

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_h(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

(Alt) In[]:=
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```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.

Read more at <http://katlas.org/wiki/KnotTheory>.

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This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

In[]:= **MakePDF** []

(Alt) In[]:=
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```
 $\delta_{ij} := \text{KroneckerDelta}[i, j];$ 
```

(Alt) In[]:=
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```
SetAttributes[{SS}, HoldAll];
SS[ $\mathcal{E}_-$ , op_] := Collect[Normal@Series[ $\mathcal{E}$ , { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[ $\mathcal{E}_-$ ] := SS[ $\mathcal{E}$ , Together];
```

(Alt) In[]:=
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```
$N = 9; (* We hereby commit that we will never go higher than sl(9) *)
x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
y[i_, j_] := y<>i<>j; b[i_] := b<>i; B[i_] := B<>i;
η[i_, j_] := η<>i<>j; β[i_] := β<>i;
τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_] := τ[s]; ℓ[s_] := ℓ[s];
```

In[]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm

Time: 0.03 seconds Out[]=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

Greek - Latin duality:

(Alt) In[]:=
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```
Do[As[i, 1, $N - 1], As[j, i + 1, $N], x[i, j]* = ξ[i, j];
  ξ[i, j]* = x[i, j]; y[i, j]* = η[i, j];
  η[i, j]* = y[i, j]]
Do[As[i, 1, $N - 1], A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i];
  b[i]* = β[i]; β[i]* = b[i]]
(vs_List)* := (v ↦ v*) /@ vs;
(u_{αβ})* := (u*)_{αβ};
```

(Alt) In[]:=
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```
LatinVars = Flatten@{Table[As[i, 1, $N - 1], As[j, i + 1, $N], {x[i, j], y[i, j]}],
  Table[As[i, 1, $N - 1], {a[i], b[i]}]};
GreekVars = LatinVars*;
LatinVarPatten = (Alternatives @@ LatinVars)_;
GreekVarPatten = (Alternatives @@ GreekVars)_;
VarPattern = (Alternatives @@ (LatinVars ∪ GreekVars))_;
```

(Alt) In[]:=
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```
CF[ $\mathcal{E}$ _] := PPCF@Module[{s1},
  s1 = If[$k ==  $\infty$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor@}p}$ ,
    Expand[ $\mathcal{E}$ ] /.  $e^{p-} \rightarrow e^{\text{Factor@}p}$  /.  $q^{n-} \rightarrow \sum_{k=0}^{\$k} \frac{(\hbar \in \gamma n)^k}{k!}$  /.  $e^{k-} \rightarrow 0$  ;  $k > \$k \rightarrow 0$ 
  ]
  (*Expand@Collect[s1,Cases[s1,VarPattern, $\infty$ ],CCF] /. CCF→Factor*)
]
```

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Weights: (Need confirmation!!!)

(Alt) In[]:=

```
Do[As[i, 1, $N - 1], As[j, i + 1, $N],
  Wt[x[i, j]] = {1, i - j}; Wt[y[i, j]] = {0, i - j};
  Wt[ $\xi$ [i, j]] = {0, j - i}; Wt[ $\eta$ [i, j]] = {1, j - i};
];
Do[As[i, 1, $N - 1],
  Wt[a[i]] = {1, 0}; Wt[b[i]] = {0, 0};
  Wt[ $\alpha$ [i]] = {0, 0}; Wt[ $\beta$ [i]] = {1, 0};
];
Wt[u_] := Wt[u];
```

(Alt) In[]:=
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```
(* q~ $e^{\hbar \gamma \epsilon}$ ; Ai[n]~ $e^{\hbar \in ai}$ ; *)
```

(Alt) In[]:=
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```
$n = 5; $k =  $\infty$ ; $E := {$n, $k}
```

(Alt) In[]:=
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```
$EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n], x[i, j]],
  Table[As[i, 1, $n - 1], {a[i], A[i][_]}]
}
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, gi → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y}] /. $EPBWRule; x_ < y_ := ! OrderedQ[{y, x}] /. $EPBWRule;
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

In[]:= \$PBWGens*

Time: 0.01 seconds Out[]:=

```
{ $\xi$ 12,  $\xi$ 13,  $\xi$ 14,  $\xi$ 15,  $\xi$ 23,  $\xi$ 24,  $\xi$ 25,  $\xi$ 34,  $\xi$ 35,  $\xi$ 45,  $\alpha$ 1,  $\alpha$ 2,  $\alpha$ 3,  $\alpha$ 4}
```

(Alt) In[]:=
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```
UCF[ $\mathcal{E}$ _] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] :=
      Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]]
  }],
  U[____], F
] /. F → CF]
```

(Alt) In[]:=
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```
ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j__[ $\mathcal{E}$ _] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k__[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k__[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k__[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k__[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}$ _] := ms[UCF[ $\mathcal{E}$ ]];
```

(Alt) In[]:=
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```
Supp[ $\mathcal{E}$ _] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] :=> i, ∞];
```

(Alt) In[]:=
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```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σi[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. ; k > $k :=> 0];
  Do[z = mσi,i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];
```

(Alt) In[]:=
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```

 $\Delta_{i \rightarrow j, k}[\mathcal{E}_-] := \text{UCF@Module}[\{j1, k1\}, \mathcal{E} /. \{$ 
 $U_i[] \rightarrow U_j[] U_k[],$ 
 $U_i[f_-, fs\_]\Rightarrow m_{k, k1 \rightarrow k} @ m_{j, j1 \rightarrow j} [\Delta_{j, k}[f] \Delta_{i \rightarrow j1, k1}[U_i[fs]]]$ 
 $\}];$ 
 $\Delta_{i \rightarrow j, k, l\_}[\mathcal{E}_-] := \mathcal{E} // \Delta_{i \rightarrow j, k} // \Delta_{k \rightarrow l, l}$ 

```

(Alt) In[]:=
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```

 $S_i[\mathcal{E}_-] := \text{UCF@Module}[\{i1\}, \mathcal{E} /. \{$ 
 $U_i[f_-, fs\_]\Rightarrow m_{i1, i \rightarrow i} [S_i[f] S_{i1}[U_{i1}[fs]]]$ 
 $\}];$ 

```

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The $\mathbb{A}$ -algebra

(Alt) In[]:=
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```

RRk[ai_, aj_] /;  $\tau[ai] == \tau[aj] == "a" := U_k[aj, ai];$ 
RRk[ai_, Aj_[n_]] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[Aj[n], ai];$ 
RRk[Aj_[n_], ai_] /;  $\tau[ai] == "a" \wedge \tau[Aj] == "A" := U_k[ai, Aj[n]];$ 
RRk[Ai_[m_], Aj_[n_]] /;  $\tau[Ai] == \tau[Aj] == "A" := U_k[Aj[n], Ai[m]];$ 
RRk[am_, xij_] /;  $\tau[am] == "a" \wedge \tau[xij] == "x" := \text{Module}[\{i, j, m\},$ 
 $\{i, j\} = \mathcal{L} @ xij; \{m\} = \mathcal{L} @ am;$ 
 $U_k[xij, am] + \gamma (\delta_{m, i} - \delta_{m+1, i} - \delta_{m, j} + \delta_{m+1, j}) U_k[xij] / 2$ 
 $];$ 
RRk[Am_[n_], xij_] /;  $\tau[Am] == "A" \wedge \tau[xij] == "x" := \text{Module}[\{i, j, m\},$ 
 $\{i, j\} = \mathcal{L} @ xij; \{m\} = \mathcal{L} @ Am; q^{-n (\delta_{n, i} - \delta_{n+1, i} - \delta_{n, j} + \delta_{n+1, j})} U_k[xij, Am[n]]$ 
 $]$ 

```

In[]:= RR₀[a[1], x[1, 2]]

Time: 0.05 seconds Out[]:=

$\gamma U_0[x12] + U_0[x12, a1]$

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We take the reduction rules RR for xij's from Yamane page

509:

- (I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,
- $$\begin{array}{ccc} \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$
- (IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.
- $$\begin{array}{ccc} \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} & \begin{array}{c} i \quad j \\ \hline m \quad n \end{array} \end{array}$$

$$\begin{aligned}
q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} &= 0 && \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\
[e_{ij}, e_{mn}] &= 0 && \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\
[e_{ij}, e_{mn}] &= (q^2 - q^{-2})e_{in}e_{mj} && \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\
q^2e_{ij}e_{mn} - e_{mn}e_{ij} &= qe_{in} && \text{if } ((i, j), (m, n)) \in C_{(V)}.
\end{aligned}$$

(Alt) In[*]:= pdf

```

RR_ [xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 UR[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), UR[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, UR[xij, xmn] + (q-2 - q2) UR[x[i, n], x[m, j]], (* IV *)
    j == m, q2 UR[xij, xmn] - q UR[x[i, n]] (* V *)
  ]
]

```

In[*]:= U₁[x34] U₂[x23] U₃[x12] // m_{1,2→2}

Time: 0.06 seconds Out[]=

-q U₂[x24] U₃[x12] + q² U₂[x23, x34] U₃[x12]

In[*]:= U₁[x34] U₂[x23] U₃[x12] // m_{1,2→2} // m_{2,3→0}

Time: 0.05 seconds Out[]=

q² U₀[x14] - q³ U₀[x12, x24] - q³ U₀[x13, x34] + q⁴ U₀[x12, x23, x34]

```

In[*]:= Block[{$k = ∞}, Module[{Bas = $PBWgens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]

```

Time: 1.05 seconds Out[]=

{ }

```

In[*]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]

```

» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[*]=

{ }

```

Module[{Bas = $PBWGens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 ->2 // m2,4 ->2 // m1,2 ->1) ==
      (P // m3,4 ->3 // m2,3 ->2 // m1,2 ->1) == (P // m1,2 ->1 // m3,4 ->3 // m1,3 ->1) ==
      (P // m1,2 ->1 // m1,3 ->1 // m1,4 ->1) == (P // m2,3 ->2 // m1,2 ->1 // m1,4 ->1)
  ]
]
» {x12, x13, x23, a1, a2}

Out[8]=
{ }

```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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According to DeltaUpTo9.nb, the following formulas produce an easy to see pattern.

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```

Do[As[i, 1, $n - 1],
  Δj,k[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj,k[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj,k[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj,k[x[i, n]] = mk,k1 ->k[mj,j1 ->j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
  ]]
]

```

```

In[*]:= Block[{ $k = ∞, $n = 5}, Column@Module[{ Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]]

Time: 0.11 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{ Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]]
]

» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}

Out[*]=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

(Alt) In[]:=
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```
Do[As[i, 1, 8],
  Sj-[a[i]] = -Uj[a[i]];
  Sj-[A[i][n]] = Uj[A[i][-n]];
  Sj-[x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, 7], As[n, i + 2, 9],
  Sj-[x[i, n]] = mj1,j→j[
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

In[]:= Block[{ \$k = ∞, \$n = 5}, Column@Module[{ Bas = \$PBWGens, P},
 Table[As[P, U_i /@ Bas], P → S_i[P]]
]]

Time: 0.05 seconds Out[]:=

$$\begin{aligned}
 U_i[x_{12}] &\rightarrow -\frac{U_i[x_{12}]}{q^2} \\
 U_i[x_{13}] &\rightarrow -\frac{U_i[x_{13}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{23}] \\
 U_i[x_{14}] &\rightarrow -\frac{U_i[x_{14}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{24}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{34}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{34}] \\
 U_i[x_{15}] &\rightarrow -\frac{U_i[x_{15}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{25}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{35}] + \\
 &\quad \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{14}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{23}, x_{35}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{12}, x_{24}, x_{45}] + \\
 &\quad \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{13}, x_{34}, x_{45}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q\right) U_i[x_{12}, x_{23}, x_{34}, x_{45}] \\
 U_i[x_{23}] &\rightarrow -\frac{U_i[x_{23}]}{q^2} \\
 U_i[x_{24}] &\rightarrow -\frac{U_i[x_{24}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{34}] \\
 U_i[x_{25}] &\rightarrow -\frac{U_i[x_{25}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{23}, x_{35}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{24}, x_{45}] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4}\right) U_i[x_{23}, x_{34}, x_{45}] \\
 U_i[x_{34}] &\rightarrow -\frac{U_i[x_{34}]}{q^2} \\
 U_i[x_{35}] &\rightarrow -\frac{U_i[x_{35}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{34}, x_{45}] \\
 U_i[x_{45}] &\rightarrow -\frac{U_i[x_{45}]}{q^2} \\
 U_i[a_1] &\rightarrow -U_i[a_1] \\
 U_i[a_2] &\rightarrow -U_i[a_2] \\
 U_i[a_3] &\rightarrow -U_i[a_3] \\
 U_i[a_4] &\rightarrow -U_i[a_4]
 \end{aligned}$$

In[]:= Block[{ \$k = ∞, \$n = 9}, S_i[U_i[x₁₉]]]

Time: 0.09 seconds Out[]:=

$$\begin{aligned}
 &-\frac{U_i[x_{19}]}{q^2} + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{12}, x_{29}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{13}, x_{39}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{14}, x_{49}] + \\
 &\quad \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{15}, x_{59}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{16}, x_{69}] + \left(-\frac{1}{q^5} + \frac{1}{q}\right) U_i[x_{17}, x_{79}] +
 \end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{q^5} + \frac{1}{q} \right) U_i[x18, x89] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x23, x39] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x24, x49] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x25, x59] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x26, x69] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x27, x79] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x12, x28, x89] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x13, x34, x49] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x13, x35, x59] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x13, x36, x69] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x13, x37, x79] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x13, x38, x89] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x14, x45, x59] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x14, x46, x69] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x14, x47, x79] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x14, x48, x89] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x15, x56, x69] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x15, x57, x79] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x15, x58, x89] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x16, x67, x79] + \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x16, x68, x89] + \\
& \left(-1 - \frac{1}{q^8} + \frac{2}{q^4} \right) U_i[x17, x78, x89] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x23, x34, x49] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x23, x35, x59] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x23, x36, x69] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x23, x37, x79] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x23, x38, x89] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x24, x45, x59] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x24, x46, x69] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x24, x47, x79] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x24, x48, x89] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x25, x56, x69] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x25, x57, x79] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x25, x58, x89] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x26, x67, x79] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x26, x68, x89] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x12, x27, x78, x89] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x13, x34, x45, x59] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x13, x34, x46, x69] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x13, x34, x47, x79] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i[x13, x34, x48, x89] +
\end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{35}, x_{56}, x_{69}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{35}, x_{57}, x_{79}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{35}, x_{58}, x_{89}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{36}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{36}, x_{68}, x_{89}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{13}, x_{37}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{45}, x_{56}, x_{69}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{45}, x_{57}, x_{79}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{45}, x_{58}, x_{89}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{46}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{46}, x_{68}, x_{89}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{14}, x_{47}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{15}, x_{56}, x_{67}, x_{79}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{15}, x_{56}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{15}, x_{57}, x_{78}, x_{89}] + \left(-\frac{1}{q^{11}} + \frac{3}{q^7} - \frac{3}{q^3} + q \right) U_i [x_{16}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{34}, x_{45}, x_{59}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{34}, x_{46}, x_{69}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{34}, x_{47}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{34}, x_{48}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{35}, x_{56}, x_{69}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{35}, x_{57}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{35}, x_{58}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{36}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{36}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{23}, x_{37}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{45}, x_{56}, x_{69}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{45}, x_{57}, x_{79}] +
\end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{45}, x_{58}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{46}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{46}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{24}, x_{47}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{25}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{25}, x_{56}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{25}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{12}, x_{26}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{45}, x_{56}, x_{69}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{45}, x_{57}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{45}, x_{58}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{46}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{46}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{34}, x_{47}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{35}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{35}, x_{56}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{35}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{13}, x_{36}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{14}, x_{45}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{14}, x_{45}, x_{56}, x_{68}, x_{89}] +
\end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{14}, x_{45}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{14}, x_{46}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{14}} + \frac{4}{q^{10}} - \frac{6}{q^6} + \frac{4}{q^2} - q^2 \right) U_i [x_{15}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{45}, x_{56}, x_{69}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{45}, x_{57}, x_{79}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{45}, x_{58}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{46}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{46}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{34}, x_{47}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{35}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{35}, x_{56}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{35}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{23}, x_{36}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{24}, x_{45}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{24}, x_{45}, x_{56}, x_{68}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{24}, x_{45}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{24}, x_{46}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{12}, x_{25}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{13}, x_{34}, x_{45}, x_{56}, x_{67}, x_{79}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i [x_{13}, x_{34}, x_{45}, x_{56}, x_{68}, x_{89}] +
\end{aligned}$$

$$\begin{aligned}
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i[x_{13}, x_{34}, x_{45}, x_{57}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i[x_{13}, x_{34}, x_{46}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i[x_{13}, x_{35}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{17}} + \frac{5}{q^{13}} - \frac{10}{q^9} + \frac{10}{q^5} - \frac{5}{q} + q^3 \right) U_i[x_{14}, x_{45}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{23}, x_{34}, x_{45}, x_{56}, x_{67}, x_{79}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{23}, x_{34}, x_{45}, x_{56}, x_{68}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{23}, x_{34}, x_{45}, x_{57}, x_{78}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{23}, x_{34}, x_{46}, x_{67}, x_{78}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{23}, x_{35}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{12}, x_{24}, x_{45}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(6 - \frac{1}{q^{20}} + \frac{6}{q^{16}} - \frac{15}{q^{12}} + \frac{20}{q^8} - \frac{15}{q^4} - q^4 \right) U_i[x_{13}, x_{34}, x_{45}, x_{56}, x_{67}, x_{78}, x_{89}] + \\
& \left(-\frac{1}{q^{23}} + \frac{7}{q^{19}} - \frac{21}{q^{15}} + \frac{35}{q^{11}} - \frac{35}{q^7} + \frac{21}{q^3} - 7q + q^5 \right) U_i[x_{12}, x_{23}, x_{34}, x_{45}, x_{56}, x_{67}, x_{78}, x_{89}]
\end{aligned}$$

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ -> True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X -> (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A2[-1], A1[1], A2[-1], A1[-2], A2[1], A1[2], A2[0], A2[2], A2[0]}

```

Out[]:=

```

{x12 -> 0, x13 -> 0, x23 -> 0, a1 -> 0, a2 -> 0, A1[-2] -> U1[],
A2[-1] -> U1[], A1[1] -> U1[], A2[-1] -> U1[], A1[-2] -> U1[],
A2[1] -> U1[], A1[2] -> U1[], A2[0] -> U1[], A2[2] -> U1[], A2[0] -> U1[]}

```

pdf

The Generating Functions Engine

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Canonical Forms:

(Alt) In[]:=
pdf

```
LogReduce[ $\mathcal{E}$ _] :=
 $\mathcal{E} /. c\_ * \text{Log}[a\_] \Rightarrow \text{Log@Factor}[a^c] /. \text{Log}[a\_] + \text{Log}[b\_] \Rightarrow \text{Log@Factor}[a b];$ 
CF[ $\mathcal{E}_E$ ] := CF /@ MapAt[LogReduce,  $\mathcal{E}$ , 1];
CF[ $\mathcal{E} : \mathbb{E}_{--}[\_\_\_]$ ] := CF /@  $\mathcal{E}$ ;
```

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$\mathbb{E}$ operations:

(Alt) In[]:=
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```
 $\mathcal{E}_E[\$] := \text{Length}[\mathcal{E}] - 1; \mathbb{E}_{[\mathcal{ES}_{--}]}[\$] := \mathbb{E}[\mathcal{ES}][\$];$ 
 $\mathcal{E}_E[k\_Integer] := \mathcal{E}[[k + 1]]; \mathbb{E}_{[\mathcal{ES}_{--}]}[k\_Integer] := \{\mathcal{ES}\}[[k + 1]];$ 
 $\mathbb{E} /: \mathcal{E1}_E \equiv \mathcal{E2}_E := \text{Inner}[\text{CF}\@ \#1 == \text{CF}\@ \#2 \ \&, \mathcal{E1}, \mathcal{E2}, \text{And}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}_{--}] \equiv \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}_{--}] \wedge := (d1 == d2) \wedge (r1 == r2) \wedge (\mathbb{E}[\mathcal{E1S}] \equiv \mathbb{E}[\mathcal{E2S}]);$ 
 $\mathbb{E} /: \mathcal{E1}_E * \mathcal{E2}_E := \mathbb{E} @@ \text{Table}[\text{CF}[\mathcal{E1}[\text{kk}] + \mathcal{E2}[\text{kk}]], \{\text{kk}, 0, \text{Min}[\mathcal{E1}[\$], \mathcal{E2}[\$]]\}];$ 
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}_{--}] \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}_{--}] \wedge := \mathbb{E}_{(d1 \cup d2) \rightarrow (r1 \cup r2)} @@ (\mathbb{E}[\mathcal{E1S}] \mathbb{E}[\mathcal{E2S}]);$ 
```

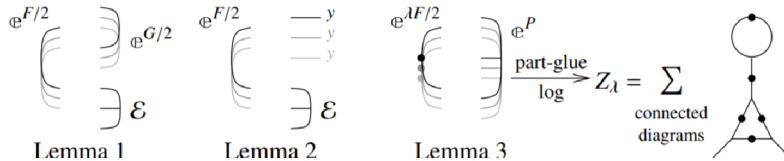
(Alt) In[]:=
pdf

```
 $\mathbb{E}_{d1 \rightarrow r1}[\mathcal{E1S}_{--}] // \mathbb{E}_{d2 \rightarrow r2}[\mathcal{E2S}_{--}] := \text{Module}[\{\text{is} = r1 \cap d2, \text{lvs}, \text{llvs}\},$ 
 $\text{lvs} = \text{Cases}[\{\mathcal{E1S}\}, (\text{Alternatives} @@ \text{LatinVars})_i /; \text{MemberQ}[\text{is}, i], \infty] \cup$ 
 $\text{Cases}[\{\mathcal{E2S}\}, (\text{Alternatives} @@ \text{GreekVars})_i /; \text{MemberQ}[\text{is}, i], \infty]^*;$ 
 $\text{llvs} = \text{lvs} /. u\_i \Rightarrow u_{\$i};$ 
 $\mathbb{E}_{(d1 \cup \text{Complement}[d2, \text{is}]) \rightarrow (r2 \cup \text{Complement}[r1, \text{is}])} @@ (\text{Zip}[\text{llvs} \cup \text{llvs}^*][\{(\text{llvs})^* . (\text{llvs}), \text{Times}[\mathbb{E}[\mathcal{E1S}] /. \text{Thread}[\text{lvs} \rightarrow \text{llvs}], \mathbb{E}[\mathcal{E2S}] /. \text{Thread}[\text{lvs}^* \rightarrow \text{llvs}^*]\}]])$ 
 $]$ 
```

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Ziping! Lemmas 2 and 3 are combined, yet they must be applied first to the middle weight variables and then to the heavy and light variables.

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Comment. Zip3 of the outer variables must occur after all other operations are completed, because we must allow for gluings of the weight n variables in perturbations with the weight 0 variables in the coefficients of Q .

dvs stands for “Diagonal Variables”. In gl_n they are the x_{ij} ’s and the y_{ij} ’s. They have weights $\{1,0\}$ and $\{0,0\}$.

(Alt) In[]:=
pdf

```
Zipvs[_{F_, E_}] := Module[{dvs = Select[vs, (Wt[#] == {0, 0} ∨ Wt[#] == {1, 0}) &]},
  {F, E} // Zip1vs( * // Zip2Complement[vs, dvs] * ) // Zip2dvs //
  Zip3Complement[vs, dvs] // Zip3dvs
] // Last
```

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Getting rid of the quadratic.

Lemma 1. With convergences left to the reader,

$$\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B$$

(Alt) In[]:=
pdf

```
Zip1{} = Identity;
Zip1vs[_{F_, E[Q_, P___]}] := Module[{I, F, G, u, v},
  I = IdentityMatrix@Length@vs;
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂u,vF, 0], {u, vs}, {v, vs}];
  G = Table[If[Wt[u] + Wt[v] == {1, 0}, ∂u,vQ, 0], {u, vs}, {v, vs}];
  {CF[(vs).(F.Inverse[I - G.F]).(vs)/2],
  E[CF[Q - PowerExpand@Log[Det[I - G.F]]/2 - vs.G.vs/2], P]}
]
```

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Getting rid of linear terms.

Lemma 2. $\left\langle F: \mathcal{E} \otimes^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \right\rangle_B = \mathbb{E}^{\frac{1}{2}} \sum_{i,j \in B} F_{ij} y_i y_j \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$.

(Alt) In[]:=
pdf

```
Zip2{} = Identity;
Zip2vs[_{F_, E[Q_, P___]}] := Module[{F, Y, u, v},
  F = Table[If[Wt[u] + Wt[v] == {1, 0}, CF[∂u,vF], 0], {u, vs}, {v, vs}];
  Y = Table[∂vQ, {v, vs}] /. Table[v → 0, {v, vs}];
  CF / @ {F, E[Q - Y.vs + Y.F.Y/2, P] /. Thread[vs → vs + F.Y]}
]
```

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Dealing with Feynman diagrams.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: \mathbb{E}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

Note that the power m of λ is at most $k - 1 + \frac{2k+2}{2} = 2k$. We write $Z_\lambda = \sum Z[m] \lambda^m$.

(Alt) In[]:=
pdf

```
Zip3vs_@{ $\mathcal{F}_-$ ,  $\mathcal{E}_-\mathcal{E}$ } := Module[{F, u, v, Z, kk, jj, $m = 0, m, n},
  Do[Z[0, kk] =  $\mathcal{E}$ [[kk + 1]], {kk, 0,  $\mathcal{E}@\$$ }]];
  F[u_, v_] := F[u, v] = CF@If[Wt[u] + Wt[v] == {1, 0},  $\partial_{u,v}\mathcal{F}$ , 0];
  Z[m_, kk_, u_] := Z[m, kk, u] =  $\partial_u Z[m, kk]$ ;
  Z[m_, kk_, u_, v_] := Z[m, kk, u, v] =  $\partial_v Z[m, kk, u]$ ;
  For[m = 0, m ≤ 2 $m, ++m, For[kk = 0, kk ≤  $\mathcal{E}@\$$ , ++kk,
    Z[m + 1, kk] = CF@Sum[
      If[F[u, v] == 0, 0,  $\frac{F[u, v]}{2 (m + 1)}$  (Z[m, kk, u, v] +
        Sum[Z[n, jj, u] * Z[m - n, kk - jj, v], {n, 0, m}, {jj, 0, kk}])],
      {u, vs}, {v, vs}];
    If[Z[m + 1, kk] != 0, $m = m + 1]
  ]];
  CF /@ ({
     $\mathcal{F}$  - Sum[F[u, v] * u * v / 2, {u, vs}, {v, vs}],
     $\mathcal{E}@@$ Table[Sum[Z[m, kk], {m, 0, $m}], {kk, 0,  $\mathcal{E}@\$$ }]
  } /. Table[v → 0, {v, vs}])
]
```

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“Define” Code

(Alt) In[]:=
pdf

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs]);
Define[op_is__ =  $\mathcal{E}_-$ ] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_nisp, $E, PPBoot@Block[{i, j, k}, op_isp, $E =  $\mathcal{E}$ ; op_nisp, $E]];
    SD[op_isp, op_{is, $E}]; SD[op_sis__, op_{sis}];
  ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
  } ] ]]
```

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The Generating Functions for the $\mathbb{A}$ operations

(Alt) In[]:=

BeginProfile[]

Time: 0.06 seconds Out[]:=

ProfileRoot

(Alt) In[]:=

\$k = 1;

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Ad, TT, Seepage

(Alt) In[]:=

pdf

```
PotQ[0] = True;
PotQ[c_. U_i_[aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

(Alt) In[]:=

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```
Ad[x_][y_] /; PotQ[x] ∧ (Head[y] != Plus) := If[y === 0, 0, PPAdPoty[eUB[y,x]/y y]];
Ad[x_][y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

(Alt) In[]:=

pdf

```
Ad[x_][y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

In[]:= \$k = 2

Time: 0.00 seconds Out[]:=

2

In[]:= Ad[ε ħ U_j[x₁₂] U_k[]][U_j[x₂₃] U_k[a₄]]

Time: 0.06 seconds Out[]:=

$$(-\epsilon \hbar - \gamma \epsilon^2 \hbar^2) U_j[x_{13}] U_k[a_4] + U_j[x_{23}] U_k[a_4] + 2 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, x_{23}] U_k[a_4]$$
In[]:= PotQ[7 ħ U_j[x₁₂] U_k[a₃]]

Time: 0.02 seconds Out[]:=

PotQ[7 ħ U_j[x₁₂] U_k[a₃]]

In[*]:= **Ad**[7 $\hbar$ U_j [**x12**] U_k [**a3**]] [**U_j**[**x23**] U_k [**a4**]]

Time: 0.09 seconds Out[]=

$$U_j[x_{23}] U_k[a_4] + \left(-7\hbar - 7\gamma \in \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3 \right) U_j[x_{13}] U_k[a_3, a_4] + \\ \left(14\gamma \in \hbar^2 + 14\gamma^2 \epsilon^2 \hbar^3 \right) U_j[x_{12}, x_{23}] U_k[a_3, a_4] - 98\gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{13}] U_k[a_3, a_3, a_4] + \\ 98\gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{12}, x_{23}] U_k[a_3, a_3, a_4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x_{12}, x_{12}, x_{13}] U_k[a_3, a_3, a_3, a_4]$$

In[*]:= **Ad**[π $\mathbf{i}$ U_j [**a1**]] [**U_j**[**x23**] U_k [**a4**]]

Time: 0.05 seconds Out[]=

$$e^{\frac{i\pi\gamma}{2}} U_j[x_{23}] U_k[a_4]$$

(Alt) In[]:=
pdf

```
TTλ [] = 0; (* Translated Tangent *)
TTλ [xs___, x_] := PPTT@UCF[Ad[x][TTλ[xs]] + ∂λ x];
TTλ [xs_List] := TTλ @@ xs
```

In[*]:= **TT_λ**[λ ξ_{12} U_0 [**x12**] U_1 [], λ ξ_{23} U_0 [**x23**] U_1 []]

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[x_{12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[x_{13}] U_1[] + \xi_{23} U_0[x_{23}] U_1[]$$

(Alt) In[]:=
pdf

```
Oi [poly_] := Total [
  CoefficientRules[poly, #i & /@ $PBWGens] /.
  (p_ -> c_) :-> c Ui @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
];
Oi ,is__ [poly_] := Oi [Ois [poly]];
Pi [ε_] := ε /. Ui [xs___] :-> Times @@ (#i & /@ {xs});
Pi ,is__ [poly_] := Pi [Pis [poly]];
```

In[*]:= **O_i**[$x_{12}^2 x_{13} + 7 x_{23}^{17}$]

Time: 0.06 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] + \\ 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$$

In[*]:= **O_{i,j}**[$x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3$]

Time: 0.03 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] U_j[x_{24}] + \\ 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}] \\ U_j[x_{25}, x_{25}, x_{25}]$$

In[*]:= **P_{i,j}**@**O_{i,j}**[$x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3$]

Time: 0.06 seconds Out[]=

$$x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3$$

(Alt) In[]:=
pdf

```
Seepagei[{ }, c_] = c;
Seepagei[ξS_List, P_] := Module[{lx},
  lx = $PBWgens[Length@ξS];
  CF@Collect[P, lxi, Pi[Ad[Last[ξS] Ui[lx]]][0i[Seepagei[Most@ξS, #]]]]
  &]
];
Seepagei, is[ξS_List, ξSS_List, P_] := Seepagei[ξS, Seepageis[ξSS, P]]
```

In[]:= \$k = 1
Time: 0.00 seconds Out[]:=
1

In[]:= \$PBWgens*
Time: 0.01 seconds Out[]:=
{ξ₁₂, ξ₁₃, ξ₁₄, ξ₁₅, ξ₂₃, ξ₂₄, ξ₂₅, ξ₃₄, ξ₃₅, ξ₄₅, α₁, α₂, α₃, α₄}

In[]:= Seepage_i[\$PBWgens*, x_{35i}]
Time: 0.16 seconds Out[]:=
$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35i} + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \gamma \in \xi_{45} \hbar x_{35i} x_{45i}$$

In[]:= Seepage_{i,j}[#_i & /@ \$PBWgens*, #_j & /@ \$PBWgens*, x_{35i} x_{45j}]
Time: 0.25 seconds Out[]:=
$$e^{\frac{1}{2} \gamma (\alpha_{2i} - \alpha_{3i} + \alpha_{3j} - \alpha_{4i} - 2\alpha_{4j})} x_{35i} x_{45j} + 2 e^{-\frac{1}{2} \gamma (-\alpha_{2i} - \alpha_{3j} + 3\alpha_{4i} + 2\alpha_{4j})} \gamma \gamma \in \hbar x_{35i} x_{45i} x_{45j} \xi_{45i}$$

(Alt) In[]:=
pdf

```
Invertis[_] := Module[{A, B, U1, j}, (* ε=A U[] + ε B *)
  U1 = Times@@(U#[] & /@ {is});
  A = Coefficient[ε, U1] /. ε → 0;
  B = UCF[(ε - A U1)];
  UCF[
    A-1 (U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply@@ConstantArray[B/A, j]])]
]
```

In[]:= \$k = 3;
Invert_{i,j}[7 U_i[] U_j[] + ε U_i[x₁₂] U_j[x₂₃]]
Time: 0.09 seconds Out[]:=
$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \epsilon U_i[x_{12}] U_j[x_{23}] +$$

$$\frac{1}{343} \epsilon^2 U_i[x_{12}, x_{12}] U_j[x_{23}, x_{23}] - \frac{\epsilon^3 U_i[x_{12}, x_{12}, x_{12}] U_j[x_{23}, x_{23}, x_{23}]}{2401}$$

{\$k = 1, \$n = 3}
Time: 0.01 seconds Out[]:=
{2, 3}

(Alt) In[]:=
pdf

```
TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
  PPTriangularDSolve@Module[{ss = {}, reqns = eqns, rinit = inits, rfs = fs, s, p},
    While[rfs != {},
      {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
      {{s}} = DSolve[reqns[[p]] ∧ rinit[[p]], rfs[[p]], λ];
      AppendTo[ss, s /. F_Function → ReplacePart[F, 2 → CF[F[[2]]]]];
      reqns = Delete[reqns, p] /. s /. e_ == 0 → CF[e] == 0;
      rinit = Delete[rinit, p];
      rfs = Delete[rfs, p];
    ];
    ss
  ]
```

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$$\begin{aligned}
 L_\lambda &= \mathbb{C}\langle \lambda_{i_1} x \rangle \mathbb{C}\langle \lambda_{i_2} x \rangle & \text{Seepage is defined by: } \mathbb{C}\langle f_i x_i \rangle \mathcal{O}(\text{Seepage}(\{f_i\}, A)) &= \mathcal{O}(e^{f_i x_i} A) \\
 \text{TT}(L_\lambda) &= L' \partial_\lambda L_\lambda & S(B) &= \text{Seepage}(\{f_i\}, B). \\
 R_\lambda &= \mathbb{C}\langle f+g \rangle, \quad f = f_i x_i, \quad g = \sum_{|w|=1} g_w w & e^f = F, \quad e^g = G, \quad R_\lambda = \mathcal{O}(FG), \quad \partial_\lambda F = F \partial_\lambda f \\
 \text{TT}(R_\lambda) &= R'_\lambda \partial_\lambda R_\lambda = \mathcal{O}(FG)' \mathcal{O}(\partial_\lambda FG) = S(G)' \mathcal{O}(F)' \mathcal{O}(F(\partial_\lambda f G + \partial_\lambda G)) = S(G)' S(\partial_\lambda f G + \partial_\lambda G)
 \end{aligned}$$

(Alt) In[] :=
pdf

```

Define[amti,j→k = PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ES[s_][ε_] := ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uk[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk/@$PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@0k[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ (ff. (#k & /@$PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i,j}→{k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], (α[m]i + α[m]j) a[m]k, 0],
    Total[Cases[fs, fp,_] /. fp,u => u (fp,u /. ss)[λ] /. λ | h | γ → 1]
  ]
]
]

```

(Alt) In[] :=
pdf

```

Define[aΔti→j,k = PP"aΔt"[$k,$n] @Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]; ε);
  TTL = ES["TTL"] @
    TTλ@Table[UCF[If[τ[g] === "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] => Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" => If[p == 0, (u*)i, 0];
  ffk = ffj /. j → k;
  fs = Flatten[Table[
    fp,ℙj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙj,k[u][λ] ℙj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj] (Uk /@ $PBWGens) ∪ Uk] (Uj /@ $PBWGens)}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
  SG = ES["SG"] @PP"SG"@S[G];
  TTR = ES["TTR"] @PP"TTR"@UCF[
    Invertj,k[SG] ⊗ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{j,k} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n-1], α[m]i (a[m]j + a[m]k)], 0],
    Total[Cases[fs, fp,-]] /. fp,- => u (fp,u /. ss)[λ] /. λ | ℏ | γ → 1]
  ]
]]

```

In[*]:= { \$k = 1, \$n = 2 }; amt_{i,j→k}

$$(\text{amt}_{1,2 \rightarrow 2} // \text{amt}_{2,3 \rightarrow 1}) \equiv (\text{amt}_{2,3 \rightarrow 2} // \text{amt}_{1,2 \rightarrow 1})$$

Time: 0.01 seconds Out[*]=

$$\mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + x_{12k} \left(\xi_{12i} + e^{\alpha_{1i}} \xi_{12j} \right), 0 \right]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; amt_{i,j→k}

$$(\text{amt}_{1,2 \rightarrow 4} // \text{amt}_{4,3 \rightarrow 5}) \equiv (\text{amt}_{2,3 \rightarrow 4} // \text{amt}_{1,4 \rightarrow 5})$$

Time: 6.86 seconds Out[*]=

$$\begin{aligned} \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ \left. x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \right. \\ \left. - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \right. \\ \left. 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i} \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 2, \$n = 3 }; amt_{i,j→k}

(amt_{1,2→4} // amt_{4,3→5}) ≡ (amt_{2,3→4} // amt_{1,4→5})

Time: 83.27 seconds Out[]=

$$\begin{aligned} & E_{\{i,j\} \rightarrow \{k\}} \left[a_{1k} (\alpha_{1i} + \alpha_{1j}) + a_{2k} (\alpha_{2i} + \alpha_{2j}) + x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) + \right. \\ & \quad x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right), \\ & \quad - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{13k} \xi_{12j} \xi_{13i} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} - 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} x_{13k} x_{23k} \xi_{13j} \xi_{23i}, \\ & \quad - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} + \\ & \quad 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + \\ & \quad x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \quad x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \quad x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \quad x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \quad x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \quad x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \quad \left. x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) \right] \end{aligned}$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 2 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

$$\begin{aligned} & \gg \text{TTL } e^{-\gamma \alpha_{1i}} \xi_{12i} U_j [x_{12}] U_k [] - 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j [x_{12}] U_k [a_1] + \\ & \quad e^{-\gamma \alpha_{1i}} \xi_{12i} U_j [] U_k [x_{12}] + 2 e^{-\gamma \alpha_{1i}} \in \hbar \xi_{12i} U_j [a_1] U_k [x_{12}] \\ & \gg \text{SG } U_j [] U_k [] + e^{-\gamma \alpha_{1i}} \in U_j [x_{12}] U_k [a_1] f_{1,a_{1k} x_{12j}} [\lambda] + e^{-\gamma \alpha_{1i}} \in U_j [a_1] U_k [x_{12}] f_{1,a_{1j} x_{12k}} [\lambda] \\ & \gg \text{TTR } U_j [x_{12}] U_k [x_{12}] \left(-e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1j} x_{12k}} [\lambda] f_{0,x_{12j}'} [\lambda] - e^{-2\gamma \alpha_{1i}} \gamma \in f_{1,a_{1k} x_{12j}} [\lambda] f_{0,x_{12k}'} [\lambda] \right) + \\ & \quad \ll 3 \gg + e^{-\gamma \alpha_{1i}} \in U_j [a_1] U_k [x_{12}] f_{1,a_{1j} x_{12k}'} [\lambda] \end{aligned}$$

Out[*]=

$$E_{\{i\} \rightarrow \{j,k\}} [(a_{1j} + a_{1k}) \alpha_{1i} + x_{12j} \xi_{12i} + x_{12k} \xi_{12i}, -2 a_{1k} x_{12j} \xi_{12i} + 2 a_{1j} x_{12k} \xi_{12i}]$$

Out[*]=

True

In[*]:= { \$k = 1, \$n = 3 }; aΔt_{i→j,k}

(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

» **TTL** $e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_i} U_j[x_{12}] U_k[] +$
 $\left(e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \xi_{13_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \lambda \xi_{12_i} \xi_{23_i} + e^{-\frac{1}{2}\gamma(\alpha_1+\alpha_2)} \gamma \in \lambda \hbar \xi_{12_i} \xi_{23_i} \right) U_j[x_{13}] U_k[] +$
 $\ll 31 \gg + 2 e^{-\frac{3\gamma\alpha_2}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_j[] U_k[x_{13}, x_{23}]$

» **SG** $U_j[] U_k[] + e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \in U_j[x_{12}] U_k[a_1] f_{1,a_1 x_{12}}[\lambda] + \ll 21 \gg +$
 $U_j[] U_k[x_{13}, x_{23}] \left(e^{-\frac{3\gamma\alpha_2}{2}} \in f_{0,x_{23}}[\lambda] f_{1,x_{12} x_{23}}[\lambda] + e^{-\frac{3\gamma\alpha_2}{2}} \in f_{1,x_{13} x_{23}}[\lambda] \right)$

» **TTR** $\ll 1 \gg$

Out[*]=

$E_{\{i\} \rightarrow \{j,k\}} [$
 $(a_{1j} + a_{1k}) \alpha_{1i} + (a_{2j} + a_{2k}) \alpha_{2i} + x_{12j} \xi_{12_i} + x_{12k} \xi_{12_i} + x_{13j} \xi_{13_i} + x_{13k} \xi_{13_i} + x_{23j} \xi_{23_i} + x_{23k} \xi_{23_i},$
 $-2 a_{1k} x_{12j} \xi_{12_i} + 2 a_{1j} x_{12k} \xi_{12_i} - 2 a_{1k} x_{13j} \xi_{13_i} - 2 a_{2k} x_{13j} \xi_{13_i} + 2 a_{1j} x_{13k} \xi_{13_i} +$
 $2 a_{2j} x_{13k} \xi_{13_i} + x_{12k} x_{13j} \xi_{12_i} \xi_{13_i} - x_{12j} x_{13k} \xi_{12_i} \xi_{13_i} - 2 a_{2k} x_{23j} \xi_{23_i} + 2 a_{2j} x_{23k} \xi_{23_i} -$
 $x_{12k} x_{23j} \xi_{12_i} \xi_{23_i} + x_{13k} x_{23j} \xi_{13_i} \xi_{23_i} - x_{13j} x_{23k} \xi_{13_i} \xi_{23_i} + x_{12j} x_{23k} (4 \xi_{13_i} + \xi_{12_i} \xi_{23_i})]$

Out[*]=

True

In[*]:= { \$k = 2, \$n = 3 }; aΔt_{i→j,k}
(aΔt_{1→1,3} // aΔt_{1→1,2}) ≡ (aΔt_{1→1,2} // aΔt_{2→2,3})

» **TTL** $e^{-\frac{1}{2}\gamma(2\alpha_1-\alpha_2)} \xi_{12_i} U_j[x_{12}] U_k[] + \ll 125 \gg +$
 $\left(2 e^{\frac{1}{2}\gamma(\alpha_1-5\alpha_2)} \gamma^2 \epsilon^2 \lambda^2 \hbar^2 \xi_{13_i} \xi_{23_i}^2 + \frac{2}{3} e^{\ll 1 \gg} \gamma^2 \epsilon^2 \ll 1 \gg \hbar^2 \xi_{12_i} \xi_{23_i}^3 \right) \ll 1 \gg \ll 1 \gg$

» **SG** $U_j[] U_k[] + \ll 332 \gg$

In[*]:= { \$k = 1, \$n = 2 }; (amt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})

Time: 0.05 seconds Out[*]=

True

(Alt) In[*]:=

Block[{ \$k = 1, \$n = 3 }, Echo@ { amt_{1,2→1}, aΔt_{1→1,2} };
(aamt_{1,2→1} // aΔt_{1→1,2}) ≡ (aΔt_{2→3,4} // aΔt_{1→1,2} // amt_{1,3→1} // amt_{2,4→2})]

» $\left\{ E_{\{1,2\} \rightarrow \{1\}} \left[a_{11} (\alpha_{11} + \alpha_{12}) + a_{21} (\alpha_{21} + \alpha_{22}) + x_{12_1} \left(\xi_{12_1} + e^{\frac{1}{2}(2\alpha_1-\alpha_2)} \xi_{12_2} \right) + \right. \right.$
 $x_{13_1} \left(\xi_{13_1} + e^{\frac{1}{2}(\alpha_1+\alpha_2)} \xi_{13_2} - e^{\frac{1}{2}(2\alpha_1-\alpha_2)} \xi_{12_2} \xi_{23_1} \right) + x_{23_1} \left(\xi_{23_1} + e^{\frac{1}{2}(-\alpha_1+2\alpha_2)} \xi_{23_2} \right),$
 $-2 e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{12_1} x_{13_1} \xi_{12_2} \xi_{13_1} - e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{13_1} \xi_{12_2} \xi_{23_1} +$
 $2 e^{\frac{1}{2}(2\alpha_1-\alpha_2)} x_{12_1} x_{23_1} \xi_{12_2} \xi_{23_1} - 2 e^{\frac{1}{2}(\alpha_1+\alpha_2)} x_{13_1} x_{23_1} \xi_{13_2} \xi_{23_1} \left. \right], E_{\{1\} \rightarrow \{1,2\}} [$
 $(a_{11} + a_{12}) \alpha_{11} + (a_{21} + a_{22}) \alpha_{21} + x_{12_1} \xi_{12_1} + x_{12_2} \xi_{12_1} + x_{13_1} \xi_{13_1} + x_{13_2} \xi_{13_1} + x_{23_1} \xi_{23_1} + x_{23_2} \xi_{23_1},$
 $-2 a_{12} x_{12_1} \xi_{12_1} + 2 a_{11} x_{12_2} \xi_{12_1} - 2 a_{12} x_{13_1} \xi_{13_1} - 2 a_{22} x_{13_1} \xi_{13_1} + 2 a_{11} x_{13_2} \xi_{13_1} +$
 $2 a_{21} x_{13_2} \xi_{13_1} + x_{12_2} x_{13_1} \xi_{12_1} \xi_{13_1} - x_{12_1} x_{13_2} \xi_{12_1} \xi_{13_1} - 2 a_{22} x_{23_1} \xi_{23_1} + 2 a_{21} x_{23_2} \xi_{23_1} -$
 $x_{12_2} x_{23_1} \xi_{12_1} \xi_{23_1} + x_{13_2} x_{23_1} \xi_{13_1} \xi_{23_1} - x_{13_1} x_{23_2} \xi_{13_1} \xi_{23_1} + x_{12_1} x_{23_2} (4 \xi_{13_1} + \xi_{12_1} \xi_{23_1}) \left. \right] \}$

(Alt) Out[*]=

True

(Alt) In[]:=
pdf

```

Define[aSti = Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL = ES["TTL"] @ TTλ @
    Echo@Reverse@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Si[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Ui[x___] ⇒ Ui[x], ∞];
  ff = Table[
    Sum[εp fp, ui[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp, ui[λ] /; τ[u] == "a" ⇒ If[p == 0, - (u*)i, 0];
  fs = Flatten[Table[
    fp, Pi[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pi[u][λ] Pi[u],
    {p, 0, $k}, {u, Complement[mons[p], (Ui /@ $PBWGens)]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Oi[Seepagei[ff, SS@UCF@B]]];
  SG = ES["SG"] @ S[G];
  TTR = ES["TTR"] @ UCF[Inverti[SG] ⊗ S[(∂λ (ff. (#i & /@ $PBWGens)) G + ∂λ G)]];
  diff = UCF[TTL - TTR];
  eqns = ES["eqns"] @ Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E{i}→{i} @@ Table[As[p, 0, $k], Plus[
    If[p == 0, Sum[As[m, 1, $n - 1], -α[m]i (a[m]i)], 0],
    Total[Cases[fs, fp, _]] /. fp, u ⇒ u (fp, u /. ss)[λ] /. λ | ħ | γ → 1]
  ]
]]

```

```
In[*]:= $k = 1; $n = 3; aSti
aΔt1→2,3 // aSt2 // amt2,3→1
```

Time: 0.25 seconds Out[]=

$$\begin{aligned} E_{\{i\} \rightarrow \{i\}} \left[-a_{1i} \alpha_{1i} - a_{2i} \alpha_{2i} - e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} - e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + \right. \\ \left. x_{13i} \left(-e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} - e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right), 2 e^{\frac{1}{2}(-2\alpha_{1i} + \alpha_{2i})} x_{12i} \xi_{12i} + \right. \\ \left. 2 e^{-\frac{3\alpha_{1i}}{2}} x_{12i} x_{13i} \xi_{12i} \xi_{13i} + 2 e^{\frac{1}{2}(\alpha_{1i} - 2\alpha_{2i})} x_{23i} \xi_{23i} + 2 e^{-\frac{3\alpha_{2i}}{2}} x_{13i} x_{23i} \xi_{13i} \xi_{23i} + \right. \\ \left. 2 e^{-\alpha_{1i} - \alpha_{2i}} x_{13i}^2 \xi_{12i} \xi_{13i} \xi_{23i} + x_{12i} x_{23i} \left(4 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) + \right. \\ \left. x_{13i} \left(2 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{13i} + 3 e^{\frac{1}{2}(-\alpha_{1i} - \alpha_{2i})} \xi_{12i} \xi_{23i} \right) \right] \end{aligned}$$

Out[]=

$$E_{\{1\} \rightarrow \{1\}} [\emptyset, \emptyset]$$