

Hidden Utilities and T_EX

Headers

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Pensieve header: Quantum $\mathfrak{sl}_n$ following Rosso and Yamane.

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Notes.

1. Rosso is “An Analogue of P.B.W. Theorem and the Universal R -matrix for $U_{\hbar}(\mathfrak{sl}(N+1))$ ”, Yamane is “A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N ”.
2. We call our algebra $U_q(\mathfrak{sl}_n)$, not $U_{\hbar}(\mathfrak{sl}(N+1))$.
3. We do not have a name for Rosso’s X_i and Y_j . Rosso’s $E_{\epsilon_i - \epsilon_{j+1}}$ is our $x[i, j] \leftrightarrow x_{ij}$. Likewise Rosso’s $\eta_{\epsilon_i - \epsilon_{j+1}}$ is our $y[i, j] \leftrightarrow y_{ij}$.
4. Rosso’s H_i is our a_i and Rosso’s ξ_i is our b_i .

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```
In[ ]:= SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at <http://katlas.org/wiki/KnotTheory>.

pdf

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

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This version: April 2020. Original version: July 1994.

```
In[ ]:= MakePDF[]
```

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```
In[ ]:=  $\delta_{ij}$  := KroneckerDelta[i, j];
```

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```
In[ ]:= SetAttributes[{SS}, HoldAll];
SS[_ $\mathcal{E}$ _, op_] := Collect[Normal@Series[_ $\mathcal{E}$ _, { $\epsilon$ , 0, $k}],  $\epsilon$ , op];
SS[_ $\mathcal{E}$ _] := SS[_ $\mathcal{E}$ _, Together];
```

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```
In[ ]:= CF[_ $\mathcal{E}$ _] := PPCF@If[$k ==  $\infty$ ,
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^{op}}$ ],
  Expand[Expand[_ $\mathcal{E}$ ] /.  $e^{p-} \mapsto e^{\text{Factor}^{op}}$  /.  $q^{n-} \mapsto \sum_{k=0}^{k} \frac{(\hbar \in \gamma n)^k}{k!}$ ] /.  $e^{k-} \mapsto$  ;  $k > $k \mapsto 0$ 
];
```

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```
In[ ]:= x[i_, j_] := x<>i<>j; a[i_] := a<>i; A[i_] := A<>i;
ξ[i_, j_] := ξ<>i<>j; α[i_] := α<>i; ℳ[i_] := ℳ<>i;
```

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```
In[ ]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
ℓ[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
τ[s_[]] := τ[s]; ℓ[s_[]] := ℓ[s];
```

```
In[ ]:= {τ[x[3, 4]], ℓ[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[]:=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for R at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[ ]:= (* q~e^h γ ∈; Ai[n]~e^h ∈ ai; *)
```

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```
In[ ]:= $n = 5; $k = ∞; $E := {$n, $k}
```

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```
In[ ]:= $EPBWGens := Flatten@{
  Table[As[i, 1, $n - 1], As[j, i + 1, $n],
    x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
  Table[As[i, 1, $n - 1],
    A[i]* = ℳ[i]; ℳ[i]* = A[i]; a[i]* = α[i]; α[i]* = a[i]; {a[i], A[i][_]}]
  }
ℓ_List* := #* & /@ ℓ;
$EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i → {i, k}}, {g, $EPBWGens}]];
x_ ≤ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $EPBWRule];
$PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[ ]:= $PBWGens*
```

Time: 0.01 seconds Out[]:=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34, ξ35, ξ45, α1, α2, α3, α4}
```

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```

In[*]:= UCF[ $\mathcal{E}_-$ ] := PPUCF@Module[{F, k}, Expand@Collect[
   $\mathcal{E}$  /. {Ui[L____, _[0], r____] → Ui[1, r]} /. If[$k == ∞, {}, {
    Ui[L____, AB[n_], r____] ⇒
    Sum[As[k, 0, $k],  $\frac{(-2)^k \hbar^k \epsilon^k n^k}{k!}$  Ui[L, Sequence@@Table[a@@L[AB], k], r]
  }]],
  U[____], F
] /. F → CF

```

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```

In[*]:= ms__[0] = 0;
ms__[x_Plus] := UCF[ms /@ x];
mi→j[_] :=  $\mathcal{E}$  /. Ui → Uj;
mi,j→k[c_. Ui[x____] Uj[_]] := c Uk[x];
mi,j→k[c_. Ui[_] Uj[y____]] := c Uk[y];
mi,j→k[c_. Ui[xx____, x_[n1_]] Uj[x_[n2_], yy____]] :=
  UCF[c Ui[xx, x[n1 + n2]] Uj[yy]] /. mi,j→k;
mi,j→k[c_. Ui[xx____, x_] Uj[y_, yy____]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((c Ui[xx] RRj[x, y] /. UCF /. mi,j→i) Uj[yy] /. UCF /. mi,j→k) /. UCF
];
ms__[ $\mathcal{E}_-$ ] := ms[UCF[ $\mathcal{E}$ ]];

```

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```

In[*]:= Supp[ $\mathcal{E}_-$ ] := Union@Cases[{ $\mathcal{E}$ }, Ui[____] ⇒ i, ∞];

```

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```

In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⋈ y_ := PPNCM@Module[{i, is = Supp[x] ∩ Supp[y], σ, z, k},
  z = x; Do[z = mi→σi[z], {i, is}];
  z = Expand[y z];
  If[$k < ∞, z = z /.  $\epsilon^{k-}$  /. k > $k ⇒ 0];
  Do[z = mσi,i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⋈ y - y ⋈ x];

```

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```

In[*]:= Δi→j,k[_] := UCF@Module[{j1, k1},  $\mathcal{E}$  /. {
  Ui[_] → Uj[_] Uk[_],
  Ui[f_, fs____] ⇒ mk,k1→k@mj,j1→j[Δj,k[f] Δi→j1,k1[Ui[fs]]]
}];
Δi→j,k,l[_] :=  $\mathcal{E}$  /. Δi→j,k /. Δk→k,l

```

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```
In[*]:= Si[_ε_] := UCF@Module[{i1}, ε /. {
    Ui[f_, fs___] => mi1,i[Si[f] Si1[Ui1[fs]]]
}];
```

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The $\mathbb{A}$ -algebra

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```
In[*]:= RRk[ai_, aj_] /; τ[ai] == τ[aj] == "a" := Uk[aj, ai];
RRk[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[Aj[n], ai];
RRk[Aj_[n_], ai_] /; τ[ai] == "a" ∧ τ[Aj] == "A" := Uk[ai, Aj[n]];
RRk[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := Uk[Aj[n], Ai[m]];
RRk[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@am;
    Uk[xij, am] + γ (δm,i - δm+1,i - δm,j + δm+1,j) Uk[xij] / 2
];
RRk[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
    {i, j} = L@xij; {m} = L@Am; q-n (δn,i - δn+1,i - δn,j + δn+1,j) Uk[xij, Am[n]]
]
```

```
In[*]:= RR0[a[1], x[1, 2]]
```

Time: 0.05 seconds Out[]:=

```
γ U0[x12] + U0[x12, a1]
```

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We take the reduction rules RR for xij's from Yamane page

509:

$$(I) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array} \quad (II) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array} \quad (III) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array}$$

$$(IV) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array} \quad (V) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array} \quad (VI) \quad \begin{array}{c} i & \dots & j \\ \hline m & \dots & n \end{array}$$

$$\begin{array}{ll} q^{-2} e_{ij} e_{mn} - e_{mn} e_{ij} = 0 & \text{if } ((i, j), (m, n)) \in C_{(I)} \cup C_{(III)}. \\ [e_{ij}, e_{mn}] = 0 & \text{if } ((i, j), (m, n)) \in C_{(II)} \cup C_{(VI)}. \\ [e_{ij}, e_{mn}] = (q^2 - q^{-2}) e_{in} e_{mj} & \text{if } ((i, j), (m, n)) \in C_{(IV)}. \\ q^2 e_{ij} e_{mn} - e_{mn} e_{ij} = q e_{in} & \text{if } ((i, j), (m, n)) \in C_{(V)}. \end{array}$$

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```

In[ ]:= RRk[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = L@xij; {m, n} = L@xmn;
  Which[
    (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q-2 Uk[xij, xmn], (* IVIII *)
    (i < m < j ∧ n < j) ∨ (j < m), Uk[xij, xmn], (* IIIVI *)
    i < m < j ∧ n > j, Uk[xij, xmn] + (q-2 - q2) Uk[x[i, n], x[m, j]], (* IV *)
    j == m, q2 Uk[xij, xmn] - q Uk[x[i, n]] (* V *)
  ]
]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2
Time: 0.06 seconds Out:=
-q U2[x24] U3[x12] + q2 U2[x23, x34] U3[x12]

In[ ]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
Time: 0.05 seconds Out:=
q2 U0[x14] - q3 U0[x12, x24] - q3 U0[x13, x34] + q4 U0[x12, x23, x34]

In[ ]:= Block[{$k = ∞}, Module[{Bas = $PBWGens, P},
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
  ]
]]
Time: 1.05 seconds Out:=
{}

In[ ]:= Block[{$k = ∞, $n = 3}, Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]];
  DeleteCases[_ → 0] [Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → UCF[(P // m1,2→2 // m2,3→0) - (P // m2,3→2 // m1,2→0)]
  ]
]]
» {x12, x13, x23, a1, a2, A2[0], A2[-2], A2[-2], A2[2], A2[2], A1[-1], A2[-2], A1[0], A2[-2], A2[1]}

Out[ ]:=
{}

```

```

Module[{Bas = $PBWgens, P},
  DeleteCases[_ -> True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} -> (P // m2,3 -> 2 // m2,4 -> 2 // m1,2 -> 1) ==
      (P // m3,4 -> 3 // m2,3 -> 2 // m1,2 -> 1) == (P // m1,2 -> 1 // m3,4 -> 3 // m1,3 -> 1) ==
      (P // m1,2 -> 1 // m1,3 -> 1 // m1,4 -> 1) == (P // m2,3 -> 2 // m1,2 -> 1 // m1,4 -> 1)
  ]
]
» {x12, x13, x23, a1, a2}

```

Out[8]=

{ }

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[8]:=

```

Do[As[i, 1, $n - 1],
  (* Warning: these initialization fail if $n is increased later on *)
  Δj_,k_ [a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_ [A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_ [x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Δj_,k_ [x[i, n]] = m[k, k1 -> k] [m[j, j1 -> j] [
    q Δj,k [x[i, n - 1]] Δj1,k1 [x[n - 1, n]] - q^-1 Δj,k [x[n - 1, n]] Δj1,k1 [x[i, n - 1]]
  ]
]

```

```

In[*]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
]

Time: 0.06 seconds Out[*]=
Ui[x12] → Uj[A1[-1]] Uk[x12] + Uj[x12] Uk[A1[1]]
Ui[x13] → Uj[A1[-1], A2[-1]] Uk[x13] +
  (-1/q2 + q2) Uj[x12, A2[-1]] Uk[x23, A1[1]] + Uj[x13] Uk[A1[1], A2[1]]
Ui[x14] →
  Uj[A1[-1], A2[-1], A3[-1]] Uk[x14] + (-1/q2 + q2) Uj[x12, A2[-1], A3[-1]] Uk[x24, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1]] Uk[x34, A1[1], A2[1]] + Uj[x14] Uk[A1[1], A2[1], A3[1]]
Ui[x15] → Uj[A1[-1], A2[-1], A3[-1], A4[-1]] Uk[x15] +
  (-1/q2 + q2) Uj[x12, A2[-1], A3[-1], A4[-1]] Uk[x25, A1[1]] +
  (-1/q2 + q2) Uj[x13, A3[-1], A4[-1]] Uk[x35, A1[1], A2[1]] +
  (-1/q2 + q2) Uj[x14, A4[-1]] Uk[x45, A1[1], A2[1], A3[1]] +
  Uj[x15] Uk[A1[1], A2[1], A3[1], A4[1]]
Ui[x23] → Uj[A2[-1]] Uk[x23] + Uj[x23] Uk[A2[1]]
Ui[x24] → Uj[A2[-1], A3[-1]] Uk[x24] +
  (-1/q2 + q2) Uj[x23, A3[-1]] Uk[x34, A2[1]] + Uj[x24] Uk[A2[1], A3[1]]
Ui[x25] →
  Uj[A2[-1], A3[-1], A4[-1]] Uk[x25] + (-1/q2 + q2) Uj[x23, A3[-1], A4[-1]] Uk[x35, A2[1]] +
  (-1/q2 + q2) Uj[x24, A4[-1]] Uk[x45, A2[1], A3[1]] + Uj[x25] Uk[A2[1], A3[1], A4[1]]
Ui[x34] → Uj[A3[-1]] Uk[x34] + Uj[x34] Uk[A3[1]]
Ui[x35] → Uj[A3[-1], A4[-1]] Uk[x35] +
  (-1/q2 + q2) Uj[x34, A4[-1]] Uk[x45, A3[1]] + Uj[x35] Uk[A3[1], A4[1]]
Ui[x45] → Uj[A4[-1]] Uk[x45] + Uj[x45] Uk[A4[1]]
Ui[a1] → Uj[a1] Uk[] + Uj[] Uk[a1]
Ui[a2] → Uj[a2] Uk[] + Uj[] Uk[a2]
Ui[a3] → Uj[a3] Uk[] + Uj[] Uk[a3]
Ui[a4] → Uj[a4] Uk[] + Uj[] Uk[a4]

In[*]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]]], {10}]]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
  ]
]
» {x12, x13, x23, a1, a2, A2[1], A1[0], A2[-2], A1[1], A2[-2], A2[1], A2[2], A1[1], A1[-1], A2[0]}

Out[*]=
{ }

```

```

In[ ]:= Module[{Bas = $PBWGens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ -> 0] [Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} -> UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A1[2], A2[-1], A2[1], A1[2], A2[-2], A2[1], A1[-2], A1[-1], A2[1]}
Out[ ]:=
{ }

```

Is Δ a quantization of the co-bracket?

```

In[ ]:= $k = 1
Time: 0.02 seconds Out[ ]:=
1

In[ ]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U1 /@ Bas],
    P -> UCF[(Δ1→1,2[P] - Δ1→2,1[P]) // .
      {qn- -> 1 + n ħ / 2, Ui[xs___, A-[n_], as___] -> Ui[xs] + n ħ Ui[xs, a@@L[A], as] / 2}
    ] /. ħn- /; n > 1 -> 0
  ]
]
Time: 0.05 seconds Out[ ]:=
U1[x12] -> -4 ∈ ħ U1[x12] U2[a1] + 4 ∈ ħ U1[a1] U2[x12]
U1[x13] -> -4 ∈ ħ U1[x13] U2[a1] - 4 ∈ ħ U1[x13] U2[a2] - 4 ∷ ∈ ħ U1[x23] U2[x12] +
  4 ∈ ħ U1[a1] U2[x13] + 4 ∈ ħ U1[a2] U2[x13] + 4 ∷ ∈ ħ U1[x12] U2[x23]
U1[x23] -> -4 ∈ ħ U1[x23] U2[a2] + 4 ∈ ħ U1[a2] U2[x23]
U1[a1] -> 0
U1[a2] -> 0

```

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The Antipode

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```
In[*]:= Do[As[i, 1, $n - 1],
  Sj_ [a[i]] = -Uj[a[i]];
  Sj_ [A[i][n_]] = Uj[A[i][-n]];
  Sj_ [x[i, i + 1]] = -q-2 Uj[x[i, i + 1]] (*?*)
];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  Sj_ [x[i, n]] = mj1,j→j [
    q Sj[x[i, n - 1]] Sj1[x[n - 1, n]] - q-1 Sj[x[n - 1, n]] Sj1[x[i, n - 1]]
  ]
]
```

```
In[*]:= $k = 2
```

```
Time: 0.00 seconds Out[ ]=
2
```

```
In[*]:= Module[{Bas = $PBWgens, P},
  Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]] [RandomInteger[{-2, 2}]], {10}]];
  DeleteCases[_ → True] [Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→2,3 // S2 // m2,3→1)
  ]
]
» {x12, x13, x23, a1, a2, A1[-2], A2[-1], A1[1], A2[-1], A1[-2], A2[1], A1[2], A2[0], A2[2], A2[0]}
```

```
Out[*]=
```

```
{x12 → 0, x13 → 0, x23 → 0, a1 → 0, a2 → 0, A1[-2] → U1[],
  A2[-1] → U1[], A1[1] → U1[], A2[-1] → U1[], A1[-2] → U1[],
  A2[1] → U1[], A1[2] → U1[], A2[0] → U1[], A2[2] → U1[], A2[0] → U1[]}
```

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The Generating Functions for the $\mathbb{A}$ operations

```
In[*]:= BeginProfile[]
```

```
Time: 0.05 seconds Out[ ]=
ProfileRoot
```

```
In[*]:= $k = 1;
```

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Ad, TT, Seepage

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```
In[*]:= PotQ[0] = True;
PotQ[c_. U_ [aj_] (EUs : U_[] ...)] /; (τ[aj] == "a") ∧ FreeQ[c, U] = True;
PotQ[x_Plus] := PotQ /@ (And @@ x);
```

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```
In[*]:= Ad[x_] [y_] /; PotQ[x] ∧ (Head[y] != Plus) := PPAdPoty[eUB[y,x]/y y];
Ad[x_] [y_Plus] /; PotQ[x] := PPAdPotPlus[Ad[x] /@ y];
```

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```
In[*]:= Ad[x_] [y_] /; ¬ TrueQ[PotQ[x]] := PPAdNil@Module[{k = 1, s = y, t = y},
  While[(t = UB[t, x]) != 0, s += t / k!; ++k];
  UCF[s]
]
```

```
In[*]:= $k = 2
```

Time: 0.00 seconds Out:=

2

```
In[*]:= Ad[ε ħ Uj[x12] Uk[]] [Uj[x23] Uk[a4]]
```

Time: 0.06 seconds Out:=

$$\left(-\epsilon \hbar - \gamma \epsilon^2 \hbar^2\right) U_j[x_{13}] U_k[a_4] + U_j[x_{23}] U_k[a_4] + 2 \gamma \epsilon^2 \hbar^2 U_j[x_{12}, x_{23}] U_k[a_4]$$

```
In[*]:= PotQ[7 ħ Uj[x12] Uk[a3]]
```

Time: 0.02 seconds Out:=

$$\text{PotQ}[7 \hbar U_j[x_{12}] U_k[a_3]]$$

```
In[*]:= Ad[7 ħ Uj[x12] Uk[a3]] [Uj[x23] Uk[a4]]
```

Time: 0.09 seconds Out:=

$$U_j[x_{23}] U_k[a_4] + \left(-7 \hbar - 7 \gamma \epsilon \hbar^2 - \frac{7}{2} \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{13}] U_k[a_3, a_4] + \\ \left(14 \gamma \epsilon \hbar^2 + 14 \gamma^2 \epsilon^2 \hbar^3\right) U_j[x_{12}, x_{23}] U_k[a_3, a_4] - 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{13}] U_k[a_3, a_3, a_4] + \\ 98 \gamma^2 \epsilon^2 \hbar^4 U_j[x_{12}, x_{12}, x_{23}] U_k[a_3, a_3, a_4] - \frac{686}{3} \gamma^2 \epsilon^2 \hbar^5 U_j[x_{12}, x_{12}, x_{13}] U_k[a_3, a_3, a_3, a_4]$$

```
In[*]:= Ad[π ĩ Uj[a1]] [Uj[x23] Uk[a4]]
```

Time: 0.05 seconds Out:=

$$e^{\frac{i \pi \gamma}{2}} U_j[x_{23}] U_k[a_4]$$

pdf

```
In[*]:= TTλ[] = 0; (* Translated Tangent *)
TTλ[xs___, x_] := PPTT@UCF[Ad[x] [TTλ[xs]] + ∂λ x];
TTλ[xs_List] := TTλ@@xs
```

In[*]:= $\text{TT}_\lambda[\lambda \xi_{12} U_0[x_{12}] U_1[], \lambda \xi_{23} U_0[x_{23}] U_1[]]$

Time: 0.08 seconds Out[]=

$$\xi_{12} U_0[x_{12}] U_1[] + \lambda \xi_{12} \xi_{23} U_0[x_{13}] U_1[] + \xi_{23} U_0[x_{23}] U_1[]$$

pdf

```
In[*]:= O_i_[poly_] := Total[
  CoefficientRules[poly, #i & /@ $PBWGens] /.
  (p_ -> c_) -> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
];
O_i_,is_[poly_] := O_i[O_is[poly]];
P_i_[e_] := e /. U_i[xs___] -> Times @@ (#i & /@ {xs});
P_i_,is_[poly_] := P_i[P_is[poly]];
```

In[*]:= $O_i[x_{12}^2 x_{13} + 7 x_{23}^{17}]$

Time: 0.06 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] + 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}]$$

In[*]:= $O_{i,j}[x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3]$

Time: 0.03 seconds Out[]=

$$U_i[x_{12}, x_{12}, x_{13}] U_j[x_{24}] + 7 U_i[x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}, x_{23}] U_j[x_{25}, x_{25}, x_{25}]$$

In[*]:= $P_{i,j} @ O_{i,j}[x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3]$

Time: 0.06 seconds Out[]=

$$x_{12}^2 x_{13} x_{24} + 7 x_{23}^{17} x_{25}^3$$

pdf

```
In[*]:= Seepage_i_[{ }, c_] = c;
Seepage_i_[eS_List, P_] := Module[{lx},
  lx = $PBWGens[[Length@eS]];
  CF@Collect[P, lx_i, P_i[Ad[Last[eS] U_i[lx]]][O_i[Seepage_i[Most@eS, #]]]]
  &];
Seepage_i_,is_[eS_List, eSs_List, P_] := Seepage_i[eS, Seepage_is[eSs, P]]
```

In[*]:= $\$k = 1$

Time: 0.00 seconds Out[]=

$$1$$

In[*]:= $\$PBWGens^*$

Time: 0.01 seconds Out[]=

$$\{\xi_{12}, \xi_{13}, \xi_{14}, \xi_{15}, \xi_{23}, \xi_{24}, \xi_{25}, \xi_{34}, \xi_{35}, \xi_{45}, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

In[*]:= $\text{Seepage}_i[\$PBWGens^*, x_{35}_i]$

Time: 0.16 seconds Out[]=

$$e^{\frac{1}{2}(\alpha_2 - \alpha_3 - \alpha_4)} \gamma x_{35}_i + 2 e^{\frac{1}{2}(\alpha_2 - 3\alpha_4)} \gamma \gamma \in \xi_{45} \hbar x_{35}_i x_{45}_i$$

In[]:= **Seepage**_{i,j}[**#i** & /@ **\$PBWGens**^{*}, **#j** & /@ **\$PBWGens**^{*}, **x35_i x45_j**]

Time: 0.25 seconds Out[]=

$$e^{\frac{1}{2} \gamma (\alpha_2 - \alpha_3 + \alpha_3 - \alpha_4 - 2 \alpha_4)} x_{35_i} x_{45_j} + 2 e^{-\frac{1}{2} \gamma (-\alpha_2 - \alpha_3 + 3 \alpha_4 + 2 \alpha_4)} \gamma \in \hbar x_{35_i} x_{45_i} x_{45_j} \xi_{45_i}$$

pdf

```
In[ ]:= Invertis__[ℰ_] := Module[{A, B, U1, j}, (* ℰ=A U[ ] + ℰ B *)
U1 = Times@@ (U#[ ] & /@ {is});
A = Coefficient[ℰ, U1] /. ℰ → 0;
B = UCF[ (ℰ - A U1) ];
UCF[
  A-1 (U1 + Sum[As[j, 1, $k], (-1)j NonCommutativeMultiply @@ ConstantArray[B / A, j]])
]
```

In[]:= **\$k** = 3;

Invert_{i,j}[7 **U_i**[] **U_j**[] + **ℰ** **U_i**[**x12**] **U_j**[**x23**]]

Time: 0.09 seconds Out[]=

$$\frac{1}{7} U_i[] U_j[] - \frac{1}{49} \in U_i[x12] U_j[x23] + \frac{1}{343} \in^2 U_i[x12, x12] U_j[x23, x23] - \frac{\in^3 U_i[x12, x12, x12] U_j[x23, x23, x23]}{2401}$$

{**\$k** = 1, **\$n** = 3}

Time: 0.01 seconds Out[]=

{2, 3}

pdf

```
In[ ]:= TriangularDSolve[eqns_, inits_, fs_, λ_, CF_] :=
PPTriangularDSolve@Module[{ss = {}, reqns = eqns, rinits = inits, rfs = fs, s, p},
While[rfs != {},
  {p} = FirstPosition[Count[#, Alternatives @@ rfs, ∞, Heads → True] & /@ reqns, 1];
  {{s}} = DSolve[reqns[p] ∧ rinits[p], rfs[p], λ];
  AppendTo[ss, s = s /. F_Function ⇒ ReplacePart[F, 2 → CF[F[2]]]];
  reqns = Delete[reqns, p] /. s /. e_ == 0 ⇒ CF[e] == 0;
  rinits = Delete[rinits, p];
  rfs = Delete[rfs, p];
];
ss
]
```

pdf

$$L_\lambda = \mathbb{C}\langle \lambda_{i_1}^x \rangle \mathbb{C}\langle \lambda_{i_2}^x \rangle \quad \text{Seepage is defined by: } \mathbb{C}\langle \lambda_{i_1}^x \rangle \mathbb{C}\langle \text{Seepage}(\{f_i\}, A) \rangle = \mathbb{C}\langle e^{f_i x_i} A \rangle$$

$$\text{TT}(L_\lambda) = L' \partial_\lambda L_\lambda \quad S(B) = \text{Seepage}(\{f_i\}, B).$$

$$R_\lambda = \mathbb{C}\langle f+g \rangle, \quad f = f_i x_i, \quad g = \sum_{|w|=1} g_w w, \quad e^f = F, \quad e^g = G, \quad R_s = \mathbb{C}\langle FG \rangle, \quad \partial_\lambda F = F \partial_\lambda f$$

$$\text{TT}(R_\lambda) = R'_\lambda \partial_\lambda R_\lambda = \mathbb{C}\langle FG \rangle' \mathbb{C}\langle \partial_\lambda FG \rangle = S(G)' \mathbb{C}\langle F \rangle' \mathbb{C}\langle F(\partial_\lambda f G + \partial_\lambda G) \rangle = S(G)' S(\partial_\lambda f G + \partial_\lambda G).$$

pdf

```

In[*]:= amti,j→k := PPamt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε;
  TTL = ES["TTL"]@PPTTL@TTλ@
    Join[l1 = Table[If[τ[g] == "x", λ, 1] (g*)i Uk[g], {g, $PBWGens}], l1 /. i → j];
  mons[-1] = {};
  mons[p_] := mons[p - 1] ∪ Union@Cases[Coefficient[TTL, ε, p], Uk[____], ∞];
  ff = Table[
    Sum[εp fp,uk[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp-,u-k[λ] /; τ[u] == "a" => If[p == 0, (u*)i + (u*)j, 0];
  fs = Flatten[Table[
    fp,ℙk[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp,ℙk[u][λ] ℙk[u],
    {p, 0, $k}, {u, Complement[mons[p], Uk/@$PBWGens]}
  ];
  G = Series[egg, {ε, 0, $k}];
  S[B_] := UCF[SS@Ok[Seepagek[ff, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[Invertk[SG] ⊗ S[(∂λ(ff.(#k & /@$PBWGens)) G + ∂λG)]];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E[Total[fs] /. fp-,u- => εp u (fp,u /. ss)[λ]] /. λ | ħ | γ → 1
]

```

```

In[*]:= Block[{$k = 1, $n = 3}, amti,j→k]

```

$$\begin{aligned}
& \gg \text{TTL} \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) U_k[x_{12}] + \ll 5 \gg + (\ll 1 \gg) U_k[x_{13}, x_{23}] \\
& \gg \text{SG} U_k[] + \ll 3 \gg + U_k[x_{13}, x_{23}] \left(e^{-\frac{3}{2} \gamma (\alpha_2 + \alpha_2)} \in f_{0, x_{23_k}}[\lambda] f_{1, x_{12_k} x_{23_k}}[\lambda] + e^{-\frac{3}{2} \gamma (\ll 1 \gg)} \in f_{1, \ll 1 \gg}[\lambda] \right) \\
& \gg \text{TTR} \\
& U_k[x_{12}] \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{0, \ll 3 \gg_k'}[\lambda] + e^{-\frac{1}{2} \gamma (\ll 1 \gg)} \in f_{1, \ll 1 \gg \ll 1 \gg'}[\lambda] \right) + \ll 5 \gg + U_k[x_{13}, x_{23}] (\ll 1 \gg)
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& E \left[x_{12_k} \left(\xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{13_k} \xi_{12_j} \xi_{13_i} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{13_k} \xi_{12_j} \xi_{23_i} + \right. \\
& 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{23_k} \xi_{12_j} \xi_{23_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \in x_{13_k} x_{23_k} \xi_{13_j} \xi_{23_i} + \\
& \left. x_{13_k} \left(\xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + x_{23_k} \left(\xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2)} \xi_{23_j} \right) \right]
\end{aligned}$$

In[*]:= Block[{ \$k = 1, \$n = 4}, amt_{i,j→k}]

$$\begin{aligned}
& \gg \text{TTL} \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} \xi_{12_i} + e^{-\frac{1}{2} \gamma (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) U_k[x_{12}] + \ll 21 \gg + (\ll 1 \gg) U_k[x_{24}, x_{34}] \\
& \gg \text{SG} \\
& U_k[] + \ll 16 \gg + U_k[x_{24}, x_{34}] \left(e^{\frac{1}{2} \gamma (\alpha_1 + \alpha_1 - 3\alpha_3 - 3\alpha_3)} \in f_{0, \ll 3 \gg_k}[\lambda] f_{1, x_{23_k} \ll 3 \gg_k}[\lambda] + e^{\ll 1 \gg} \in f_{1, \ll 1 \gg \ll 1 \gg}[\lambda] \right) \\
& \gg \text{TTR} U_k[x_{12}] \left(e^{-\frac{1}{2} \gamma (2\alpha_1 + 2\alpha_1 - \alpha_2 - \alpha_2)} f_{0, \ll 3 \gg_k'}[\lambda] + e^{-\frac{1}{2} \gamma (\ll 1 \gg)} \in f_{1, \ll 1 \gg \ll 1 \gg'}[\lambda] \right) + \\
& \ll 21 \gg + U_k[x_{24}, x_{34}] (\ll 1 \gg)
\end{aligned}$$

Out[*]=

$$\begin{aligned}
& E \left[x_{12_k} \left(\xi_{12_i} + e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \right) - 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{13_k} \xi_{12_j} \xi_{13_i} - \right. \\
& 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{14_k} \xi_{12_j} \xi_{14_i} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{13_k} \xi_{12_j} \xi_{23_i} + \\
& 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{23_k} \xi_{12_j} \xi_{23_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \in x_{13_k} x_{23_k} \xi_{13_j} \xi_{23_i} + \\
& x_{13_k} \left(\xi_{13_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \right) + \\
& x_{23_k} \left(\xi_{23_i} + e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \right) + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \in x_{12_k} x_{24_k} \xi_{12_j} \xi_{24_i} - \\
& 4 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \in x_{14_k} x_{23_k} \xi_{13_j} \xi_{24_i} - 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \in x_{23_k} x_{24_k} \xi_{23_j} \xi_{24_i} + \in x_{13_k} x_{14_k} \\
& \left(-2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{14_i} + 2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{14_i} \xi_{23_i} + 2 e^{\frac{1}{2} (3\alpha_1 - \alpha_3)} \xi_{12_j} \xi_{13_j} \xi_{24_i} \right) + \\
& 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \in x_{13_k} x_{34_k} \xi_{13_j} \xi_{34_i} - 2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \in x_{14_k} x_{34_k} \xi_{14_j} \xi_{34_i} - \\
& e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \in x_{24_k} \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \in x_{23_k} x_{34_k} \xi_{23_j} \xi_{34_i} - \\
& 2 e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} \in x_{24_k} x_{34_k} \xi_{24_j} \xi_{34_i} + \in x_{14_k} \left(-e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\
& x_{14_k} \left(\xi_{14_i} + e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} - e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{24_i} - e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{34_i} \right) + \\
& x_{24_k} \left(\xi_{24_i} + e^{\frac{1}{2} (-\alpha_1 + \alpha_2 + \alpha_3)} \xi_{24_j} - e^{\frac{1}{2} (-\alpha_1 + 2\alpha_2 - \alpha_3)} \xi_{23_j} \xi_{34_i} \right) + \\
& \in x_{13_k} x_{24_k} \left(-2 e^{\frac{1}{2} (2\alpha_1 - \alpha_2)} \xi_{12_j} \xi_{23_i} \xi_{24_i} - 2 e^{\frac{1}{2} (3\alpha_2 - 2\alpha_3)} \xi_{13_j} \xi_{23_j} \xi_{34_i} \right) + \\
& \in x_{14_k} x_{24_k} \left(-2 e^{\frac{1}{2} (\alpha_1 + \alpha_3)} \xi_{14_j} \xi_{24_i} + 2 e^{\alpha_2} \xi_{14_j} \xi_{23_j} \xi_{34_i} + 2 e^{\frac{1}{2} (\alpha_1 + \alpha_2 - \alpha_3)} \xi_{13_j} \xi_{24_i} \xi_{34_i} \right) + \\
& x_{34_k} \left(\xi_{34_i} + e^{\frac{1}{2} (-\alpha_2 + 2\alpha_3)} \xi_{34_j} \right) \left. \right]
\end{aligned}$$

In[*]:= **Block**[{**\$k** = 2, **\$n** = 3}, **amt**_{i,j→k}]

$$\gg \text{TTL} \left(e^{-\frac{1}{2} \gamma (2\alpha_{1i} + 2\alpha_{1j} - \alpha_{2i} - \alpha_{2j})} \xi_{12i} + e^{-\frac{1}{2} \gamma (2\alpha_{1j} - \alpha_{2j})} \xi_{12j} \right) U_k[x_{12}] + \ll 14 \gg$$

$$\gg \text{SG} U_k[] + \ll 20 \gg +$$

$$U_k[x_{13}, x_{23}, x_{23}] \left(2 e^{\frac{1}{2} \gamma (\ll 1 \gg)} \gamma \epsilon^2 \hbar f_{\theta, x_{23k}}[\lambda] f_{1, x_{13k} x_{23k}}[\lambda] + e^{\ll 1 \gg} \ll 3 \gg + e^{\ll 1 \gg} \epsilon^2 f_{2, \ll 1 \gg}[\lambda] \right)$$

$$\gg \text{TTR} \ll 1 \gg$$

Out[*]=

$$\begin{aligned} & \mathbb{E} \left[x_{12k} \left(\xi_{12i} + e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \right) - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{12k} x_{13k} \xi_{12j} \xi_{13i} - \right. \\ & e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{13k} \xi_{12j} \xi_{23i} - \frac{1}{2} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{13k} \xi_{12j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon x_{12k} x_{23k} \xi_{12j} \xi_{23i} + \\ & 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{12k} x_{23k} \xi_{12j} \xi_{23i} + 2 e^{2\alpha_{1i} - \alpha_{2i}} \epsilon^2 x_{12k}^2 x_{23k} \xi_{12j}^2 \xi_{23i} - \\ & 2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \epsilon x_{13k} x_{23k} \xi_{13j} \xi_{23i} + 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{12k} x_{23k}^2 \xi_{12j} \xi_{23i}^2 + e^{2\alpha_{1i} - \alpha_{2i}} \epsilon^2 x_{13k}^2 \xi_{12j}^2 \xi_{23i}^2 - \\ & \frac{2}{9} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \epsilon^2 x_{13k}^3 \xi_{12j}^3 \xi_{23i}^3 + x_{13k} \left(\xi_{13i} + e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} - e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i} \right) + \\ & \epsilon^2 x_{12k} x_{13k} \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} - 2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i} \right) + \\ & \epsilon^2 x_{12k}^2 x_{13k} \left(2 e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{13i} - \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i} \right) + \\ & \epsilon^2 x_{12k} x_{13k} x_{23k} \left(-4 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i} \xi_{23i} - 4 e^{\frac{3\alpha_{1i}}{2}} \xi_{12j} \xi_{13j} \xi_{23i} \right) + \\ & \epsilon^2 x_{13k} x_{23k} \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i} - 2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^2 \right) + \\ & \epsilon^2 x_{12k} x_{13k}^2 \left(2 e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{13i}^2 + \frac{2}{3} e^{\frac{3}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j}^3 \xi_{23i}^2 \right) + \\ & \epsilon^2 x_{13k} x_{23k}^2 \left(2 e^{\frac{1}{2} (\alpha_{1i} + \alpha_{2i})} \xi_{13j} \xi_{23i}^2 - \frac{2}{3} e^{\frac{1}{2} (2\alpha_{1i} - \alpha_{2i})} \xi_{12j} \xi_{23i}^3 \right) + \\ & \epsilon^2 x_{13k}^2 x_{23k} \left(2 e^{\alpha_{1i} + \alpha_{2i}} \xi_{13j}^2 \xi_{23i} + \frac{2}{3} e^{2\alpha_{1i} - \alpha_{2i}} \xi_{12j}^2 \xi_{23i}^3 \right) + x_{23k} \left(\xi_{23i} + e^{\frac{1}{2} (-\alpha_{1i} + 2\alpha_{2i})} \xi_{23j} \right) \end{aligned}$$

pdf

```

In[*]:= aΔti→j,k := PPaΔt[$k,$n]@Module[{(*TTL, *)},
  ES[s_][ε_] := (EchoLabel[s][Short[ε, 2]]); ε);
  TTL =
    ES["TTL"]@TTλ@Table[UCF[If[τ[g] == "x", λ, 1] (g*)i Δi→j,k[Ui[g]]], {g, $PBWGens}];
  mons[-1] = {};
  mons[p_] := mons[p-1] ∪
    Union@Cases[Coefficient[TTL, ε, p], c_. Uj[x___] Uk[y___] ⇒ Uj[x] Uk[y], ∞];
  ffj = Table[
    Sum[εp fp,uj[λ], {p, 0, $k}],
    {u, $PBWGens}
  ] /. fp,uj[λ] /; τ[u] == "a" ⇒ If[p == 0, (u*)i, 0];
  ffk = ffj /. j → k;
  fs = Flatten[Table[
    fp, Pj,k[m],
    {p, 0, $k}, {m, mons[p]}
  ]];
  gg = Sum[
    εp fp, Pj,k[u][λ] Pj,k[u],
    {p, 0, $k}, {u, Complement[mons[p], Uj[] (Uk /@ $PBWGens) ∪ Uk[] (Uj /@ $PBWGens)]}
  ];
  G = Series[eGG, {ε, 0, $k}];
  S[B_] := UCF[SS@Oj,k[Seepagej,k[ffj, ffk, SS@UCF@B]]];
  SG = ES["SG"]@PPSG@S[G];
  TTR = ES["TTR"]@PPTTR@UCF[
    Invertj,k[SG] ⓧ S[(∂λ (ffj. (#j & /@ $PBWGens) + ffk. (#k & /@ $PBWGens)) G + ∂λ G)];
  diff = UCF[TTL - TTR];
  eqns = Flatten[Table[
    Coefficient[Coefficient[diff, m], ε, p] == 0,
    {p, 0, $k}, {m, mons[p]}
  ]];
  inits = (#[0] == 0) & /@ fs;
  ss = TriangularDSolve[eqns, inits, fs, λ, CF];
  E[Total[fs] /. fp,u ⇒ εp u (fp,u /. ss)[λ]] /. λ | ħ | γ → 1
]

```

```

In[*]:= Block[{$k = 1, $n = 3}, aΔti→j,k]

```

$$\gg \text{TTL } e^{-\frac{1}{2} \gamma (2\alpha_{1i} - \alpha_{2i})} \xi_{12_i} U_j [x_{12}] U_k [] + \ll 32 \gg + 2 e^{-\frac{3 \ll 1 \gg \ll 1 \gg \ll 1 \gg}{2}} \gamma \in \lambda \hbar \xi_{13_i} \xi_{23_i} U_j [] U_k [x_{13}, x_{23}]$$

$$\gg \text{SG } U_j [] U_k [] + \ll 1 \gg + \ll 20 \gg + \ll 1 \gg +$$

$$U_j [] U_k [x_{13}, x_{23}] \left(e^{-\frac{3 \gamma \alpha_{2i}}{2}} \in f_{0, x_{23_k}} [\lambda] f_{1, x_{12_k} x_{23_k}} [\lambda] + e^{-\frac{\ll 1 \gg}{2}} \in f_{1, \ll 1 \gg \ll 1 \gg} [\lambda] \right)$$

$$\gg \text{TTR } \ll 1 \gg$$

Out[*]=

$$\begin{aligned} & \mathbb{E} [x_{12_j} \xi_{12_i} - 2 \in a_{1_k} x_{12_j} \xi_{12_i} + x_{12_k} \xi_{12_i} + 2 \in a_{1_j} x_{12_k} \xi_{12_i} + \\ & x_{13_j} \xi_{13_i} - 2 \in a_{1_k} x_{13_j} \xi_{13_i} - 2 \in a_{2_k} x_{13_j} \xi_{13_i} + x_{13_k} \xi_{13_i} + 2 \in a_{1_j} x_{13_k} \xi_{13_i} + \\ & 2 \in a_{2_j} x_{13_k} \xi_{13_i} + \in x_{12_k} x_{13_j} \xi_{12_i} \xi_{13_i} - \in x_{12_j} x_{13_k} \xi_{12_i} \xi_{13_i} + x_{23_j} \xi_{23_i} - \\ & 2 \in a_{2_k} x_{23_j} \xi_{23_i} + x_{23_k} \xi_{23_i} + 2 \in a_{2_j} x_{23_k} \xi_{23_i} - \in x_{12_k} x_{23_j} \xi_{12_i} \xi_{23_i} + \\ & \in x_{13_k} x_{23_j} \xi_{13_i} \xi_{23_i} - \in x_{13_j} x_{23_k} \xi_{13_i} \xi_{23_i} + \in x_{12_j} x_{23_k} (4 \xi_{13_i} + \xi_{12_i} \xi_{23_i})] \end{aligned}$$

$$\begin{aligned} \text{In[*]} := & \mathbb{E} [x_{12_j} \xi_{12_i} - 2 \in a_{1_k} x_{12_j} \xi_{12_i} + x_{12_k} \xi_{12_i} + 2 \in a_{1_j} x_{12_k} \xi_{12_i} + \\ & x_{13_j} \xi_{13_i} - 2 \in a_{1_k} x_{13_j} \xi_{13_i} - 2 \in a_{2_k} x_{13_j} \xi_{13_i} + x_{13_k} \xi_{13_i} + 2 \in a_{1_j} x_{13_k} \xi_{13_i} + \\ & 2 \in a_{2_j} x_{13_k} \xi_{13_i} + \in x_{12_k} x_{13_j} \xi_{12_i} \xi_{13_i} - \in x_{12_j} x_{13_k} \xi_{12_i} \xi_{13_i} + x_{23_j} \xi_{23_i} - \\ & 2 \in a_{2_k} x_{23_j} \xi_{23_i} + x_{23_k} \xi_{23_i} + 2 \in a_{2_j} x_{23_k} \xi_{23_i} - \in x_{12_k} x_{23_j} \xi_{12_i} \xi_{23_i} + \\ & \in x_{13_k} x_{23_j} \xi_{13_i} \xi_{23_i} - \in x_{13_j} x_{23_k} \xi_{13_i} \xi_{23_i} + \in x_{12_j} x_{23_k} (4 \xi_{13_i} + \xi_{12_i} \xi_{23_i})] /. \epsilon \rightarrow 0 \end{aligned}$$

Time: 0.06 seconds Out[*]=

$$\mathbb{E} [x_{12_j} \xi_{12_i} + x_{12_k} \xi_{12_i} + x_{13_j} \xi_{13_i} + x_{13_k} \xi_{13_i} + x_{23_j} \xi_{23_i} + x_{23_k} \xi_{23_i}]$$