

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij} := \text{KroneckerDelta}[i, j];$ 
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[ $\mathcal{E}$ _, op_] := Collect[Normal@Series[ $\mathcal{E}$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\mathcal{E}$ ] := SS[ $\mathcal{E}$ , Together];
```

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```
In[ ]:= CF[ $\mathcal{E}$ ] := If[$k ===  $\infty$ ,
  Expand[ $\mathcal{E}$ ],
  Expand[Normal@Series[0[ $\epsilon$ ] $k+1$  +  $\mathcal{E}$  /.  $q \rightarrow e^{\hbar \epsilon}$ , { $\epsilon$ , 0, $k}]]
];
```

```
In[ ]:= Series[(1 + x)-1 + 0[x]^5, {x, 0, 20}]
```

Time: 0.03 seconds Out[]:=

$$1 - x + x^2 - x^3 + x^4 + 0[x]^5$$

```
In[ ]:= $k
```

Time: 0.02 seconds Out[]:=

$$1$$

```
In[*]:= CF[ $\epsilon^1$ ]
```

Time: 0.00 seconds Out[]=

€

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```
In[*]:= x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; ℳ[i_] := ℳ <> i;
```

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```
In[*]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[*]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]=

List["x", List[3, 4]]

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[*]:= (* q=eh ∇ €; Ai[n]=eh € ai; Bi[n]=eh ∇ bi *)
```

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```
In[*]:= $n = 3; $k = 1; $E := {$n, $k}
```

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```
In[*]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; A[i][_]],
  Table[As[i, 1, $n - 1],
    a[i]* = α[i]; α[i]* = a[i]; a[i]]
}
(#u = U0@#) & /@ $EPBWGens;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi_ → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[*]:= #* & /@ $EPBWGens
```

Time: 0.01 seconds Out[]=

{ξ12, ξ13, ξ23, A1[_]*, A2[_]*, α1, α2}

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```

In[ ]:= UCF[ $\mathcal{E}$ ] := Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {U_ $i$ [L____, _[0], r____]  $\rightarrow$  U_ $i$ [1, r]} /. If[$k ==  $\infty$ , {}, {
    U_ $i$ [L____, AB_ $n$ ], r____]  $\Rightarrow$ 
    Sum[As[k, 0, $k],  $\frac{\hbar^k \epsilon^k}{k!}$  U_ $i$ [L, Sequence@@Table[a@@L[AB], k], r]]
  }],
  U_ [____], F
] /. F  $\rightarrow$  CF]

```

```

In[ ]:= CF[ $\epsilon^2$ ]

```

```

Time: 0.00 seconds Out[ ]:=
 $\epsilon^2$ 

```

```

In[ ]:= $k

```

```

Time: 0.00 seconds Out[ ]:=
1

```

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```

In[ ]:= m $s_{--}$ [0] = 0;
m $s_{--}$ [x_Plus] := UCF[m $s_{--}$  /@ x];
m $i_{\rightarrow j_{--}}$ [ $\mathcal{E}$ ] :=  $\mathcal{E}$  /. U $i$   $\rightarrow$  U $j$ ;
m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ [c_. U $i_{--}$ [xx____] U $j_{--}$ [ ] ] := c U $k$ [x];
m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ [c_. U $i_{--}$ [ ] U $j_{--}$ [yy____] ] := c U $k$ [y];
m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ [c_. U $i_{--}$ [xx____, x_ $n1_{--}$ ] U $j_{--}$ [x_ $n2_{--}$ , yy____] ] :=
  UCF[c U $i$ [xx, x[ $n1 + n2$ ] ] U $j$ [yy] ] /. m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ ;
m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ [c_. U $i_{--}$ [xx____, x_] U $j_{--}$ [y_, yy____] ] := If[x  $\leq$  y,
  c U $k$ [xx, x, y, yy],
  ((c U $i$ [xx] RR $j$ [x, y] /. UCF /. m $i_{\rightarrow j_{--} \rightarrow i_{--}}$ ) U $j$ [yy] /. UCF /. m $i_{\rightarrow j_{--} \rightarrow k_{--}}$ ) /. UCF
];
m $s_{--}$ [ $\mathcal{E}$ ] := m $s_{--}$ [UCF[ $\mathcal{E}$ ]];

```

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```

In[ ]:=  $\Delta_{i_{\rightarrow j_{--} \rightarrow k_{--}}}$ [ $\mathcal{E}$ ] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  U $i$ [ ]  $\rightarrow$  U $j$ [ ] U $k$ [ ],
  U $i$ [f_, fs____]  $\Rightarrow$  m $k, k1 \rightarrow k$  @ m $j, j1 \rightarrow j$  [ $\Delta_{j, k}$ [f]  $\Delta_{i \rightarrow j1, k1}$ [U $i$ [fs]]]
}];
 $\Delta_{i_{\rightarrow j_{--} \rightarrow k_{--} \rightarrow l_{--}}}$ [ $\mathcal{E}$ ] :=  $\mathcal{E}$  /.  $\Delta_{i \rightarrow j, k}$  /.  $\Delta_{k \rightarrow l, l}$ 

```

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```

In[ ]:= S $i_{--}$ [ $\mathcal{E}$ ] := UCF@Module[{i1},  $\mathcal{E}$  /. {
  U $i$ [f_, fs____]  $\Rightarrow$  m $i1, i \rightarrow i$  [S $i$ [f] S $i1$ [U $i1$ [fs]]]
}];

```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RR_k[ai_, aj_] /;  $\tau[ai] == \tau[aj] == "a" := U_k[aj, ai];$ 
RR_k[ai_, Aj_[n_]] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[Aj[n], ai];$ 
RR_k[Ai_[m_], Aj_[n_]] /;  $\tau[Ai] == \tau[Aj] == "A" := U_k[Aj[n], Ai[m]];$ 
RR_k[am_, xij_] /;  $\tau[am] == "a" \wedge \tau[xij] == "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@am$ ;
   $U_k[xij, am] + \gamma(\delta_{m,i} - \delta_{m+1,i} - \delta_{m,j} + \delta_{m+1,j}) U_k[xij] / 2$ 
];
RR_k[Am_[n_], xij_] /;  $\tau[Am] == "A" \wedge \tau[xij] == "x" := Module[{i, j, m},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m} =  $\mathcal{L}@Am$ ;  $q^{-n}(\delta_{n,i} - \delta_{n+1,i} - \delta_{n,j} + \delta_{n+1,j}) U_k[xij, Am[n]]$ 
]
```

```
In[*]:= RR_0[a[1], x[1, 2]]
```

Time: 0.00 seconds Out[*]=

```
 $\gamma U_0[x12] + U_0[x12, a1]$ 
```

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We take the reduction rules RR for xij's from Yamane page

509:

$$\begin{array}{lll} \text{(I)} & i < m < j < n, & \text{(II)} \quad i < m < n < j, & \text{(III)} \quad i < m < j = n, \\ & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \\ \text{(IV)} & i < m < j < n, & \text{(V)} \quad i < j = m < n, & \text{(VI)} \quad i < j < m < n, \\ & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$

$$\begin{array}{ll} q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2e_{ij}e_{mn} - e_{mn}e_{ij} = qe_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

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```
In[*]:= RR_k[xmn_, xij_] /;  $\tau[xij] == \tau[xmn] == "x" := Module[{i, j, m, n},$ 
  {i, j} =  $\mathcal{L}@xij$ ; {m, n} =  $\mathcal{L}@xmn$ ;
  Which[
     $(i == m \wedge j < n) \vee (i < m < j \wedge n == j), q^{-2} U_k[xij, xmn], (* \text{IVIII} *)$ 
     $(i < m < j \wedge n < j) \vee (j < m), U_k[xij, xmn], (* \text{II} \vee \text{VI} *)$ 
     $i < m < j \wedge n > j, U_k[xij, xmn] + (q^{-2} - q^2) U_k[x[i, n], x[m, j]], (* \text{IV} *)$ 
     $j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* \text{V} *)$ 
  ]
]
```

```

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.03 seconds Out[*]=
(-1 - γ ∈ ħ) U2[x24] U3[x12] + (1 + 2 γ ∈ ħ) U2[x23, x34] U3[x12]

In[*]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.02 seconds Out[*]=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[*]:= Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
Time: 0.77 seconds Out[*]=
{}

In[*]:= Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]]
]
» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}

Out[*]=
{}

Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
  ]]
]
» {x12, x13, x23, a1, a2}

Out[*]=
{}

```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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```
In[*]:= Do[As[i, 1, $n - 1],
  Δj, k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj, k[A[i][n]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj, k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj, k[x[i, n]] = mk, k1→k[mj, j1→j[
    q Δj, k[x[i, n - 1]] Δj1, k1[x[n - 1, n]] - q-1 Δj, k[x[n - 1, n]] Δj1, k1[x[i, n - 1]]
  ]
]
```

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui / @ Bas], P → Δi→j, k[P]]
]
```

Time: 0.05 seconds Out:=

```
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
```

```
In[*]:= Short[{Δ4,5[a2], Δ7,9[A2[77]], Δ1,2[x12], Δ1,2[x15]}, 8]
```

Time: 0.01 seconds Out:=

```
{U4[a2] U5[] + U4[] U5[a2], U7[A2[77]] U9[A2[77]],
  U1[A1[-1]] U2[x12] + U1[x12] U2[A1[1]], Δ1,2[x15]}
```

```
In[*]:= Δ1→1,2[U1[x13, a2] U0[x34]]
```

Time: 0.01 seconds Out:=

```
U0[x34] U1[A1[-1], A2[-1], a2] U2[x13] + U0[x34] U1[A1[-1], A2[-1]] U2[x13, a2] +
  (-1/q2 + q2) U0[x34] U1[x12, A2[-1], a2] U2[x23, A1[1]] + U0[x34] U1[x13, a2] U2[A1[1], A2[1]] +
  (-1/q2 + q2) U0[x34] U1[x12, A2[-1]] U2[x23, A1[1], a2] + U0[x34] U1[x13] U2[A1[1], A2[1], a2]
```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[-1], A1[0], A1[-2], A1[1], A2[-2], A1[2], A2[2], A1[-1], A1[2], A2[1]}
Out[ ]=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> True] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> (P // m1,2→1 // Δ1→1,2) == (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
  ]]
]
» {x12, x13, x23, a1, a2, A2[2], A2[1], A1[0], A2[1], A2[1], A2[-1], A1[-1], A2[-1], A2[2], A2[-2]}
Out[ ]=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.08 seconds Out[ ]=
U1[x12] -> ħ U1[x12] U2[a1] - ħ U1[a1] U2[x12]
U1[x13] -> ħ U1[x13] U2[a1] - 2 ħ U1[x23] U2[x12] - ħ U1[a1] U2[x13] + 2 ħ U1[x12] U2[x23]
U1[x23] -> ħ U1[x23] U2[a2] - ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

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```

In[ ]:= Do[As[i, 1, $n - 1],
  Sj_ [a[i]] = -Uj[a[i]];
  Sj_ [A[i][n_]] = Uj[A[i][-n]];
  Sj_ [x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj_ [x[i, n]] = mj1, j→j [
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]

» {x12, x13, x23, a1, a2, A2[1], A2[-2], A1[-1], A1[2], A1[-2], A1[-1], A1[1], A1[2], A1[-1], A1[-1]}

Out[ ]:=
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A2[1] → U1[],
  A2[-2] → U1[], A1[-1] → U1[], A1[2] → U1[], A1[-2] → U1[],
  A1[-1] → U1[], A1[1] → U1[], A1[2] → U1[], A1[-1] → U1[], A1[-1] → U1[]}

```

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The Generating Functions for the $\mathbb{A}$ operations

```
In[ ]:= $k = 1;
```

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NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

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```

In[ ]:= Supp[ $\mathcal{E}$ _] := Union@Cases[{ $\mathcal{E}$ }, Ui[_]] :> i, ∞];

In[ ]:= Supp[2 ħ U1[x45] U2[x34]]

Time: 0.00 seconds Out[ ]:=
{1, 2}

```

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```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := Module[{i, is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σ[i][z], {i, is}];
  z = Expand[y z]; Do[z = mσ[i], i→i[z], {i, is}]; z];
UB[x_, y_] := UCF[x ⋈ y - y ⋈ x];
```

```
In[*]:= (2 ħ1 U1[x45] U2[x34]) ⋈ (2 ħ2 U1[x45] U2[x34])
```

Time: 0.03 seconds Out[]:=

```
4 ħ1 ħ2 U1[x45, x45] U2[x34, x34]
```

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```
In[*]:= (*adPower[x_, k_, Env_] [c_*y_QU] := UCF[c adPower[x, k, Env] [y]]];
adPower[x_, k_, Env_] [y_Plus] := adPower[x, k, Env] /@ y; *)
adPower[_ , 0, Env_] [y_] := y;
(*adPower[x_, k_, Env_] [0] = 0; *)
adPower[x_, k_, Env_] [y_] :=
  (*adPower[x, k, Env] [y] = *) UCF@UB[adPower[x, k - 1, Env] [y], x];
adPower[x_, k_] [y_] := adPower[x, k, $E] [y];
```

```
In[*]:= Table[As[k, 0, 5], adPower[U0[x12], k] [U0[x23]]]
```

Time: 0.06 seconds Out[]:=

```
{U0[x23], (-1 - γ ∈ ħ) U0[x13] + 2 γ ∈ ħ U0[x12, x23], 0, 0, 0, 0}
```

```
In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j, k[P]]
]
```

Time: 0.08 seconds Out[]:=

```
Ui [x12] → Uj [x12] Uk [] + ∈ ħ Uj [x12] Uk [a1] + Uj [] Uk [x12] + ∈ ħ Uj [a1] Uk [x12]
Ui [x13] → Uj [x13] Uk [] + ∈ ħ Uj [x13] Uk [a1] + ∈ ħ Uj [x13] Uk [a2] +
  Uj [] Uk [x13] + ∈ ħ Uj [a1] Uk [x13] + ∈ ħ Uj [a2] Uk [x13] + 4 γ ∈ ħ Uj [x12] Uk [x23]
Ui [x23] → Uj [x23] Uk [] + ∈ ħ Uj [x23] Uk [a2] + Uj [] Uk [x23] + ∈ ħ Uj [a2] Uk [x23]
Ui [a1] → Uj [a1] Uk [] + Uj [] Uk [a1]
Ui [a2] → Uj [a2] Uk [] + Uj [] Uk [a2]
```

```
In[*]:= Ad[222 Uj [x12] Uk []] [Uj [x23] Uk [a4]]
Add[222 ħ Uj [x12] Uk []] [Uj [x23] Uk [a4]]
```

Time: 0.06 seconds Out[]:=

```
(-1 - γ ∈ ħ) Uj [x13] Uk [a4] + Uj [x23] Uk [a4] + 2 γ ∈ ħ Uj [x12, x23] Uk [a4]
```

Out[]:=

```
(-222 ħ - 222 γ ∈ ħ2) Uj [x13] Uk [a4] + Uj [x23] Uk [a4] + 444 γ ∈ ħ2 Uj [x12, x23] Uk [a4]
```

pdf

```
In[*]:= Wtaj [x_] := Wtaj [x] = UB[U0 [aj], U0 [x]] / U0 [x]
```

```
In[*]:= Table[Wta[j] [x23], {j, 1, $n}]
```

Time: 0.00 seconds Out[]=

$$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}\right\}$$

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```
In[*]:= Ad[c_. Ui[aj_] (EUs : U-[...] [y_] /; τ[aj] == "a" :=  
(y /. Ui[gs___] => Ui[gs] ec Total[Wtaj/@{gs}])
```

```
Ad[Uj[a1] U23[a2]] [Uj[x23] Uk[a4]]
```

Time: 0.00 seconds Out[]=

$$e^{-\gamma/2} U_j[x23] U_k[a4]$$

```
In[*]:= Ad[π i Uj[a1]] [Uj[x23] Uk[a4]]
```

Time: 0.02 seconds Out[]=

$$e^{-\frac{1}{2} i \pi \gamma} U_j[x23] U_k[a4]$$

pdf

```
In[*]:= Ad[0] = Identity;
```

pdf

```
In[*]:= (*Ad[c_. Ui[xjk_] (EUs:U-[...] [y_] /; τ[xjk]=="x":= Module[{k=0,s=0,t},  
While[(t=adPower[Ui[xjk],k][y])!=0,  
s+=t/k!;  
++k  
];  
UCF[s**EUs]  
]*) (*c independent??*)  
  
Ad[xx_][y_] := Module[{k=0,s=0,t},  
While[(t=adPower[xx,k][y])!=0,  
s+=t/k!;  
++k  
];  
UCF[s]  
]
```

pdf

```
In[*]:= TTλ[] = 0; (* Translated Tangent *)  
TTλ[xs___, x_] := UCF[Ad[x][TTλ[xs]] + ∂λ x];  
TTλ[xs_List] := TTλ@@xs
```

```
In[*]:= TTλ[λ ξ12 U0[x12] U1[], λ ξ23 U0[x23] U1[]]
```

Time: 0.03 seconds Out[]=

$$\xi_{12} U_0[x12] U_1[] + (\xi_{12} + \gamma \in \xi_{12} \hbar) U_0[x13] U_1[] + \xi_{23} U_0[x23] U_1[] - 2 \gamma \in \xi_{12} \hbar U_0[x12, x23] U_1[]$$

In[*]:= **\$PBWgens**

Time: 0.00 seconds Out[=

{x12, x13, x23, a1, a2}

In[*]:= **TTL = TT_λ@Take[Join[**

Cases[\$PBWgens, g_ => Which[

τ[g] === "x", λ (g*)_i U_k[g],

τ[g] === "a", (g*)_i U_k[g]

]],

Cases[\$PBWgens, g_ => Which[

τ[g] === "x", λ (g*)_j U_k[g],

τ[g] === "a", (g*)_j U_k[g]

]]

], All]

Time: 7.51 seconds Out[=

$$\begin{aligned}
& \left(e^{\gamma \alpha_{1i} + \gamma \alpha_{1j} - \frac{\gamma \alpha_{2i}}{2} - \frac{\gamma \alpha_{2j}}{2}} \xi_{12i} + e^{\gamma \alpha_{1j} - \frac{\gamma \alpha_{2j}}{2}} \xi_{12j} \right) U_k[x12] + \\
& \left(e^{\gamma \alpha_{1i} + \frac{\gamma \alpha_{1j}}{2} - \frac{1}{2} \gamma (\alpha_{1i} - \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \xi_{13i} + e^{\frac{\gamma \alpha_{1j}}{2} + \frac{\gamma \alpha_{2j}}{2}} \xi_{13j} + e^{\gamma \alpha_{1i} + \frac{\gamma \alpha_{1j}}{2} - \frac{1}{2} \gamma (\alpha_{1i} - \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \lambda \xi_{12i} \xi_{23i} + \right. \\
& e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} + \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12i} \xi_{23i} - e^{\frac{\gamma \alpha_{1j}}{2} - \frac{1}{2} \gamma (\alpha_{1i} - \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2} + \frac{\gamma \alpha_{2j}}{2}} \lambda \xi_{12j} \xi_{23i} - \\
& e^{\frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} + 2 \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12j} \xi_{23i} + e^{\gamma \alpha_{1i} + \frac{\gamma \alpha_{1j}}{2} - \frac{\gamma \alpha_{2i}}{2} + \frac{\gamma \alpha_{2j}}{2}} \lambda \xi_{12i} \xi_{23j} + \\
& \left. e^{\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{1j} - \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12i} \xi_{23j} + e^{\frac{\gamma \alpha_{1j}}{2} + \frac{\gamma \alpha_{2j}}{2}} \lambda \xi_{12j} \xi_{23j} + e^{\frac{\gamma \alpha_{1j}}{2} + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12j} \xi_{23j} \right) U_k[x13] + \\
& \left(e^{\gamma \alpha_{2i} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 2 \alpha_{2j})} \xi_{23i} + e^{\frac{\gamma \alpha_{1i}}{2} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 2 \alpha_{2j})} \xi_{23j} \right) U_k[x23] + \\
& \left(2 e^{\frac{3}{2} \gamma (\alpha_{1i} + \alpha_{1j})} \gamma \in \lambda \hbar \xi_{12i} \xi_{13i} - 2 e^{\frac{1}{2} \gamma (\alpha_{1i} + 3 \alpha_{1j} + \alpha_{2i})} \gamma \in \lambda \hbar \xi_{12j} \xi_{13i} + \right. \\
& 2 e^{\frac{1}{2} \gamma (2 \alpha_{1i} + 3 \alpha_{1j} - \alpha_{2i})} \gamma \in \lambda \hbar \xi_{12i} \xi_{13j} + 2 e^{\frac{3 \gamma \alpha_{1j}}{2}} \gamma \in \lambda \hbar \xi_{12j} \xi_{13j} \left. \right) U_k[x12, x13] + \\
& \left(-2 e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} + \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12i} \xi_{23i} + 2 e^{\frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} + 2 \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12j} \xi_{23i} - \right. \\
& 2 e^{\frac{1}{2} \gamma (2 \alpha_{1i} + \alpha_{1j} - \alpha_{2i}) + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12i} \xi_{23j} - 2 e^{\frac{\gamma \alpha_{1j}}{2} + \frac{\gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{12j} \xi_{23j} \left. \right) U_k[x12, x23] + \\
& \left(2 e^{\gamma \alpha_{1i} + \gamma \alpha_{1j} + \gamma \alpha_{2i} + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} - 2 e^{\gamma \alpha_{1j} + \frac{3 \gamma \alpha_{2i}}{2} + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23i} + \right. \\
& 2 e^{\gamma \alpha_{1i} + \gamma \alpha_{1j} - \frac{\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i}) + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23j} - 2 e^{\gamma \alpha_{1j} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i}) + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12j} \\
& \xi_{13i} \xi_{23j} + 2 e^{\gamma \alpha_{1i} + \gamma \alpha_{1j} - \frac{\gamma \alpha_{2i}}{2} + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12i} \xi_{13j} \xi_{23j} + 2 e^{\gamma \alpha_{1j} + \gamma \alpha_{2j}} \gamma \in \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23j} \left. \right) \\
& U_k[x13, x13] + \left(2 e^{\frac{3 \gamma \alpha_{2i}}{2} + \frac{3 \gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{13i} \xi_{23i} - 2 e^{-\frac{\gamma \alpha_{1i}}{2} + \gamma \alpha_{2i} + \frac{3 \gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{13j} \xi_{23i} + \right. \\
& 2 e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i}) + \frac{3 \gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{13i} \xi_{23j} + 2 e^{\frac{3 \gamma \alpha_{2j}}{2}} \gamma \in \lambda \hbar \xi_{13j} \xi_{23j} \left. \right) U_k[x13, x23]
\end{aligned}$$

pdf

```
In[*]:= mons[p_] := Union@Cases[Coefficient[TTL,  $\epsilon$ , p], U_k[___],  $\infty$ ];
mons /@ Range[0, $k]
```

Time: 0.03 seconds Out[]=

pdf

```
{ {U_k[x12], U_k[x13], U_k[x23]},
  {U_k[x13], U_k[x12, x13], U_k[x12, x23], U_k[x13, x13], U_k[x13, x23]} }
```

```
In[*]:= U_k /@ $PBWGens
```

Time: 0.02 seconds Out[]=

```
{U_k[x12], U_k[x13], U_k[x23], U_k[a1], U_k[a2]}
```

pdf

```
In[*]:= ff = Table[
  Sum[ $\epsilon^p$  f_{p,u}[\lambda], {p, 0, $k}],
  {u, $PBWGens}
] /. f_{0,u}[\lambda] /;  $\tau[u] = "a" \Rightarrow (u^*)_i + (u^*)_j$ 
```

Time: 0.01 seconds Out[]=

pdf

```
{f_{0,x12}[\lambda] +  $\epsilon$  f_{1,x12}[\lambda], f_{0,x13}[\lambda] +  $\epsilon$  f_{1,x13}[\lambda],
  f_{0,x23}[\lambda] +  $\epsilon$  f_{1,x23}[\lambda],  $\alpha 1_i + \alpha 1_j + \epsilon$  f_{1,a1}[\lambda],  $\alpha 2_i + \alpha 2_j + \epsilon$  f_{1,a2}[\lambda]}
```

```
In[*]:= fs = Flatten[Table[
  f_{k,Times@@m},
  {k, 0, $k}, {m, mons[k]}
]]
```

Time: 0.03 seconds Out[]=

```
{f_{0,x12}, f_{0,x13}, f_{0,x23}, f_{1,x13}, f_{1,x12 x13}, f_{1,x12 x23}, f_{1,x13^2}, f_{1,x13 x23}}
```

```
In[*]:= gg = Sum[
   $\epsilon^p$  f_{p,Times@@u}[\lambda] Times@@u,
  {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
]
```

Time: 0.02 seconds Out[]=

```
x12 x13  $\epsilon$  f_{1,x12 x13}[\lambda] + x13^2  $\epsilon$  f_{1,x13^2}[\lambda] + x12 x23  $\epsilon$  f_{1,x12 x23}[\lambda] + x13 x23  $\epsilon$  f_{1,x13 x23}[\lambda]
```

```
In[*]:= gg2 = Sum[
   $\epsilon^p$  f_{p,Times@@u}[\lambda] Times@@(#k & /@ u),
  {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
]
```

Time: 0.02 seconds Out[]=

```
 $\epsilon$  x12_k x13_k f_{1,x12 x13}[\lambda] +  $\epsilon$  x13_k^2 f_{1,x13^2}[\lambda] +  $\epsilon$  x12_k x23_k f_{1,x12 x23}[\lambda] +  $\epsilon$  x13_k x23_k f_{1,x13 x23}[\lambda]
```

```
In[*]:= G = Series[ $e^{gg}$ , { $\epsilon$ , 0, $k}]
```

Time: 0.03 seconds Out[]=

```
1 + (x12 x13 f_{1,x12 x13}[\lambda] + x13^2 f_{1,x13^2}[\lambda] + x12 x23 f_{1,x12 x23}[\lambda] + x13 x23 f_{1,x13 x23}[\lambda])  $\epsilon$  + O[ $\epsilon$ ]^2
```

```
In[ ]:= O_i[poly_] := Total[
  CoefficientRules[poly, $PBWGens] /.
    (p_ -> c_) -> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
]
```

```
In[ ]:= O_i[x12^2 x13 + 7 x23^17]
```

Time: 0.02 seconds Out[]=

```
U_i[x12, x12, x13] +
  7 U_i[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]
```

pdf

$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i,j} x \rangle \mathbb{C}\langle \lambda_{i,j} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i x_i \rangle \mathbb{C}\langle \text{Seepage}(\{f_i\}, A) \rangle &= \mathbb{C}\langle e^{f_i x_i} A \rangle \\
 \text{Tr}(L_\lambda) &= L'_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f + g \rangle, \quad f = f_i x_i, \quad g = \sum_{i,j} g_{i,j} x_{i,j} & e^f = F, \quad e^g = G, \quad R_\lambda = \mathbb{C}\langle FG \rangle, \quad \partial_\lambda F = F \partial_\lambda f \\
 \text{Tr}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle \mathbb{C}\langle F(\partial_\lambda FG + \lambda G) \rangle = S(G)' S(\partial_\lambda FG + \lambda G)
 \end{aligned}$$

pdf

```
In[ ]:= Seepage_i[{}, c_] = c;
Seepage_i[ξs_List, P_] := Module[{lx},
  lx = $PBWGens[[Length@ξs]];
  CF@Collect[P, lx, (Ad[Last[ξs] U_i[lx]] [O_i[Seepage_i[Most@ξs, #]]] /. U_i -> Times) &]
]
```

```
In[ ]:= $PBWGens
```

Time: 0.02 seconds Out[]=

```
{x12, x13, x23, a1, a2}
```

```
In[ ]:= Seepage_i[#* & /@ $PBWGens, x12]
```

Time: 0.20 seconds Out[]=

$$\begin{aligned}
 & e^{a_1 \gamma - \frac{a_2 \gamma}{2}} x_{12} + e^{\frac{1}{2} (a_1 + a_2) \gamma} x_{13} \xi_{23} + 2 e^{\frac{3 a_1 \gamma}{2}} x_{12} x_{13} \gamma \in \xi_{13} \hbar + \\
 & e^{\frac{1}{2} (a_1 + a_2) \gamma} x_{13} \gamma \in \xi_{23} \hbar - 2 e^{\frac{1}{2} (a_1 + a_2) \gamma} x_{12} x_{23} \gamma \in \xi_{23} \hbar + 2 e^{(a_1 + a_2) \gamma} x_{13}^2 \gamma \in \xi_{13} \xi_{23} \hbar
 \end{aligned}$$

```
In[ ]:= ff
```

Time: 0.00 seconds Out[]=

$$\begin{aligned}
 & \{f_{0,x12}[\lambda] + \epsilon f_{1,x12}[\lambda], f_{0,x13}[\lambda] + \epsilon f_{1,x13}[\lambda], \\
 & f_{0,x23}[\lambda] + \epsilon f_{1,x23}[\lambda], \alpha_1 i + \alpha_1 j + \epsilon f_{1,a1}[\lambda], \alpha_2 i + \alpha_2 j + \epsilon f_{1,a2}[\lambda]\}
 \end{aligned}$$

```
In[ ]:= SS@UCF@G
```

Time: 0.03 seconds Out[]=

$$1 + \epsilon (x_{12} x_{13} f_{1,x12 x13}[\lambda] + x_{13}^2 f_{1,x13^2}[\lambda] + x_{12} x_{23} f_{1,x12 x23}[\lambda] + x_{13} x_{23} f_{1,x13 x23}[\lambda])$$

In[*]:= **Seepage_i**[ff, SS@UCF@G]

Time: 0.55 seconds Out[]=

$$\begin{aligned} 1 + e^{\frac{3}{2} \gamma (\alpha_1 + \alpha_1)} x_{12} x_{13} &\in f_{1, x_{12} x_{13}}[\lambda] + e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} x_{13}^2 \in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{13}}[\lambda] + \\ e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} x_{13}^2 &\in f_{1, x_{13}^2}[\lambda] + e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} x_{12} x_{23} \in f_{1, x_{12} x_{23}}[\lambda] + \\ e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} x_{13} x_{23} &\in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{23}}[\lambda] + e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} x_{13} x_{23} \in f_{1, x_{13} x_{23}}[\lambda] \end{aligned}$$

In[*]:= **O_i@Seepage_i**[ff, SS@UCF@G]

Time: 0.50 seconds Out[]=

$$\begin{aligned} U_i[] + e^{\frac{3}{2} \gamma (\alpha_1 + \alpha_1)} &\in U_i[x_{12}, x_{13}] f_{1, x_{12} x_{13}}[\lambda] + \\ U_i[x_{13}, x_{13}] &\left(e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} \in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{13}}[\lambda] + e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} \in f_{1, x_{13}^2}[\lambda] \right) + \\ e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} &\in U_i[x_{12}, x_{23}] f_{1, x_{12} x_{23}}[\lambda] + \\ U_i[x_{13}, x_{23}] &\left(e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{23}}[\lambda] + e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{1, x_{13} x_{23}}[\lambda] \right) \end{aligned}$$

pdf

In[*]:= **S_i_[B_] := UCF[SS@O_i[Seepage_i[ff, SS@UCF@B]]];**

In[*]:= **Short[SG = S_k[G], 10]**

Time: 0.67 seconds Out[]=

$$\begin{aligned} U_k[] + e^{\frac{3}{2} \gamma (\alpha_1 + \alpha_1)} &\in U_k[x_{12}, x_{13}] f_{1, x_{12} x_{13}}[\lambda] + \\ U_k[x_{13}, x_{13}] &\left(e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} \in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{13}}[\lambda] + e^{\gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} \in f_{1, x_{13}^2}[\lambda] \right) + \\ e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 + \alpha_2 + \alpha_2)} &\in U_k[x_{12}, x_{23}] f_{1, x_{12} x_{23}}[\lambda] + \\ U_k[x_{13}, x_{23}] &\left(e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{0, x_{23}}[\lambda] f_{1, x_{12} x_{23}}[\lambda] + e^{\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{1, x_{13} x_{23}}[\lambda] \right) \end{aligned}$$

In[*]:= **Invert_i[_[E_]] := Module[{A, B, β, j}, (* E=A U[] + E B *)**
A = Coefficient[E, U_i[]] /. e → 0;
B = UCF[(E - A U_i[])];
UCF[A⁻¹ (U_i[] + Sum[As[j, 1, \$k], (-1)^j NonCommutativeMultiply@@ConstantArray[B, j]])]
]

In[*]:= **Invert_k**[U_k[] + e ^{$\frac{3}{2} \gamma (\alpha_1 + \alpha_1)$} ∈ U_k[x₁₂, x₁₃] f_{1, x₁₂ x₁₃}[λ]]

Time: 0.00 seconds Out[]=

$$U_k[] - e^{\frac{3}{2} \gamma (\alpha_1 + \alpha_1)} \in U_k[x_{12}, x_{13}] f_{1, x_{12} x_{13}}[\lambda]$$

In[*]:= **UCF**[(U_k[] - e ^{$\frac{3}{2} \gamma (\alpha_1 + \alpha_1)$} ∈ U_k[x₁₂, x₁₃] f_{1, x₁₂ x₁₃}[λ]) ⊗
 (U_k[] + e ^{$\frac{3}{2} \gamma (\alpha_1 + \alpha_1)$} ∈ U_k[x₁₂, x₁₃] f_{1, x₁₂ x₁₃}[λ])]

Time: 0.02 seconds Out[]=

$$U_k[]$$

In[*]:= **TTR** = **UCF**[**Invert**_k[**SG**] $\otimes$ **S**_k[(∂_λ (**ff**.**\$PBWgens**) **G** + ∂_λ **G**)]]

Time: 14.44 seconds Out[]=

$$\begin{aligned}
& \in U_k[a1] f_{1,a1}'[\lambda] + \in U_k[a2] f_{1,a2}'[\lambda] + \\
& U_k[x12] \left(e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} f_{\theta,x12}'[\lambda] + e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,a1}[\lambda] f_{\theta,x12}'[\lambda] - \right. \\
& \quad \left. \frac{1}{2} e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,a2}[\lambda] f_{\theta,x12}'[\lambda] + e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,x12}'[\lambda] \right) + \\
& U_k[x13] \left(e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} f_{\theta,x23}[\lambda] f_{\theta,x12}'[\lambda] + e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in \hbar f_{\theta,x23}[\lambda] f_{\theta,x12}'[\lambda] + \right. \\
& \quad \frac{1}{2} e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] f_{1,a1}[\lambda] f_{\theta,x12}'[\lambda] + \frac{1}{2} e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] \\
& \quad f_{1,a2}[\lambda] f_{\theta,x12}'[\lambda] + e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,x23}[\lambda] f_{\theta,x12}'[\lambda] + e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} f_{\theta,x13}'[\lambda] + \\
& \quad \frac{1}{2} e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,a1}[\lambda] f_{\theta,x13}'[\lambda] + \frac{1}{2} e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,a2}[\lambda] f_{\theta,x13}'[\lambda] + \\
& \quad \left. e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] f_{1,x12}'[\lambda] + e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,x13}'[\lambda] \right) + \\
& U_k[x12, x13] \left(2 e^{\frac{3}{2}\gamma(a1+a1j)} \gamma \in \hbar f_{\theta,x13}[\lambda] f_{\theta,x12}'[\lambda] + \right. \\
& \quad \left. e^{\frac{3}{2}\gamma(a1+a1j)} \gamma \in f_{1,x12x23}[\lambda] f_{\theta,x12}'[\lambda] + e^{\frac{3}{2}\gamma(a1+a1j)} \gamma \in f_{1,x12x13}'[\lambda] \right) + \\
& U_k[x13, x13] \left(2 e^{\gamma(a1+a1j+a2+a2j)} \gamma \in \hbar f_{\theta,x13}[\lambda] f_{\theta,x23}[\lambda] f_{\theta,x12}'[\lambda] + \right. \\
& \quad e^{\gamma(a1+a1j+a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] f_{1,x12x23}[\lambda] f_{\theta,x12}'[\lambda] + e^{\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,x13x23}[\lambda] f_{\theta,x12}'[\lambda] + \\
& \quad \left. e^{\gamma(a1+a1j+a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] f_{1,x12x13}'[\lambda] + e^{\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,x13x23}'[\lambda] \right) + \\
& U_k[x23] \left(e^{-\frac{1}{2}\gamma(a1+a1j-2(a2+a2j))} f_{\theta,x23}'[\lambda] - \frac{1}{2} e^{-\frac{1}{2}\gamma(a1+a1j-2(a2+a2j))} \gamma \in f_{1,a1}[\lambda] f_{\theta,x23}'[\lambda] + \right. \\
& \quad \left. e^{-\frac{1}{2}\gamma(a1+a1j-2(a2+a2j))} \gamma \in f_{1,a2}[\lambda] f_{\theta,x23}'[\lambda] + e^{-\frac{1}{2}\gamma(a1+a1j-2(a2+a2j))} \gamma \in f_{1,x23}'[\lambda] \right) + \\
& U_k[x12, x23] \left(-2 e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in \hbar f_{\theta,x23}[\lambda] f_{\theta,x12}'[\lambda] + e^{\frac{1}{2}\gamma(a1+a1j+a2+a2j)} \gamma \in f_{1,x12x23}'[\lambda] \right) + \\
& U_k[x13, x23] \left(2 e^{\frac{3}{2}\gamma(a2+a2j)} \gamma \in \hbar f_{\theta,x23}[\lambda] f_{\theta,x13}'[\lambda] + \right. \\
& \quad \left. e^{\frac{3}{2}\gamma(a2+a2j)} \gamma \in f_{\theta,x23}[\lambda] f_{1,x12x23}'[\lambda] + e^{\frac{3}{2}\gamma(a2+a2j)} \gamma \in f_{1,x13x23}'[\lambda] \right)
\end{aligned}$$

In[*]:= **diff** = **UCF**[**TTL** - **TTR**]

Time: 5.39 seconds Out[]=

$$\begin{aligned}
& - \in U_k[a1] f_{1,a1}'[\lambda] - \in U_k[a2] f_{1,a2}'[\lambda] + \\
& U_k[x12] \left(e^{\gamma a1 + \frac{1}{2}\gamma(2a1j - a2i - a2j)} \xi_{12i} + e^{\frac{\gamma a2i}{2} + \frac{1}{2}\gamma(2a1j - a2i - a2j)} \xi_{12j} - \right. \\
& \quad e^{\gamma a1 + \frac{1}{2}\gamma(2a1j - a2i - a2j)} f_{\theta,x12}'[\lambda] - e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,a1}[\lambda] f_{\theta,x12}'[\lambda] + \\
& \quad \left. \frac{1}{2} e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,a2}[\lambda] f_{\theta,x12}'[\lambda] - e^{\gamma(a1+a1j)-\frac{1}{2}\gamma(a2+a2j)} \gamma \in f_{1,x12}'[\lambda] \right) + \\
& U_k[x13] \left(e^{\gamma(a1+a2i) + \frac{1}{2}\gamma(-a1i+a1j-a2i+a2j)} \xi_{13i} + e^{\frac{1}{2}\gamma(a1+a2i) + \frac{1}{2}\gamma(-a1i+a1j-a2i+a2j)} \xi_{13j} + \right. \\
& \quad e^{\gamma(a1+a2i) + \frac{1}{2}\gamma(-a1i+a1j-a2i+a2j)} \lambda \xi_{12i} \xi_{23i} + e^{\gamma(a1+a2i)-\frac{1}{2}\gamma(a1i-a1j+a2i-a2j)} \gamma \in \lambda \hbar \xi_{12i} \xi_{23i} - \\
& \quad e^{\frac{3\gamma a2i}{2} + \frac{1}{2}\gamma(-a1i+a1j-a2i+a2j)} \lambda \xi_{12j} \xi_{23i} - e^{\frac{3\gamma a2i}{2} - \frac{1}{2}\gamma(a1i-a1j+a2i-a2j)} \gamma \in \lambda \hbar \xi_{12j} \xi_{23i} + \\
& \quad \left. e^{\frac{3\gamma a1i}{2} + \frac{1}{2}\gamma(-a1i+a1j-a2i+a2j)} \lambda \xi_{12i} \xi_{23j} + e^{\frac{3\gamma a1i}{2} - \frac{1}{2}\gamma(a1i-a1j+a2i-a2j)} \gamma \in \lambda \hbar \xi_{12i} \xi_{23j} + \right)
\end{aligned}$$

$$\begin{aligned}
& e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \lambda \xi_{12_j} \xi_{23_j} + e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in \lambda \hbar \xi_{12_j} \xi_{23_j} - \\
& e^{\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} f_{\theta,x23}[\lambda] f_{\theta,x12'}[\lambda] - e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in \hbar \\
& f_{\theta,x23}[\lambda] f_{\theta,x12'}[\lambda] - \frac{1}{2} e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in f_{\theta,x23}[\lambda] f_{1,a1}[\lambda] f_{\theta,x12'}[\lambda] - \\
& \frac{1}{2} e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in f_{\theta,x23}[\lambda] f_{1,a2}[\lambda] f_{\theta,x12'}[\lambda] - \\
& e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \in f_{1,x23}[\lambda] f_{\theta,x12'}[\lambda] - e^{\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} f_{\theta,x13'}[\lambda] - \\
& \frac{1}{2} e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in f_{1,a1}[\lambda] f_{\theta,x13'}[\lambda] - \\
& \frac{1}{2} e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \gamma \in f_{1,a2}[\lambda] f_{\theta,x13'}[\lambda] - \\
& e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \in f_{\theta,x23}[\lambda] f_{1,x12'}[\lambda] - e^{\gamma(\alpha_1+\alpha_2)-\frac{1}{2}\gamma(\alpha_1-\alpha_1+\alpha_2-\alpha_2)} \in f_{1,x13'}[\lambda] \Big) + \\
& U_k[x12, x13] \left(2 e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{13_i} - 2 e^{\frac{1}{2}\gamma(\alpha_1+3\alpha_1+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_j} \xi_{13_i} + \right. \\
& 2 e^{\frac{1}{2}\gamma(2\alpha_1+3\alpha_1-\alpha_2)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{13_j} + 2 e^{\frac{3\gamma\alpha_1}{2}} \gamma \in \lambda \hbar \xi_{12_j} \xi_{13_j} - 2 e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \gamma \in \hbar \\
& f_{\theta,x13}[\lambda] f_{\theta,x12'}[\lambda] - e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \in f_{1,x12\ x23}[\lambda] f_{\theta,x12'}[\lambda] - e^{\frac{3}{2}\gamma(\alpha_1+\alpha_1)} \in f_{1,x12\ x13'}[\lambda] \Big) + \\
& U_k[x13, x13] \left(2 e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \gamma \in \lambda^2 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_i} - 2 e^{\gamma\alpha_1+\frac{3\gamma\alpha_1}{2}+\gamma\alpha_2} \gamma \in \lambda^2 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_i} + \right. \\
& 2 e^{\frac{3\gamma\alpha_1}{2}+\gamma(\alpha_1+\alpha_2)} \gamma \in \lambda^2 \hbar \xi_{12_i} \xi_{13_i} \xi_{23_j} - 2 e^{\frac{1}{2}\gamma(\alpha_1+2\alpha_1+\alpha_2+2\alpha_2)} \gamma \in \lambda^2 \hbar \xi_{12_j} \xi_{13_i} \xi_{23_j} + \\
& 2 e^{\gamma\alpha_1+\gamma\alpha_1-\frac{\gamma\alpha_2}{2}+\gamma\alpha_2} \gamma \in \lambda^2 \hbar \xi_{12_i} \xi_{13_j} \xi_{23_j} + 2 e^{\gamma(\alpha_1+\alpha_2)} \gamma \in \lambda^2 \hbar \xi_{12_j} \xi_{13_j} \xi_{23_j} - \\
& 2 e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \gamma \in \hbar f_{\theta,x13}[\lambda] f_{\theta,x23}[\lambda] f_{\theta,x12'}[\lambda] - \\
& e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{\theta,x23}[\lambda] f_{1,x12\ x23}[\lambda] f_{\theta,x12'}[\lambda] - e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{1,x13\ x23}[\lambda] f_{\theta,x12'}[\lambda] - \\
& e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{\theta,x23}[\lambda] f_{1,x12\ x13'}[\lambda] - e^{\gamma(\alpha_1+\alpha_1+\alpha_2+\alpha_2)} \in f_{1,x13^2}[\lambda] \Big) + \\
& U_k[x23] \left(e^{\gamma\alpha_2-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2\alpha_2)} \xi_{23_i} + e^{\frac{\gamma\alpha_1}{2}-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2\alpha_2)} \xi_{23_j} - \right. \\
& e^{\gamma\alpha_2-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2\alpha_2)} f_{\theta,x23'}[\lambda] + \frac{1}{2} e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2(\alpha_2+\alpha_2))} \gamma \in f_{1,a1}[\lambda] f_{\theta,x23'}[\lambda] - \\
& e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2(\alpha_2+\alpha_2))} \gamma \in f_{1,a2}[\lambda] f_{\theta,x23'}[\lambda] - e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_1-2(\alpha_2+\alpha_2))} \in f_{1,x23'}[\lambda] \Big) + \\
& U_k[x12, x23] \left(-2 e^{\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_i} + \right. \\
& 2 e^{\frac{3\gamma\alpha_2}{2}+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_j} \xi_{23_i} - 2 e^{\frac{3\gamma\alpha_1}{2}+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_j} - \\
& 2 e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_j} \xi_{23_j} + 2 e^{\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \\
& \gamma \in \hbar f_{\theta,x23}[\lambda] f_{\theta,x12'}[\lambda] - e^{\gamma(\alpha_1+\alpha_2)+\frac{1}{2}\gamma(-\alpha_1+\alpha_1-\alpha_2+\alpha_2)} \in f_{1,x12\ x23'}[\lambda] \Big) + \\
& U_k[x13, x23] \left(2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1-2\alpha_2-3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_i} + \right. \\
& 2 e^{\frac{1}{2}\gamma(\alpha_1+\alpha_2+3\alpha_2)} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_j} + 2 e^{\frac{3\gamma\alpha_2}{2}} \gamma \in \lambda \hbar \xi_{13_j} \xi_{23_j} - 2 e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \gamma \in \hbar \\
& f_{\theta,x23}[\lambda] f_{\theta,x13'}[\lambda] - e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{\theta,x23}[\lambda] f_{1,x12\ x23'}[\lambda] - e^{\frac{3}{2}\gamma(\alpha_2+\alpha_2)} \in f_{1,x13\ x23'}[\lambda] \Big)
\end{aligned}$$

```
In[*]:= Mons[p_] := Union@Cases[Coefficient[diff, e, p], Uk[___], ∞];
Mons /@ Range[0, $k]
```

Time: 0.05 seconds Out:=

```
{ {Uk[x12], Uk[x13], Uk[x23]}, {Uk[a1], Uk[a2], Uk[x12],
  Uk[x13], Uk[x23], Uk[x12, x13], Uk[x12, x23], Uk[x13, x13], Uk[x13, x23]} }
```

```
In[*]:= diff /. e -> 0
```

Time: 0.00 seconds Out:=

$$\begin{aligned} & U_k[x12] \left(e^{\gamma \alpha_{1i} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \xi_{12i} + e^{\frac{\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \xi_{12j} - e^{\gamma \alpha_{1i} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} f_{\theta, x12}'[\lambda] \right) + \\ & U_k[x13] \left(e^{\gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \xi_{13i} + e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \xi_{13j} + \right. \\ & \quad e^{\gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \lambda \xi_{12i} \xi_{23i} - e^{\frac{3\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \lambda \xi_{12j} \xi_{23i} + \\ & \quad e^{\frac{3\gamma \alpha_{1i}}{2} + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \lambda \xi_{12i} \xi_{23j} + e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} \lambda \xi_{12j} \xi_{23j} - \\ & \quad \left. e^{\gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} f_{\theta, x23}[\lambda] f_{\theta, x12}'[\lambda] - e^{\gamma (\alpha_{1i} + \alpha_{2i}) + \frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} - \alpha_{2i} + \alpha_{2j})} f_{\theta, x13}'[\lambda] \right) + \\ & U_k[x23] \left(e^{\gamma \alpha_{2i} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 2\alpha_{2j})} \xi_{23i} + e^{\frac{\gamma \alpha_{1i}}{2} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 2\alpha_{2j})} \xi_{23j} - e^{\gamma \alpha_{2i} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} - 2\alpha_{2j})} f_{\theta, x23}'[\lambda] \right) \end{aligned}$$

```
In[*]:= mons[0][[1, 1]]
```

Time: 0.00 seconds Out:=

x12

```
In[*]:= {sol0} =
```

```
DSolve[And@@Table[As[m, Mons[0]], Coefficient[diff /. e -> 0, m] == 0 && f_{\theta, m[[1]]}[0] == 0],
  Table[As[m, Mons[0]], f_{\theta, m[[1]]}, \lambda]
```

Time: 1.59 seconds Out:=

$$\begin{aligned} & \left\{ \left\{ f_{\theta, x12} \rightarrow \text{Function}[\{\lambda\}, e^{-\gamma \alpha_{1i}} \lambda \left(e^{\gamma \alpha_{1i}} \xi_{12i} + e^{\frac{\gamma \alpha_{2i}}{2}} \xi_{12j} \right) \right\}, \right. \\ & \quad f_{\theta, x13} \rightarrow \text{Function}[\{\lambda\}, -e^{-\gamma \alpha_{1i} - \gamma \alpha_{2i} - \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \lambda \left(-e^{\gamma \alpha_{1i} + \gamma \alpha_{2i} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \xi_{13i} - \right. \\ & \quad e^{\gamma \alpha_{1i} + \gamma \alpha_{2i}} \xi_{13j} + e^{\frac{3\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \lambda \xi_{12j} \xi_{23i} - e^{2\gamma \alpha_{1i} + \frac{\gamma \alpha_{2i}}{2}} \lambda \xi_{12i} \xi_{23j} + \\ & \quad \left. e^{\frac{3\gamma \alpha_{1i}}{2} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \lambda \xi_{12i} \xi_{23j} - e^{\gamma \alpha_{1i} + \gamma \alpha_{2i}} \lambda \xi_{12j} \xi_{23j} + e^{\frac{\gamma \alpha_{1i}}{2} + \frac{\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (\alpha_{1i} + \alpha_{2i})} \lambda \xi_{12j} \xi_{23j} \right) \right\}, \\ & \quad \left. f_{\theta, x23} \rightarrow \text{Function}[\{\lambda\}, e^{-\gamma \alpha_{2i}} \lambda \left(e^{\gamma \alpha_{2i}} \xi_{23i} + e^{\frac{\gamma \alpha_{1i}}{2}} \xi_{23j} \right) \right] \right\} \end{aligned}$$

```
In[*]:= diff1 = UCF[diff / e /. sol0]
```

Time: 7.34 seconds Out:=

$$\begin{aligned} & -U_k[a1] f_{1, a1}'[\lambda] - U_k[a2] f_{1, a2}'[\lambda] + \\ & U_k[x12] \left(-e^{\gamma \alpha_{1i} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \gamma \xi_{12i} f_{1, a1}[\lambda] - e^{\frac{\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \gamma \xi_{12j} f_{1, a1}[\lambda] + \right. \\ & \quad \frac{1}{2} e^{\gamma \alpha_{1i} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \gamma \xi_{12i} f_{1, a2}[\lambda] + \\ & \quad \left. \frac{1}{2} e^{\frac{\gamma \alpha_{2i}}{2} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \gamma \xi_{12j} f_{1, a2}[\lambda] - e^{\gamma \alpha_{1i} + \frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} f_{1, x12}'[\lambda] \right) + \\ & U_k[x13] \left(-e^{\gamma \alpha_{1i} + \gamma \alpha_{2i} - \frac{1}{2} \gamma (\alpha_{1i} - \alpha_{1j} + \alpha_{2i} - \alpha_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} + e^{\frac{1}{2} \gamma (\alpha_{1i} + \alpha_{1j} + \alpha_{2i} + \alpha_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} - \right. \\ & \quad \left. e^{\frac{3\gamma \alpha_{2i}}{2} - \frac{1}{2} \gamma (\alpha_{1i} - \alpha_{1j} + \alpha_{2i} - \alpha_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} - e^{\frac{1}{2} \gamma (-\alpha_{1i} + \alpha_{1j} + 2\alpha_{2i} + \alpha_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} - \right. \end{aligned}$$

$$\begin{aligned}
& e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23j} + e^{\frac{1}{2}\gamma(2a_{1i}+a_{1j}-a_{2i}+a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23j} - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23j} + e^{\frac{1}{2}\gamma(a_{1j}+a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23j} - \\
& \frac{1}{2} e^{\gamma a_{1i} + \frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \xi_{13i} f_{1,a1}[\lambda] - \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \xi_{13j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \gamma a_{2i} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} f_{1,a1}[\lambda] + e^{\gamma a_{2i} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a1}[\lambda] - \frac{1}{2} e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12i} \xi_{23j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \xi_{13i} f_{1,a2}[\lambda] - \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \xi_{13j} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \gamma a_{2i} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} f_{1,a2}[\lambda] + e^{\gamma a_{2i} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a2}[\lambda] - \frac{1}{2} e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12i} \xi_{23j} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}+a_{2i}-a_{2j})} \gamma \lambda \xi_{12j} \xi_{23j} f_{1,a2}[\lambda] - e^{\gamma a_{1i} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2i}-a_{2j})} \xi_{12i} f_{1,x23}[\lambda] - \\
& e^{\frac{\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-a_{1j}-a_{2i}-a_{2j})} \xi_{12j} f_{1,x23}[\lambda] - e^{\gamma a_{2i} + \frac{1}{2}\gamma(a_{1i}+a_{1j}-a_{2i}+a_{2j})} \lambda \xi_{23i} f_{1,x12'}[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{1}{2}\gamma(a_{1i}+a_{1j}-a_{2i}+a_{2j})} \lambda \xi_{23j} f_{1,x12'}[\lambda] - e^{\frac{1}{2}\gamma(a_{1i}+a_{1j}+a_{2i}+a_{2j})} f_{1,x13'}[\lambda] \Big) + U_k[x_{12}, x_{13}] \\
& \left(-4 e^{\gamma(a_{1i}+a_{2i}) - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} \gamma \lambda \hbar \xi_{12j} \xi_{13i} + 2 e^{\gamma(a_{1i}+a_{2i}) - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{12j} \xi_{23i} + \right. \\
& 2 e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} \gamma \lambda^2 \hbar \xi_{12j}^2 \xi_{23i} - e^{\frac{1}{2}\gamma(4a_{1i}+a_{2i}) - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} \xi_{12i} f_{1,x12x23}[\lambda] - \\
& \left. e^{\gamma(a_{1i}+a_{2i}) - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} \xi_{12j} f_{1,x12x23}[\lambda] - e^{\frac{1}{2}\gamma(4a_{1i}+a_{2i}) - \frac{1}{2}\gamma(a_{1i}-3a_{1j}+a_{2i})} f_{1,x12x13'}[\lambda] \right) + \\
& U_k[x_{13}, x_{13}] \left(2 e^{\gamma(a_{1i}+a_{1j}+a_{2i}+a_{2j})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} - 2 e^{\gamma a_{1i} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} - \right. \\
& 4 e^{\gamma a_{1j} + \frac{3\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23i} - 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13j} \xi_{23i} - \\
& 2 e^{-\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23i} + 2 e^{\gamma a_{1j} + \frac{3\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12i} \xi_{12j} \xi_{23i}^2 + \\
& 2 e^{-\gamma a_{1i} + \gamma a_{1j} + 2\gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12j}^2 \xi_{23i}^2 - 2 e^{\frac{3\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23j} + \\
& 2 e^{\frac{3\gamma a_{1i}}{2} + \gamma(a_{1j}+a_{2j})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23j} - 2 e^{\frac{1}{2}\gamma(a_{1i}+2a_{1j}+a_{2i}+2a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23j} - \\
& 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23j} + 2 e^{\gamma(a_{1j}+a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23j} - \\
& 2 e^{\gamma a_{1j} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23j} + 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12i} \xi_{12j} \xi_{23i} \xi_{23j} + \\
& 2 e^{-\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12j}^2 \xi_{23i} \xi_{23j} - e^{\gamma a_{1i} + \gamma a_{2i} + \gamma(a_{1j}+a_{2j})} \lambda \xi_{12i} \xi_{23i} f_{1,x12x23}[\lambda] - \\
& e^{\frac{3\gamma a_{2i}}{2} + \gamma(a_{1j}+a_{2j})} \lambda \xi_{12j} \xi_{23i} f_{1,x12x23}[\lambda] - e^{\frac{3\gamma a_{1i}}{2} + \gamma(a_{1j}+a_{2j})} \lambda \xi_{12i} \xi_{23j} f_{1,x12x23}[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} + \gamma(a_{1j}+a_{2j})} \lambda \xi_{12j} \xi_{23j} f_{1,x12x23}[\lambda] - e^{\gamma a_{1i} + \gamma(a_{1j}+a_{2i}+a_{2j})} \xi_{12i} f_{1,x13x23}[\lambda] - \\
& e^{\frac{\gamma a_{2i}}{2} + \gamma(a_{1j}+a_{2i}+a_{2j})} \xi_{12j} f_{1,x13x23}[\lambda] - e^{\gamma a_{2i} + \gamma(a_{1i}+a_{1j}+a_{2j})} \lambda \xi_{23i} f_{1,x12x13'}[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \gamma(a_{1i}+a_{1j}+a_{2j})} \lambda \xi_{23j} f_{1,x12x13'}[\lambda] - e^{\gamma(a_{1i}+a_{1j}+a_{2i}+a_{2j})} f_{1,x13^2}[\lambda] \Big) +
\end{aligned}$$

$$\begin{aligned}
& U_k[x23] \left(\frac{1}{2} e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23i} f_{1,a1}[\lambda] + \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23j} f_{1,a1}[\lambda] - \right. \\
& e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23i} f_{1,a2}[\lambda] - \\
& \left. e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23j} f_{1,a2}[\lambda] - e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} f_{1,x23}'[\lambda] \right) + \\
& U_k[x12, x23] \left(4 e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23i} - e^{\gamma (a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} f_{1,x12 x23}'[\lambda] \right) + \\
& U_k[x13, x23] \left(-4 e^{\frac{1}{2} \gamma (a_{1i} + a_{2i}) + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda \hbar \xi_{13j} \xi_{23i} + \right. \\
& 4 e^{\frac{3\gamma a_{2i}}{2} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}^2 + 4 e^{\frac{1}{2} \gamma (a_{1i} + a_{2i}) + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i} \xi_{23j} - \\
& e^{\gamma a_{1i} + \gamma a_{2i} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \lambda \xi_{23i} f_{1,x12 x23}'[\lambda] - \\
& \left. e^{\frac{3\gamma a_{1i}}{2} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \lambda \xi_{23j} f_{1,x12 x23}'[\lambda] - e^{\gamma a_{1i} + \gamma a_{2i} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} f_{1,x13 x23}'[\lambda] \right)
\end{aligned}$$

In[*]:= **M1 = Join[mons[1], Table[Uk[a[j]], {j, \$n - 1}]]**

Time: 0.05 seconds Out[]=

{Uk[x13], Uk[x12, x13], Uk[x12, x23], Uk[x13, x13], Uk[x13, x23], Uk[a1], Uk[a2]}

In[*]:= **And@@Table[As[m, Mons[1]], Coefficient[diff1, m] == 0 && (m /. Uk[z___] => f1, Times[z]) [0] == 0]**

Time: 0.06 seconds Out[]=

$$\begin{aligned}
& -f_{1,a1}'[\lambda] == 0 \& f_{1,a1}[0] == 0 \& -f_{1,a2}'[\lambda] == 0 \& f_{1,a2}[0] == 0 \& \\
& -e^{\gamma a_{1i} + \frac{1}{2} \gamma (2 a_{1j} - a_{2i} - a_{2j})} \gamma \xi_{12i} f_{1,a1}[\lambda] - e^{\frac{\gamma a_{2i}}{2} + \frac{1}{2} \gamma (2 a_{1j} - a_{2i} - a_{2j})} \gamma \xi_{12j} f_{1,a1}[\lambda] + \frac{1}{2} e^{\gamma a_{1i} + \frac{1}{2} \gamma (2 a_{1j} - a_{2i} - a_{2j})} \gamma \\
& \xi_{12i} f_{1,a2}[\lambda] + \frac{1}{2} e^{\frac{\gamma a_{2i}}{2} + \frac{1}{2} \gamma (2 a_{1j} - a_{2i} - a_{2j})} \gamma \xi_{12j} f_{1,a2}[\lambda] - e^{\gamma a_{1i} + \frac{1}{2} \gamma (2 a_{1j} - a_{2i} - a_{2j})} f_{1,x12}'[\lambda] == 0 \& \\
& f_{1,x12}[0] == 0 \& -e^{\gamma a_{1i} + \gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23i} + e^{\frac{1}{2} \gamma (a_{1i} + a_{1j} + a_{2i} + a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23i} - \\
& e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23i} - e^{\frac{1}{2} \gamma (-a_{1i} + a_{1j} + 2 a_{2i} + a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23i} - \\
& e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23j} + e^{\frac{1}{2} \gamma (2 a_{1i} + a_{1j} - a_{2i} + a_{2j})} \gamma \lambda \hbar \xi_{12i} \xi_{23j} - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23j} + e^{\frac{1}{2} \gamma (a_{1j} + a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23j} - \\
& \frac{1}{2} e^{\gamma a_{1i} + \frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \xi_{13i} f_{1,a1}[\lambda] - \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \xi_{13j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} f_{1,a1}[\lambda] + e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a1}[\lambda] - \frac{1}{2} e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12i} \xi_{23j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23j} f_{1,a1}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \xi_{13i} f_{1,a2}[\lambda] - \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \xi_{13j} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\gamma a_{1i} + \gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12i} \xi_{23i} f_{1,a2}[\lambda] + e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23i} f_{1,a2}[\lambda] - \frac{1}{2} e^{\frac{3\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12i} \xi_{23j} f_{1,a2}[\lambda] - \\
& \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \xi_{12j} \xi_{23j} f_{1,a2}[\lambda] - e^{\gamma a_{1i} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2i} - a_{2j})} \xi_{12i} f_{1,x23}[\lambda] -
\end{aligned}$$

$$\begin{aligned}
& e^{\frac{\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} - a_{2i} - a_{2j})} \xi_{12j} f_{1,x23}[\lambda] - e^{\gamma a_{2i} + \frac{1}{2} \gamma (a_{1i} + a_{1j} - a_{2i} + a_{2j})} \lambda \xi_{23i} f_{1,x12}'[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{1}{2} \gamma (a_{1i} + a_{1j} - a_{2i} + a_{2j})} \lambda \xi_{23j} f_{1,x12}'[\lambda] - e^{\frac{1}{2} \gamma (a_{1i} + a_{1j} + a_{2i} + a_{2j})} f_{1,x13}'[\lambda] = 0 \& f_{1,x13}[0] = 0 \& \\
& \frac{1}{2} e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23i} f_{1,a1}[\lambda] + \frac{1}{2} e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23j} f_{1,a1}[\lambda] - \\
& e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23i} f_{1,a2}[\lambda] - e^{\frac{\gamma a_{1i}}{2} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} \gamma \xi_{23j} f_{1,a2}[\lambda] - \\
& e^{\gamma a_{2i} - \frac{1}{2} \gamma (a_{1i} + a_{1j} - 2 a_{2j})} f_{1,x23}'[\lambda] = 0 \& f_{1,x23}[0] = 0 \& \\
& -4 e^{\gamma (a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} \gamma \lambda \hbar \xi_{12j} \xi_{13i} + 2 e^{\gamma (a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{12j} \xi_{23i} + \\
& 2 e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} \gamma \lambda^2 \hbar \xi_{12j}^2 \xi_{23i} - e^{\frac{1}{2} \gamma (4 a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} \xi_{12i} f_{1,x12 \times 23}[\lambda] - \\
& e^{\gamma (a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} \xi_{12j} f_{1,x12 \times 23}[\lambda] - \\
& e^{\frac{1}{2} \gamma (4 a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - 3 a_{1j} + a_{2i})} f_{1,x12 \times 13}'[\lambda] = 0 \& f_{1,x12 \times 13}[0] = 0 \& \\
& 4 e^{\frac{3\gamma a_{2i}}{2} - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} \gamma \lambda \hbar \xi_{12j} \xi_{23i} - e^{\gamma (a_{1i} + a_{2i}) - \frac{1}{2} \gamma (a_{1i} - a_{1j} + a_{2i} - a_{2j})} f_{1,x12 \times 23}'[\lambda] = 0 \& \\
& f_{1,x12 \times 23}[0] = 0 \& \\
& 2 e^{\gamma (a_{1i} + a_{1j} + a_{2i} + a_{2j})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} - 2 e^{\gamma a_{1i} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23i} - \\
& 4 e^{\gamma a_{1j} + \frac{3\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23i} - 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13j} \xi_{23i} - \\
& 2 e^{-\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23i} + 2 e^{\gamma a_{1j} + \frac{3\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12i} \xi_{12j} \xi_{23i}^2 + \\
& 2 e^{-\gamma a_{1i} + \gamma a_{1j} + 2 \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12j}^2 \xi_{23i}^2 - 2 e^{\frac{3\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23j} + \\
& 2 e^{\frac{3\gamma a_{1i}}{2} + \gamma (a_{1j} + a_{2j})} \gamma \lambda^2 \hbar \xi_{12i} \xi_{13i} \xi_{23j} - 2 e^{\frac{1}{2} \gamma (a_{1i} + 2 a_{1j} + a_{2i} + 2 a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23j} - \\
& 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i} \xi_{23j} + 2 e^{\gamma (a_{1j} + a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23j} - \\
& 2 e^{\gamma a_{1j} + \gamma a_{2j}} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13j} \xi_{23j} + 2 e^{\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \frac{\gamma a_{2i}}{2} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12i} \xi_{12j} \xi_{23i} \xi_{23j} + \\
& 2 e^{-\frac{\gamma a_{1i}}{2} + \gamma a_{1j} + \gamma a_{2i} + \gamma a_{2j}} \gamma \lambda^3 \hbar \xi_{12j}^2 \xi_{23i} \xi_{23j} - e^{\gamma a_{1i} + \gamma a_{2i} + \gamma (a_{1j} + a_{2j})} \lambda \xi_{12i} \xi_{23i} f_{1,x12 \times 23}[\lambda] - \\
& e^{\frac{3\gamma a_{2i}}{2} + \gamma (a_{1j} + a_{2j})} \lambda \xi_{12j} \xi_{23i} f_{1,x12 \times 23}[\lambda] - e^{\frac{3\gamma a_{1i}}{2} + \gamma (a_{1j} + a_{2j})} \lambda \xi_{12i} \xi_{23j} f_{1,x12 \times 23}[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \frac{\gamma a_{2i}}{2} + \gamma (a_{1j} + a_{2j})} \lambda \xi_{12j} \xi_{23j} f_{1,x12 \times 23}[\lambda] - e^{\gamma a_{1i} + \gamma (a_{1j} + a_{2i} + a_{2j})} \xi_{12i} f_{1,x13 \times 23}[\lambda] - \\
& e^{\frac{\gamma a_{2i}}{2} + \gamma (a_{1j} + a_{2i} + a_{2j})} \xi_{12j} f_{1,x13 \times 23}[\lambda] - e^{\gamma a_{2i} + \gamma (a_{1i} + a_{1j} + a_{2j})} \lambda \xi_{23i} f_{1,x12 \times 13}'[\lambda] - \\
& e^{\frac{\gamma a_{1i}}{2} + \gamma (a_{1i} + a_{1j} + a_{2j})} \lambda \xi_{23j} f_{1,x12 \times 13}'[\lambda] - e^{\gamma (a_{1i} + a_{1j} + a_{2i} + a_{2j})} f_{1,x13}^2[\lambda] = 0 \& f_{1,x13}^2[0] = 0 \& \\
& -4 e^{\frac{1}{2} \gamma (a_{1i} + a_{2i}) + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda \hbar \xi_{13j} \xi_{23i} + 4 e^{\frac{3\gamma a_{2i}}{2} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}^2 + \\
& 4 e^{\frac{1}{2} \gamma (a_{1i} + a_{2i}) + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i} \xi_{23j} - \\
& e^{\gamma a_{1i} + \gamma a_{2i} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \lambda \xi_{23i} f_{1,x12 \times 23}'[\lambda] - e^{\frac{3\gamma a_{1i}}{2} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} \lambda \xi_{23j} f_{1,x12 \times 23}'[\lambda] - \\
& e^{\gamma a_{1i} + \gamma a_{2i} + \frac{1}{2} \gamma (-2 a_{1i} + a_{2i} + 3 a_{2j})} f_{1,x13 \times 23}'[\lambda] = 0 \& f_{1,x13 \times 23}[0] = 0
\end{aligned}$$

In[*]:= {sol1} = DSolve[And@@

Table[As[m, Mons[1]], Coefficient[diff1, m] == 0 && (m /. Uk[z___] => f1, Times[z])[0] == 0],
Table[As[m, Mons[1]], (m /. Uk[z___] => f1, Times[z])], λ]

Set: Part 4 of Asymptotics`AsymptoticRSolveValueDump`c\$32728[[Asymptotics`AsymptoticRSolveValueDump`i\$32728]] does not exist.

Set: Part 5 of Asymptotics`AsymptoticRSolveValueDump`c\$32728[[Asymptotics`AsymptoticRSolveValueDump`i\$32728]] does not exist.

Part: Part 4 of {0, 0} does not exist.

Part: Part 5 of {0, 0} does not exist.

Set: Part 4 of Asymptotics`AsymptoticRSolveValueDump`c\$32728[[Asymptotics`AsymptoticRSolveValueDump`i\$32728]] does not exist.

General: Further output of Set::partw will be suppressed during this calculation.

Part: Part 4 of {0, 0} does not exist.

General: Further output of Part::partw will be suppressed during this calculation.

Out[*]=

$$\left\{ \left\{ \begin{aligned} &f_{1,a1} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,a2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x12} \rightarrow \text{Function}[\{\lambda\}, 0], \\ &f_{1,x23} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x13} \rightarrow \text{Function}\left[\{\lambda\}, -e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}\right], \\ &f_{1,x12 \times 13} \rightarrow \text{Function}\left[\{\lambda\}, -2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{13i}\right], \\ &f_{1,x13^2} \rightarrow \text{Function}[\{\lambda\}, 0], f_{1,x12 \times 23} \rightarrow \text{Function}\left[\{\lambda\}, 2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{12j} \xi_{23i}\right], \\ &f_{1,x13 \times 23} \rightarrow \text{Function}\left[\{\lambda\}, -2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \lambda^2 \hbar \xi_{13j} \xi_{23i}\right] \end{aligned} \right\} \right\}$$

In[*]:= UCF[diff1 /. sol1]

Time: 0.12 seconds Out[*]=

0

Multiplication tensor m_A

In[*]:= Collect[Simplify[ff. (#k & /@ \$PBWGens) + gg2 /. sol0 /. sol1 /. $\lambda \rightarrow 1$], e]

Time: 0.58 seconds Out[*]=

$$\begin{aligned} &a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \xi_{12i} + e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} x_{12k} \xi_{12j} + \\ &x_{13k} \xi_{13i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} x_{13k} \xi_{13j} + x_{23k} \xi_{23i} - e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} x_{13k} \xi_{12j} \xi_{23i} + \\ &\in \left(-2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \hbar x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \hbar x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2}\gamma(-2\alpha_1+\alpha_2)} \gamma \right. \\ &\quad \left. \hbar x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \hbar x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right) + e^{\frac{1}{2}\gamma(\alpha_1-2\alpha_2)} x_{23k} \xi_{23j} \end{aligned}$$

Now Δ_A

In[*]:= $\Delta_{1 \rightarrow 2, 3}[\lambda U_1[x_{12}]]$

Time: 0.00 seconds Out[*]=

$$\lambda U_2[x_{12}] U_3[] + \epsilon \lambda \hbar U_2[x_{12}] U_3[a_1] + \lambda U_2[] U_3[x_{12}] + \epsilon \lambda \hbar U_2[a_1] U_3[x_{12}]$$

In[*]:= Ad[$\Delta_{1 \rightarrow 2, 3}[\lambda U_1[x_{12}]]$][$\Delta_{1 \rightarrow 2, 3}[\lambda U_1[x_{12}]]$]

Time: 0.05 seconds Out[*]=

$$\lambda U_2[x_{12}] U_3[] + \epsilon \lambda \hbar U_2[x_{12}] U_3[a_1] + \lambda U_2[] U_3[x_{12}] + \epsilon \lambda \hbar U_2[a_1] U_3[x_{12}]$$

In[*]:= Ad[$\pi \mathbf{i} U_j[a_1]$][$U_j[x_{23}] U_k[a_4]$]

Time: 0.00 seconds Out[*]=

$$e^{-\frac{1}{2} \mathbf{i} \pi \gamma} U_j[x_{23}] U_k[a_4]$$

In[*]:= Ad[$U_j[x_{12}]$][$U_j[x_{23}] U_k[a_4]$]

Time: 0.01 seconds Out[*]=

$$\text{Ad}[U_j[x_{12}]] [U_j[x_{23}] U_k[a_4]]$$

In[*]:= $\text{Ad}[U_j[x_{12}]] [U_j[x_{23}] U_k[x_{12}]]$

Time: 0.02 seconds Out[]:=

$\text{Ad}[U_j[x_{12}]] [U_j[x_{23}] U_k[x_{12}]]$