

Pensieve header: The PBW multiplication tensor for upper nilpotent matrices using the λ -tangent formalism. Some material from pensieve://Projects/UEA.

```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

Prolog

```
In[ ]:= BeginPackage["UEA`"];
Print["UEA` does computations in general universal enveloping
algebras and PBW algebras. It is in the public domain, available
at http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/.
Dror Bar-Natan is committed to support it within
reason until June 1, 2027. This is version 260601."];
Print["UEA` implements / extends ",
Sort@{"**", B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule,  $\delta$ },
"."];
Begin["`Private`"];
```

UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the public domain, available at <http://drorbn.net/AcademicPensieve/Projects/SolvablePBW/>. Dror Bar-Natan is committed to support it within reason until June 1, 2027. This is version 260601.
UEA` implements / extends { $**$, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, δ , \$Basis, \$PBWRule}.

Utilities

```
In[ ]:=  $\delta_{i,j}$  := If[i == j, 1, 0];
```

Implementing general universal enveloping algebras

```
In[ ]:= B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_p] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs__] => U_i[xs],
  x_ /; MemberQ[$Basis, x] => U_i[x]
};
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
In[*]:= m_s__[0] = 0;
m_s__[x_Plus] := m_s_/@x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i->j_[ε_] := ε /. U_i->U_j;
```

```
In[*]:= m_i,j->k_[c_. U_i_[x___] U_j_[_]] := c U_k[x];
m_i,j->k_[c_. U_i[_] U_j_[y___]] := c U_k[y];
m_i,j->k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

```
In[*]:= UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@
  Flatten@Table[(U_i@@@Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is, d-j}]}];
```

```
In[*]:= Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = m_i->σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = m_σ@i,i->i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ⋈ y - y ⋈ x];
```

Epilog

```
In[*]:= End[]; EndPackage[];
```

Predefined Algebras

sl(2)

```
In[*]:= Print["UEA`SetAlgebra knows \"sl2\"."];
```

UEA`SetAlgebra knows "sl2".

```
In[*]:= SetAlgebra["sl2"] := (
  Print["In sl2: <e,h,f>/([h,e]=2e, [h,f]=-2f, [e,f]=h)."];
  B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h;
  $Basis = {e, h, f};
  $PBWRule = {e → 1, h → 2, f → 3};
);
```

un_n

```
In[*]:= Print["UEA`SetAlgebra knows the nilpotent algebra un_n."];
```

UEA`SetAlgebra knows the nilpotent algebra un_n.

```
In[*]:= SetAlgebra[un_n_] := (
  $Basis = Flatten@Table[xα,β, {β, 2, n}, {α, 1, β - 1}];
  $PBWRule = Thread[$Basis → Range@Length@$Basis];
  B[xi,j, xk,l] := δj,k xi,l - δl,i xk,j;
);
```

```
In[*]:= SetAlgebra[un5];
$PBWRule
MatrixForm@Table[{b1, b2} → B[b1, b2], {b1, $Basis}, {b2, $Basis}]
```

```
Out[*]= {x1,2 → 1, x1,3 → 2, x2,3 → 3, x1,4 → 4, x2,4 → 5, x3,4 → 6, x1,5 → 7, x2,5 → 8, x3,5 → 9, x4,5 → 10}
```

```
Out[*]//MatrixForm=
```

$\{x_{1,2}, x_{1,2}\} \rightarrow 0$	$\{x_{1,2}, x_{1,3}\} \rightarrow 0$	$\{x_{1,2}, x_{2,3}\} \rightarrow x_{1,3}$	$\{x_{1,2}, x_{1,4}\} \rightarrow 0$	$\{x_{1,2}, x_{2,4}\} \rightarrow x_{1,4}$
$\{x_{1,3}, x_{1,2}\} \rightarrow 0$	$\{x_{1,3}, x_{1,3}\} \rightarrow 0$	$\{x_{1,3}, x_{2,3}\} \rightarrow 0$	$\{x_{1,3}, x_{1,4}\} \rightarrow 0$	$\{x_{1,3}, x_{2,4}\} \rightarrow 0$
$\{x_{2,3}, x_{1,2}\} \rightarrow -x_{1,3}$	$\{x_{2,3}, x_{1,3}\} \rightarrow 0$	$\{x_{2,3}, x_{2,3}\} \rightarrow 0$	$\{x_{2,3}, x_{1,4}\} \rightarrow 0$	$\{x_{2,3}, x_{2,4}\} \rightarrow 0$
$\{x_{1,4}, x_{1,2}\} \rightarrow 0$	$\{x_{1,4}, x_{1,3}\} \rightarrow 0$	$\{x_{1,4}, x_{2,3}\} \rightarrow 0$	$\{x_{1,4}, x_{1,4}\} \rightarrow 0$	$\{x_{1,4}, x_{2,4}\} \rightarrow 0$
$\{x_{2,4}, x_{1,2}\} \rightarrow -x_{1,4}$	$\{x_{2,4}, x_{1,3}\} \rightarrow 0$	$\{x_{2,4}, x_{2,3}\} \rightarrow 0$	$\{x_{2,4}, x_{1,4}\} \rightarrow 0$	$\{x_{2,4}, x_{2,4}\} \rightarrow 0$
$\{x_{3,4}, x_{1,2}\} \rightarrow 0$	$\{x_{3,4}, x_{1,3}\} \rightarrow -x_{1,4}$	$\{x_{3,4}, x_{2,3}\} \rightarrow -x_{2,4}$	$\{x_{3,4}, x_{1,4}\} \rightarrow 0$	$\{x_{3,4}, x_{2,4}\} \rightarrow 0$
$\{x_{1,5}, x_{1,2}\} \rightarrow 0$	$\{x_{1,5}, x_{1,3}\} \rightarrow 0$	$\{x_{1,5}, x_{2,3}\} \rightarrow 0$	$\{x_{1,5}, x_{1,4}\} \rightarrow 0$	$\{x_{1,5}, x_{2,4}\} \rightarrow 0$
$\{x_{2,5}, x_{1,2}\} \rightarrow -x_{1,5}$	$\{x_{2,5}, x_{1,3}\} \rightarrow 0$	$\{x_{2,5}, x_{2,3}\} \rightarrow 0$	$\{x_{2,5}, x_{1,4}\} \rightarrow 0$	$\{x_{2,5}, x_{2,4}\} \rightarrow 0$
$\{x_{3,5}, x_{1,2}\} \rightarrow 0$	$\{x_{3,5}, x_{1,3}\} \rightarrow -x_{1,5}$	$\{x_{3,5}, x_{2,3}\} \rightarrow -x_{2,5}$	$\{x_{3,5}, x_{1,4}\} \rightarrow 0$	$\{x_{3,5}, x_{2,4}\} \rightarrow 0$
$\{x_{4,5}, x_{1,2}\} \rightarrow 0$	$\{x_{4,5}, x_{1,3}\} \rightarrow 0$	$\{x_{4,5}, x_{2,3}\} \rightarrow 0$	$\{x_{4,5}, x_{1,4}\} \rightarrow -x_{1,5}$	$\{x_{4,5}, x_{2,4}\} \rightarrow -x_{2,5}$

```

In[ ]:= Clear[adPower];
adPower[x_, k_][0] = 0;
adPower[x_, k_][c_*y_] /; MemberQ[$Basis, y] := Expand[c adPower[x, k][y]];
adPower[x_, k_][y_Plus] := adPower[x, k] /@ y;
adPower[_ , 0][y_] := y;
adPower[x_, k_][y_] /; MemberQ[$Basis, y] :=
  adPower[x, k][y] = B[adPower[x, k - 1][y], x];

```

```

In[ ]:= x0 = ($Basis /. x -> ξ).$Basis;
y0 = ($Basis /. x -> η).$Basis

```

```

Out[ ]:=
x1,2 η1,2 + x1,3 η1,3 + x1,4 η1,4 + x1,5 η1,5 + x2,3 η2,3 + x2,4 η2,4 + x2,5 η2,5 + x3,4 η3,4 + x3,5 η3,5 + x4,5 η4,5

```

```

In[ ]:= adPower[x0, 0][x1,2]

```

```

Out[ ]:=
x1,2

```

```

In[ ]:= B[x1,2, x0]

```

```

Out[ ]:=
x1,3 ξ2,3 + x1,4 ξ2,4 + x1,5 ξ2,5

```

```

In[ ]:= adPower[x0, 1][x1,2]

```

```

Out[ ]:=
x1,3 ξ2,3 + x1,4 ξ2,4 + x1,5 ξ2,5

```

```

In[ ]:= adPower[x0, 2][x1,2]

```

```

Out[ ]:=
x1,4 ξ2,3 ξ3,4 + x1,5 ξ2,3 ξ3,5 + x1,5 ξ2,4 ξ4,5

```

```

In[ ]:= adPower[x0, 3][x1,2]

```

```

Out[ ]:=
x1,5 ξ2,3 ξ3,4 ξ4,5

```

```

In[ ]:= adPower[x0, 2][y0]

```

```

Out[ ]:=
x1,4 η3,4 ξ1,2 ξ2,3 + x1,5 η3,5 ξ1,2 ξ2,3 + x1,5 η4,5 ξ1,2 ξ2,4 - 2 x1,4 η2,3 ξ1,2 ξ3,4 + x1,5 η4,5 ξ1,3 ξ3,4 +
x1,4 η1,2 ξ2,3 ξ3,4 + x2,5 η4,5 ξ2,3 ξ3,4 - 2 x1,5 η2,3 ξ1,2 ξ3,5 + x1,5 η1,2 ξ2,3 ξ3,5 - 2 x1,5 η2,4 ξ1,2 ξ4,5 -
2 x1,5 η3,4 ξ1,3 ξ4,5 - 2 x2,5 η3,4 ξ2,3 ξ4,5 + x1,5 η1,2 ξ2,4 ξ4,5 + x1,5 η1,3 ξ3,4 ξ4,5 + x2,5 η2,3 ξ3,4 ξ4,5

```

```

In[ ]:= adPower[x0, 3][y0]

```

```

Out[ ]:=
-x1,5 η4,5 ξ1,2 ξ2,3 ξ3,4 + 3 x1,5 η3,4 ξ1,2 ξ2,3 ξ4,5 - 3 x1,5 η2,3 ξ1,2 ξ3,4 ξ4,5 + x1,5 η1,2 ξ2,3 ξ3,4 ξ4,5

```

```

In[*]:= adExp[x_][y_] := Expand@Module[{k = 0, s = 0},
  While[adPower[x, k][y] != 0,
    s += adPower[x, k][y] / k!;
    ++k
  ];
  s
]

```

```

In[*]:= adExp[x0][y0]

```

```

Out[*]=

```

$$\begin{aligned}
& x_{1,2} \eta_{1,2} + x_{1,3} \eta_{1,3} + x_{1,4} \eta_{1,4} + x_{1,5} \eta_{1,5} + x_{2,3} \eta_{2,3} + x_{2,4} \eta_{2,4} + x_{2,5} \eta_{2,5} + x_{3,4} \eta_{3,4} + x_{3,5} \eta_{3,5} + x_{4,5} \eta_{4,5} - \\
& x_{1,3} \eta_{2,3} \xi_{1,2} - x_{1,4} \eta_{2,4} \xi_{1,2} - x_{1,5} \eta_{2,5} \xi_{1,2} - x_{1,4} \eta_{3,4} \xi_{1,3} - x_{1,5} \eta_{3,5} \xi_{1,3} - x_{1,5} \eta_{4,5} \xi_{1,4} + \\
& x_{1,3} \eta_{1,2} \xi_{2,3} - x_{2,4} \eta_{3,4} \xi_{2,3} - x_{2,5} \eta_{3,5} \xi_{2,3} + \frac{1}{2} x_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \frac{1}{2} x_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \\
& x_{1,4} \eta_{1,2} \xi_{2,4} - x_{2,5} \eta_{4,5} \xi_{2,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + x_{1,5} \eta_{1,2} \xi_{2,5} + x_{1,4} \eta_{1,3} \xi_{3,4} + \\
& x_{2,4} \eta_{2,3} \xi_{3,4} - x_{3,5} \eta_{4,5} \xi_{3,4} - x_{1,4} \eta_{2,3} \xi_{1,2} \xi_{3,4} + \frac{1}{2} x_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \frac{1}{2} x_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} + \\
& \frac{1}{2} x_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \frac{1}{6} x_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + x_{1,5} \eta_{1,3} \xi_{3,5} + x_{2,5} \eta_{2,3} \xi_{3,5} - x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,5} + \\
& \frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + x_{1,5} \eta_{1,4} \xi_{4,5} + x_{2,5} \eta_{2,4} \xi_{4,5} + x_{3,5} \eta_{3,4} \xi_{4,5} - x_{1,5} \eta_{2,4} \xi_{1,2} \xi_{4,5} - \\
& x_{1,5} \eta_{3,4} \xi_{1,3} \xi_{4,5} - x_{2,5} \eta_{3,4} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} x_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \\
& \frac{1}{2} x_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} + \frac{1}{2} x_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} x_{1,5} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} x_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5}
\end{aligned}$$

```

In[*]:= Coefficient[adExp[x0][y0], x1,5]

```

```

Out[*]=

```

$$\begin{aligned}
& \eta_{1,5} - \eta_{2,5} \xi_{1,2} - \eta_{3,5} \xi_{1,3} - \eta_{4,5} \xi_{1,4} + \frac{1}{2} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \frac{1}{2} \eta_{4,5} \xi_{1,2} \xi_{2,4} + \eta_{1,2} \xi_{2,5} + \frac{1}{2} \eta_{4,5} \xi_{1,3} \xi_{3,4} - \\
& \frac{1}{6} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \eta_{1,3} \xi_{3,5} - \eta_{2,3} \xi_{1,2} \xi_{3,5} + \frac{1}{2} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \eta_{1,4} \xi_{4,5} - \eta_{2,4} \xi_{1,2} \xi_{4,5} - \eta_{3,4} \xi_{1,3} \xi_{4,5} + \\
& \frac{1}{2} \eta_{3,4} \xi_{1,2} \xi_{2,3} \xi_{4,5} + \frac{1}{2} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \frac{1}{2} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \frac{1}{2} \eta_{2,3} \xi_{1,2} \xi_{3,4} \xi_{4,5} + \frac{1}{6} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5}
\end{aligned}$$

```

In[*]:= λTangent[] = 0;
λTangent[xs___, x_] := adExp[x][λTangent[xs]] + ∂λ x;
λTangent[xs_List] := λTangent@@xs

```

```

In[*]:= $Basis ($Basis /. x -> ξ)

```

```

Out[*]=

```

$$\{x_{1,2} \xi_{1,2}, x_{1,3} \xi_{1,3}, x_{2,3} \xi_{2,3}, x_{1,4} \xi_{1,4}, x_{2,4} \xi_{2,4}, x_{3,4} \xi_{3,4}, x_{1,5} \xi_{1,5}, x_{2,5} \xi_{2,5}, x_{3,5} \xi_{3,5}, x_{4,5} \xi_{4,5}\}$$

In[*]:= $\lambda \text{Tangent} @ (\lambda \$\text{Basis} (\$ \text{Basis} /. \mathbf{x} \rightarrow \xi))$

Out[*]=

$$\begin{aligned} & \mathbf{x}_{1,2} \xi_{1,2} + \mathbf{x}_{1,3} \xi_{1,3} + \mathbf{x}_{1,4} \xi_{1,4} + \mathbf{x}_{1,5} \xi_{1,5} + \mathbf{x}_{2,3} \xi_{2,3} + \lambda \mathbf{x}_{1,3} \xi_{1,2} \xi_{2,3} + \\ & \mathbf{x}_{2,4} \xi_{2,4} + \lambda \mathbf{x}_{1,4} \xi_{1,2} \xi_{2,4} + \mathbf{x}_{2,5} \xi_{2,5} + \lambda \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,5} + \mathbf{x}_{3,4} \xi_{3,4} + \lambda \mathbf{x}_{1,4} \xi_{1,3} \xi_{3,4} + \\ & \lambda \mathbf{x}_{2,4} \xi_{2,3} \xi_{3,4} + \lambda^2 \mathbf{x}_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \mathbf{x}_{3,5} \xi_{3,5} + \lambda \mathbf{x}_{1,5} \xi_{1,3} \xi_{3,5} + \lambda \mathbf{x}_{2,5} \xi_{2,3} \xi_{3,5} + \\ & \lambda^2 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \mathbf{x}_{4,5} \xi_{4,5} + \lambda \mathbf{x}_{1,5} \xi_{1,4} \xi_{4,5} + \lambda \mathbf{x}_{2,5} \xi_{2,4} \xi_{4,5} + \lambda^2 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda \mathbf{x}_{3,5} \xi_{3,4} \xi_{4,5} + \lambda^2 \mathbf{x}_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 \mathbf{x}_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= $\text{lhs} = \text{Join}[\lambda \$\text{Basis} (\$ \text{Basis} /. \mathbf{x} \rightarrow \xi), \lambda \$\text{Basis} (\$ \text{Basis} /. \mathbf{x} \rightarrow \eta)] // \lambda \text{Tangent}$

Out[*]=

$$\begin{aligned} & \mathbf{x}_{1,2} \eta_{1,2} + \mathbf{x}_{1,3} \eta_{1,3} + \mathbf{x}_{1,4} \eta_{1,4} + \mathbf{x}_{1,5} \eta_{1,5} + \mathbf{x}_{2,3} \eta_{2,3} + \lambda \mathbf{x}_{1,3} \eta_{1,2} \eta_{2,3} + \mathbf{x}_{2,4} \eta_{2,4} + \lambda \mathbf{x}_{1,4} \eta_{1,2} \eta_{2,4} + \\ & \mathbf{x}_{2,5} \eta_{2,5} + \lambda \mathbf{x}_{1,5} \eta_{1,2} \eta_{2,5} + \mathbf{x}_{3,4} \eta_{3,4} + \lambda \mathbf{x}_{1,4} \eta_{1,3} \eta_{3,4} + \lambda \mathbf{x}_{2,4} \eta_{2,3} \eta_{3,4} + \lambda^2 \mathbf{x}_{1,4} \eta_{1,2} \eta_{2,3} \eta_{3,4} + \\ & \mathbf{x}_{3,5} \eta_{3,5} + \lambda \mathbf{x}_{1,5} \eta_{1,3} \eta_{3,5} + \lambda \mathbf{x}_{2,5} \eta_{2,3} \eta_{3,5} + \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,5} + \mathbf{x}_{4,5} \eta_{4,5} + \lambda \mathbf{x}_{1,5} \eta_{1,4} \eta_{4,5} + \\ & \lambda \mathbf{x}_{2,5} \eta_{2,4} \eta_{4,5} + \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda \mathbf{x}_{3,5} \eta_{3,4} \eta_{4,5} + \lambda^2 \mathbf{x}_{1,5} \eta_{1,3} \eta_{3,4} \eta_{4,5} + \lambda^2 \mathbf{x}_{2,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \\ & \lambda^3 \mathbf{x}_{1,5} \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \mathbf{x}_{1,2} \xi_{1,2} + \lambda \mathbf{x}_{1,3} \eta_{2,3} \xi_{1,2} + \lambda \mathbf{x}_{1,4} \eta_{2,4} \xi_{1,2} + \lambda \mathbf{x}_{1,5} \eta_{2,5} \xi_{1,2} + \\ & \lambda^2 \mathbf{x}_{1,4} \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda^2 \mathbf{x}_{1,5} \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 \mathbf{x}_{1,5} \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 \mathbf{x}_{1,5} \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} + \\ & \mathbf{x}_{1,3} \xi_{1,3} + \lambda \mathbf{x}_{1,4} \eta_{3,4} \xi_{1,3} + \lambda \mathbf{x}_{1,5} \eta_{3,5} \xi_{1,3} + \lambda^2 \mathbf{x}_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,3} + \mathbf{x}_{1,4} \xi_{1,4} + \lambda \mathbf{x}_{1,5} \eta_{4,5} \xi_{1,4} + \\ & \mathbf{x}_{1,5} \xi_{1,5} + \mathbf{x}_{2,3} \xi_{2,3} - \lambda \mathbf{x}_{1,3} \eta_{1,2} \xi_{2,3} + \lambda \mathbf{x}_{2,4} \eta_{3,4} \xi_{2,3} - \lambda^2 \mathbf{x}_{1,4} \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda \mathbf{x}_{2,5} \eta_{3,5} \xi_{2,3} - \\ & \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \eta_{3,5} \xi_{2,3} + \lambda^2 \mathbf{x}_{2,5} \eta_{3,4} \eta_{4,5} \xi_{2,3} - \lambda^3 \mathbf{x}_{1,5} \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda \mathbf{x}_{1,3} \xi_{1,2} \xi_{2,3} + \\ & \lambda^2 \mathbf{x}_{1,4} \eta_{3,4} \xi_{1,2} \xi_{2,3} + \lambda^2 \mathbf{x}_{1,5} \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 \mathbf{x}_{1,5} \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} + \mathbf{x}_{2,4} \xi_{2,4} - \lambda \mathbf{x}_{1,4} \eta_{1,2} \xi_{2,4} + \\ & \lambda \mathbf{x}_{2,5} \eta_{4,5} \xi_{2,4} - \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda \mathbf{x}_{1,4} \xi_{1,2} \xi_{2,4} + \lambda^2 \mathbf{x}_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,4} + \mathbf{x}_{2,5} \xi_{2,5} - \\ & \lambda \mathbf{x}_{1,5} \eta_{1,2} \xi_{2,5} + \lambda \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,5} + \mathbf{x}_{3,4} \xi_{3,4} - \lambda \mathbf{x}_{1,4} \eta_{1,3} \xi_{3,4} - \lambda \mathbf{x}_{2,4} \eta_{2,3} \xi_{3,4} + \lambda \mathbf{x}_{3,5} \eta_{4,5} \xi_{3,4} - \\ & \lambda^2 \mathbf{x}_{1,5} \eta_{1,3} \eta_{4,5} \xi_{3,4} - \lambda^2 \mathbf{x}_{2,5} \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda \mathbf{x}_{1,4} \xi_{1,3} \xi_{3,4} + \lambda^2 \mathbf{x}_{1,5} \eta_{4,5} \xi_{1,3} \xi_{3,4} + \lambda \mathbf{x}_{2,4} \xi_{2,3} \xi_{3,4} - \\ & \lambda^2 \mathbf{x}_{1,4} \eta_{1,2} \xi_{2,3} \xi_{3,4} + \lambda^2 \mathbf{x}_{2,5} \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda^3 \mathbf{x}_{1,5} \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} + \lambda^2 \mathbf{x}_{1,4} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \\ & \lambda^3 \mathbf{x}_{1,5} \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} + \mathbf{x}_{3,5} \xi_{3,5} - \lambda \mathbf{x}_{1,5} \eta_{1,3} \xi_{3,5} - \lambda \mathbf{x}_{2,5} \eta_{2,3} \xi_{3,5} + \lambda \mathbf{x}_{1,5} \xi_{1,3} \xi_{3,5} + \\ & \lambda \mathbf{x}_{2,5} \xi_{2,3} \xi_{3,5} - \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,5} + \mathbf{x}_{4,5} \xi_{4,5} - \lambda \mathbf{x}_{1,5} \eta_{1,4} \xi_{4,5} - \\ & \lambda \mathbf{x}_{2,5} \eta_{2,4} \xi_{4,5} - \lambda \mathbf{x}_{3,5} \eta_{3,4} \xi_{4,5} + \lambda \mathbf{x}_{1,5} \xi_{1,4} \xi_{4,5} + \lambda \mathbf{x}_{2,5} \xi_{2,4} \xi_{4,5} - \lambda^2 \mathbf{x}_{1,5} \eta_{1,2} \xi_{2,4} \xi_{4,5} + \\ & \lambda^2 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,4} \xi_{4,5} + \lambda \mathbf{x}_{3,5} \xi_{3,4} \xi_{4,5} - \lambda^2 \mathbf{x}_{1,5} \eta_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^2 \mathbf{x}_{2,5} \eta_{2,3} \xi_{3,4} \xi_{4,5} + \\ & \lambda^2 \mathbf{x}_{1,5} \xi_{1,3} \xi_{3,4} \xi_{4,5} + \lambda^2 \mathbf{x}_{2,5} \xi_{2,3} \xi_{3,4} \xi_{4,5} - \lambda^3 \mathbf{x}_{1,5} \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \mathbf{x}_{1,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} \end{aligned}$$

In[*]:= $\text{rhs} = \lambda \text{Tangent} @ (\$ \text{Basis} (\$ \text{Basis} /. \mathbf{x}_{\alpha\beta} \rightarrow \mathbf{f}_{\alpha\beta}[\lambda]))$

Out[*]=

$$\begin{aligned} & \mathbf{x}_{1,2} \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,3} \mathbf{f}_{2,3}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,4} \mathbf{f}_{2,4}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{2,5}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \\ & \mathbf{x}_{1,4} \mathbf{f}_{2,3}[\lambda] \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{2,3}[\lambda] \mathbf{f}_{3,5}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{2,4}[\lambda] \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \\ & \mathbf{x}_{1,5} \mathbf{f}_{2,3}[\lambda] \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{1,2}'[\lambda] + \mathbf{x}_{1,3} \mathbf{f}_{1,3}'[\lambda] + \mathbf{x}_{1,4} \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{1,3}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{3,5}[\lambda] \mathbf{f}_{1,3}'[\lambda] + \\ & \mathbf{x}_{1,5} \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{1,3}'[\lambda] + \mathbf{x}_{1,4} \mathbf{f}_{1,4}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{1,4}'[\lambda] + \mathbf{x}_{1,5} \mathbf{f}_{1,5}'[\lambda] + \mathbf{x}_{2,3} \mathbf{f}_{2,3}'[\lambda] + \\ & \mathbf{x}_{2,4} \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{2,3}'[\lambda] + \mathbf{x}_{2,5} \mathbf{f}_{3,5}[\lambda] \mathbf{f}_{2,3}'[\lambda] + \mathbf{x}_{2,5} \mathbf{f}_{3,4}[\lambda] \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{2,3}'[\lambda] + \mathbf{x}_{2,4} \mathbf{f}_{2,4}'[\lambda] + \\ & \mathbf{x}_{2,5} \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{2,4}'[\lambda] + \mathbf{x}_{2,5} \mathbf{f}_{2,5}'[\lambda] + \mathbf{x}_{3,4} \mathbf{f}_{3,4}'[\lambda] + \mathbf{x}_{3,5} \mathbf{f}_{4,5}[\lambda] \mathbf{f}_{3,4}'[\lambda] + \mathbf{x}_{3,5} \mathbf{f}_{3,5}'[\lambda] + \mathbf{x}_{4,5} \mathbf{f}_{4,5}'[\lambda] \end{aligned}$$

```

In[*]:= (Coefficient[lhs - rhs, #] == 0) & /@ $Basis
Out[*]=
{
 $\eta_{1,2} + \xi_{1,2} - f_{1,2}'[\lambda] = 0,$ 
 $\eta_{1,3} + \lambda \eta_{1,2} \eta_{2,3} + \lambda \eta_{2,3} \xi_{1,2} + \xi_{1,3} - \lambda \eta_{1,2} \xi_{2,3} + \lambda \xi_{1,2} \xi_{2,3} - f_{2,3}[\lambda] f_{1,2}'[\lambda] - f_{1,3}'[\lambda] = 0,$ 
 $\eta_{2,3} + \xi_{2,3} - f_{2,3}'[\lambda] = 0,$ 
 $\eta_{1,4} + \lambda \eta_{1,2} \eta_{2,4} + \lambda \eta_{1,3} \eta_{3,4} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,4} + \lambda \eta_{2,4} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,4} \xi_{1,2} + \lambda \eta_{3,4} \xi_{1,3} + \xi_{1,4} -$ 
 $\lambda^2 \eta_{1,2} \eta_{3,4} \xi_{2,3} + \lambda^2 \eta_{3,4} \xi_{1,2} \xi_{2,3} - \lambda \eta_{1,2} \xi_{2,4} + \lambda \xi_{1,2} \xi_{2,4} - \lambda \eta_{1,3} \xi_{3,4} + \lambda \xi_{1,3} \xi_{3,4} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,4} +$ 
 $\lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,4} - f_{2,4}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{1,2}'[\lambda] - f_{3,4}[\lambda] f_{1,3}'[\lambda] - f_{1,4}'[\lambda] = 0,$ 
 $\eta_{2,4} + \lambda \eta_{2,3} \eta_{3,4} + \lambda \eta_{3,4} \xi_{2,3} + \xi_{2,4} - \lambda \eta_{2,3} \xi_{3,4} + \lambda \xi_{2,3} \xi_{3,4} - f_{3,4}[\lambda] f_{2,3}'[\lambda] - f_{2,4}'[\lambda] = 0,$ 
 $\eta_{3,4} + \xi_{3,4} - f_{3,4}'[\lambda] = 0,$ 
 $\eta_{1,5} + \lambda \eta_{1,2} \eta_{2,5} + \lambda \eta_{1,3} \eta_{3,5} + \lambda^2 \eta_{1,2} \eta_{2,3} \eta_{3,5} + \lambda \eta_{1,4} \eta_{4,5} + \lambda^2 \eta_{1,2} \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{1,3} \eta_{3,4} \eta_{4,5} +$ 
 $\lambda^3 \eta_{1,2} \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{2,5} \xi_{1,2} + \lambda^2 \eta_{2,3} \eta_{3,5} \xi_{1,2} + \lambda^2 \eta_{2,4} \eta_{4,5} \xi_{1,2} + \lambda^3 \eta_{2,3} \eta_{3,4} \eta_{4,5} \xi_{1,2} +$ 
 $\lambda \eta_{3,5} \xi_{1,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{1,3} + \lambda \eta_{4,5} \xi_{1,4} + \xi_{1,5} - \lambda^2 \eta_{1,2} \eta_{3,5} \xi_{2,3} - \lambda^3 \eta_{1,2} \eta_{3,4} \eta_{4,5} \xi_{2,3} +$ 
 $\lambda^2 \eta_{3,5} \xi_{1,2} \xi_{2,3} + \lambda^3 \eta_{3,4} \eta_{4,5} \xi_{1,2} \xi_{2,3} - \lambda^2 \eta_{1,2} \eta_{4,5} \xi_{2,4} + \lambda^2 \eta_{4,5} \xi_{1,2} \xi_{2,4} -$ 
 $\lambda \eta_{1,2} \xi_{2,5} + \lambda \xi_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{1,3} \xi_{3,4} - \lambda^3 \eta_{1,2} \eta_{4,5} \xi_{2,3} \xi_{3,4} +$ 
 $\lambda^3 \eta_{4,5} \xi_{1,2} \xi_{2,3} \xi_{3,4} - \lambda \eta_{1,3} \xi_{3,5} + \lambda \xi_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,2} \xi_{2,3} \xi_{3,5} + \lambda^2 \xi_{1,2} \xi_{2,3} \xi_{3,5} -$ 
 $\lambda \eta_{1,4} \xi_{4,5} + \lambda \xi_{1,4} \xi_{4,5} - \lambda^2 \eta_{1,2} \xi_{2,4} \xi_{4,5} + \lambda^2 \xi_{1,2} \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{1,3} \xi_{3,4} \xi_{4,5} +$ 
 $\lambda^2 \xi_{1,3} \xi_{3,4} \xi_{4,5} - \lambda^3 \eta_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} + \lambda^3 \xi_{1,2} \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{2,5}[\lambda] f_{1,2}'[\lambda] -$ 
 $f_{2,3}[\lambda] f_{3,5}[\lambda] f_{1,2}'[\lambda] - f_{2,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] - f_{2,3}[\lambda] f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,2}'[\lambda] -$ 
 $f_{3,5}[\lambda] f_{1,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{1,3}'[\lambda] - f_{4,5}[\lambda] f_{1,4}'[\lambda] - f_{1,5}'[\lambda] = 0,$ 
 $\eta_{2,5} + \lambda \eta_{2,3} \eta_{3,5} + \lambda \eta_{2,4} \eta_{4,5} + \lambda^2 \eta_{2,3} \eta_{3,4} \eta_{4,5} + \lambda \eta_{3,5} \xi_{2,3} + \lambda^2 \eta_{3,4} \eta_{4,5} \xi_{2,3} + \lambda \eta_{4,5} \xi_{2,4} + \xi_{2,5} -$ 
 $\lambda^2 \eta_{2,3} \eta_{4,5} \xi_{3,4} + \lambda^2 \eta_{4,5} \xi_{2,3} \xi_{3,4} - \lambda \eta_{2,3} \xi_{3,5} + \lambda \xi_{2,3} \xi_{3,5} - \lambda \eta_{2,4} \xi_{4,5} + \lambda \xi_{2,4} \xi_{4,5} - \lambda^2 \eta_{2,3} \xi_{3,4} \xi_{4,5} +$ 
 $\lambda^2 \xi_{2,3} \xi_{3,4} \xi_{4,5} - f_{3,5}[\lambda] f_{2,3}'[\lambda] - f_{3,4}[\lambda] f_{4,5}[\lambda] f_{2,3}'[\lambda] - f_{4,5}[\lambda] f_{2,4}'[\lambda] - f_{2,5}'[\lambda] = 0,$ 
 $\eta_{3,5} + \lambda \eta_{3,4} \eta_{4,5} + \lambda \eta_{4,5} \xi_{3,4} + \xi_{3,5} - \lambda \eta_{3,4} \xi_{4,5} + \lambda \xi_{3,4} \xi_{4,5} - f_{4,5}[\lambda] f_{3,4}'[\lambda] - f_{3,5}'[\lambda] = 0,$ 
 $\eta_{4,5} + \xi_{4,5} - f_{4,5}'[\lambda] = 0$ 
}

In[*]:= {sol} = DSolve[
  Join[
    (Coefficient[lhs - rhs, #] == 0) & /@ $Basis,
    $Basis /. x $\alpha\beta$  -> f $\alpha\beta$ [0] == 0
  ],
  $Basis /. x $\alpha\beta$  -> f $\alpha\beta$ [λ],
  λ
]
Out[*]=
{
{
 $f_{1,2}[\lambda] \rightarrow \lambda \eta_{1,2} + \lambda \xi_{1,2},$ 
 $f_{1,3}[\lambda] \rightarrow \lambda \eta_{1,3} + \lambda \xi_{1,3} - \lambda^2 \eta_{1,2} \xi_{2,3},$ 
 $f_{2,3}[\lambda] \rightarrow \lambda \eta_{2,3} + \lambda \xi_{2,3},$ 
 $f_{1,4}[\lambda] \rightarrow -\lambda (-\eta_{1,4} - \xi_{1,4} + \lambda \eta_{1,2} \xi_{2,4} + \lambda \eta_{1,3} \xi_{3,4}),$ 
 $f_{2,4}[\lambda] \rightarrow -\lambda (-\eta_{2,4} - \xi_{2,4} + \lambda \eta_{2,3} \xi_{3,4}),$ 
 $f_{3,4}[\lambda] \rightarrow \lambda (\eta_{3,4} + \xi_{3,4}),$ 
 $f_{4,5}[\lambda] \rightarrow \lambda \eta_{4,5} + \lambda \xi_{4,5},$ 
 $f_{3,5}[\lambda] \rightarrow \lambda \eta_{3,5} + \lambda \xi_{3,5} - \lambda^2 \eta_{3,4} \xi_{4,5},$ 
 $f_{2,5}[\lambda] \rightarrow \lambda \eta_{2,5} + \lambda \xi_{2,5} - \lambda^2 \eta_{2,3} \xi_{3,5} - \lambda^2 \eta_{2,4} \xi_{4,5},$ 
 $f_{1,5}[\lambda] \rightarrow \lambda \eta_{1,5} + \lambda \xi_{1,5} - \lambda^2 \eta_{1,2} \xi_{2,5} - \lambda^2 \eta_{1,3} \xi_{3,5} - \lambda^2 \eta_{1,4} \xi_{4,5}$ 
}
}

```

In[*]:= **sol** /. $\lambda \rightarrow 1$

Out[*]=

$$\begin{aligned} &\{f_{1,2}[1] \rightarrow \eta_{1,2} + \xi_{1,2}, f_{1,3}[1] \rightarrow \eta_{1,3} + \xi_{1,3} - \eta_{1,2} \xi_{2,3}, f_{2,3}[1] \rightarrow \eta_{2,3} + \xi_{2,3}, \\ &f_{1,4}[1] \rightarrow \eta_{1,4} + \xi_{1,4} - \eta_{1,2} \xi_{2,4} - \eta_{1,3} \xi_{3,4}, f_{2,4}[1] \rightarrow \eta_{2,4} + \xi_{2,4} - \eta_{2,3} \xi_{3,4}, \\ &f_{3,4}[1] \rightarrow \eta_{3,4} + \xi_{3,4}, f_{4,5}[1] \rightarrow \eta_{4,5} + \xi_{4,5}, f_{3,5}[1] \rightarrow \eta_{3,5} + \xi_{3,5} - \eta_{3,4} \xi_{4,5}, \\ &f_{2,5}[1] \rightarrow \eta_{2,5} + \xi_{2,5} - \eta_{2,3} \xi_{3,5} - \eta_{2,4} \xi_{4,5}, f_{1,5}[1] \rightarrow \eta_{1,5} + \xi_{1,5} - \eta_{1,2} \xi_{2,5} - \eta_{1,3} \xi_{3,5} - \eta_{1,4} \xi_{4,5}\} \end{aligned}$$

$$\begin{aligned} \text{In[*]} := \text{osol} = \mathbb{E}_{\{1,2\} \rightarrow \{2\}} &[x_{1,1}[2] \xi_{1,1}[1] + x_{1,1}[2] \xi_{1,1}[2] + e^{-\xi_{1,1}[2]} x_{1,2}[2] \xi_{1,2}[1] + e^{-\xi_{2,2}[1]} x_{1,2}[2] \xi_{1,2}[2] + \\ &e^{-\xi_{1,1}[2]} x_{1,3}[2] \xi_{1,3}[1] + e^{-\xi_{3,3}[1]} x_{1,3}[2] \xi_{1,3}[2] + e^{-\xi_{1,1}[2]} x_{1,4}[2] \xi_{1,4}[1] + \\ &e^{-\xi_{4,4}[1]} x_{1,4}[2] \xi_{1,4}[2] + e^{-\xi_{1,1}[2]} x_{1,5}[2] \xi_{1,5}[1] + e^{-\xi_{5,5}[1]} x_{1,5}[2] \xi_{1,5}[2] + \\ &x_{2,2}[2] \xi_{2,2}[1] + x_{2,2}[2] \xi_{2,2}[2] + e^{-\xi_{2,2}[2]} x_{2,3}[2] \xi_{2,3}[1] - x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + \\ &e^{-\xi_{3,3}[1]} x_{2,3}[2] \xi_{2,3}[2] + e^{-\xi_{2,2}[2]} x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] + \\ &e^{-\xi_{4,4}[1]} x_{2,4}[2] \xi_{2,4}[2] + e^{-\xi_{2,2}[2]} x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + \\ &e^{-\xi_{5,5}[1]} x_{2,5}[2] \xi_{2,5}[2] + x_{3,3}[2] \xi_{3,3}[1] + x_{3,3}[2] \xi_{3,3}[2] + e^{-\xi_{3,3}[2]} x_{3,4}[2] \xi_{3,4}[1] - \\ &x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + e^{-\xi_{4,4}[1]} x_{3,4}[2] \xi_{3,4}[2] + \\ &e^{-\xi_{3,3}[2]} x_{3,5}[2] \xi_{3,5}[1] - x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + \\ &e^{-\xi_{5,5}[1]} x_{3,5}[2] \xi_{3,5}[2] + x_{4,4}[2] \xi_{4,4}[1] + x_{4,4}[2] \xi_{4,4}[2] + e^{-\xi_{4,4}[2]} x_{4,5}[2] \xi_{4,5}[1] - \\ &x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + \\ &e^{-\xi_{5,5}[1]} x_{4,5}[2] \xi_{4,5}[2] + x_{5,5}[2] \xi_{5,5}[1] + x_{5,5}[2] \xi_{5,5}[2], 0, 0] \llbracket 1 \rrbracket /. \xi_{i_}, i_ \rightarrow 0 \end{aligned}$$

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$$\begin{aligned} &x_{1,2}[2] \xi_{1,2}[1] + x_{1,2}[2] \xi_{1,2}[2] + x_{1,3}[2] \xi_{1,3}[1] + x_{1,3}[2] \xi_{1,3}[2] + \\ &x_{1,4}[2] \xi_{1,4}[1] + x_{1,4}[2] \xi_{1,4}[2] + x_{1,5}[2] \xi_{1,5}[1] + x_{1,5}[2] \xi_{1,5}[2] + x_{2,3}[2] \xi_{2,3}[1] - \\ &x_{1,3}[2] \xi_{1,2}[2] \xi_{2,3}[1] + x_{2,3}[2] \xi_{2,3}[2] + x_{2,4}[2] \xi_{2,4}[1] - x_{1,4}[2] \xi_{1,2}[2] \xi_{2,4}[1] + \\ &x_{2,4}[2] \xi_{2,4}[2] + x_{2,5}[2] \xi_{2,5}[1] - x_{1,5}[2] \xi_{1,2}[2] \xi_{2,5}[1] + x_{2,5}[2] \xi_{2,5}[2] + x_{3,4}[2] \xi_{3,4}[1] - \\ &x_{1,4}[2] \xi_{1,3}[2] \xi_{3,4}[1] - x_{2,4}[2] \xi_{2,3}[2] \xi_{3,4}[1] + x_{3,4}[2] \xi_{3,4}[2] + x_{3,5}[2] \xi_{3,5}[1] - \\ &x_{1,5}[2] \xi_{1,3}[2] \xi_{3,5}[1] - x_{2,5}[2] \xi_{2,3}[2] \xi_{3,5}[1] + x_{3,5}[2] \xi_{3,5}[2] + x_{4,5}[2] \xi_{4,5}[1] - \\ &x_{1,5}[2] \xi_{1,4}[2] \xi_{4,5}[1] - x_{2,5}[2] \xi_{2,4}[2] \xi_{4,5}[1] - x_{3,5}[2] \xi_{3,4}[2] \xi_{4,5}[1] + x_{4,5}[2] \xi_{4,5}[2] \end{aligned}$$

$$\text{In[*]} := \text{Expand@Total}[\text{sol} /. \{\xi_{\alpha\beta_} \Rightarrow \xi_{\alpha\beta}[1], \eta_{\alpha\beta_} \Rightarrow \xi_{\alpha\beta}[2], \text{Rule} \rightarrow \text{Times}, \lambda \rightarrow 1, f_{\alpha\beta_} \Rightarrow x_{\alpha\beta}[2]\}] - \text{osol}$$

Out[*]=

$$0$$