi[i]^2 \Xi[j]^2 +$$
$$4 \lambda^3 \zeta[i] \times \Xi[j] \times \Upsilon[j] - 4 t \lambda^3 \zeta[i] \times \Xi[j] \times \Upsilon[j] - 4 s \lambda^4 \eta[i] \times \xi[i] \times \Xi[j] \times \Upsilon[j] +$$
$$4 s t \lambda^4 \eta[i] \times \xi[i] \times \Xi[j] \times \Upsilon[j] - \lambda^4 \eta[i]^2 \Upsilon[j]^2 + 4 t \lambda^4 \eta[i]^2 \Upsilon[j]^2 - 3 t^2 \lambda^4 \eta[i]^2 \Upsilon[j]^2 -$$