Hidden Utilities and TEX

Headers

pdf

Pensieve header: Quantum sl_n following Rosso and Yamane.

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Notes.
1. Rosso is "An Analogue of P.B.W. Theorem and the Universal R-matrix for $U_\hbar(sl(N+1))$", Yamane is "A Poincare-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N".
2. We call our algebra $U_q(\mathrm{sl}_{\$n})$, not $U_\hbar(\mathrm{sl}(N+1))$.
3. We dot not have a name for Rosso's X_i and Y_j. Rosso's $E_{\epsilon_i-\epsilon_{j+1}}$ is our $x[i,j]\leftrightarrow$ xij. Likewise Rosso's $\eta_{\epsilon_i-\epsilon_{j+1}}$ is our $y[i,j]\leftrightarrow$ yij.
4. Rosso's H_i is our a_i and Rosso's ξ_i is our b_i.

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In[•]:=
```
SetDirectory@"C:\\drorbn\\AcademicPensieve\\Projects\\SolvablePBW";
<< KnotTheory`
<< ../Profile/Profile.m
```

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```
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.
```

pdf

```
This version: April 2020. Original version: July 1994.
```

In[•]:=
```
MakePDF[]
```

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In[•]:=
```
δij__ := KroneckerDelta[ij];
```

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In[•]:=
```
SetAttributes[{SS}, HoldAll];
SS[ℰ_, op_] := Collect[Normal@Series[ℰ, {ϵ, 0, $k}], ϵ, op];
SS[ℰ_] := SS[ℰ, Together];
```

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In[•]:=
```
CCF[ℰ_] := PPCCF@Expand[Expand[ℰ] /. e^p_ :> e^Factor@p];
CF[ℰ_] := PPCF@If[$k == ∞,
    CCF[ℰ],
    CCF[Normal@Series[ℰ /. q → e^(ħ ϵ γ), {ϵ, 0, $k}]]
   ];
```

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```
In[•]:= x[i_, j_] := x <> i <> j; a[i_] := a <> i; A[i_] := A <> i;
        ξ[i_, j_] := ξ <> i <> j; α[i_] := α <> i; 𝔄[i_] := 𝔄 <> i;
```

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```
In[•]:= τ[s_Symbol] := First[Characters[SymbolName[s]]];
        ι[s_Symbol] := FromDigits /@ Rest[Characters[SymbolName[s]]];
        τ[s_[_]] := τ[s]; ι[s_[_]] := ι[s];
```

```
In[•]:= {τ[x[3, 4]], ι[x[3, 4]]} // FullForm
```

Time: 0.03 seconds Out[•]=

```
List["x", List[3, 4]]
```

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The following ordering agrees with Rosso's; right at the start of Section 2 and then continued implicitly within the formula for *R* at the start of Section 4. Simply put, each class of variables is ordered lexicographically, and the classes are ordered xAayBb.

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```
In[•]:= (* q=ℯ^(ℏ γ ∈); Ai[n]=ℯ^(ℏ ∈ ai); Bi[n]=ℯ^(ℏ γ bi) *)
```

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```
In[•]:= $n = 3; $k = ∞; $E := {$n, $k}
```

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```
In[•]:= $EPBWGens := Flatten@{
          Table[As[i, 1, $n - 1], As[j, i + 1, $n],
           x[i, j]* = ξ[i, j]; ξ[i, j]* = x[i, j]; x[i, j]],
          Table[As[i, 1, $n - 1],
           A[i]* = 𝔄[i]; 𝔄[i]* = A[i]; A[i][_]],
          Table[As[i, 1, $n - 1],
           a[i]* = α[i]; α[i]* = a[i]; a[i]]
         }
        (#_U = U_0@#) & /@ $EPBWGens;
        $EPBWRule := Module[{k = 0}, Flatten@Table[{g → ++k, g_i_ → {i, k}}, {g, $EPBWGens}]];
        x_ ≼ y_ := OrderedQ[{x, y} /. $EPBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $EPBWRule];
        $PBWGens := DeleteCases[$EPBWGens, _[_]]
```

```
In[•]:= #* & /@ $EPBWGens
```

Time: 0.00 seconds Out[•]=

```
{ξ12, ξ13, ξ14, ξ15, ξ23, ξ24, ξ25, ξ34,
 ξ35, ξ45, A1[_]*, A2[_]*, A3[_]*, A4[_]*, α1, α2, α3, α4}
```

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In[•]:=
```
UCF[ℰ_] := PP_UCF@Module[{F, k}, Expand@Collect[
      ℰ //. {U_i_[l___, _[0], r___] → U_i[l, r]} //. If[$k == ∞, {}, {
        U_i_[l___, AB_[n_], r___] :>
          Sum[As[k, 0, $k], (-2)^k ħ^k ϵ^k n^k / k! U_i[l, Sequence @@ Table[a @@ ι[AB], k], r]]
        }],
      U_[___], F
     ] /. F → CF]
```

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In[•]:=
```
m_s__[0] = 0;
m_s__[x_Plus] := UCF[m_s /@ x];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  UCF[c U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k;
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((c U_i[xx] RR_j[x, y] // UCF // m_i,j→i) U_j[yy] // UCF // m_i,j→k) // UCF
  ];
m_s__[ℰ_] := m_s[UCF[ℰ]];
```

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In[•]:=
```
Δ_i_→j_,k_[ℰ_] := UCF@Module[{j1, k1}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[f_, fs___] :> m_k,k1→k@m_j,j1→j[Δ_j,k[f] Δ_i→j1,k1[U_i[fs]]]
     }];
Δ_i_→j_,k_,l__[ℰ_] := ℰ // Δ_i→j,k // Δ_k→k,l
```

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In[•]:=
```
S_i_[ℰ_] := UCF@Module[{i1}, ℰ /. {
      U_i[f_, fs___] :> m_i1,i→i[S_i[f] S_i1[U_i1[fs]]]
     }];
```

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The 𝔸-algebra

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In[•]:=
```
RR_k_[ai_, aj_] /; τ[ai] == τ[aj] == "a" := U_k[aj, ai];
RR_k_[ai_, Aj_[n_]] /; τ[ai] == "a" ∧ τ[Aj] == "A" := U_k[Aj[n], ai];
RR_k_[Ai_[m_], Aj_[n_]] /; τ[Ai] == τ[Aj] == "A" := U_k[Aj[n], Ai[m]];
RR_k_[am_, xij_] /; τ[am] == "a" ∧ τ[xij] == "x" := Module[{i, j, m},
   {i, j} = ι@xij; {m} = ι@am;
   U_k[xij, am] + γ (δ_{m,i} - δ_{m+1,i} - δ_{m,j} + δ_{m+1,j}) U_k[xij] / 2
  ];
RR_k_[Am_[n_], xij_] /; τ[Am] == "A" ∧ τ[xij] == "x" := Module[{i, j, m},
  {i, j} = ι@xij; {m} = ι@Am; q^(-n (δ_{m,i}-δ_{m+1,i}-δ_{m,j}+δ_{m+1,j})) U_k[xij, Am[n]]
 ]
```

In[•]:=
```
RR_0[a[1], x[1, 2]]
```

Time: 0.01 seconds Out[•]=

$\gamma\, U_0[x12] + U_0[x12, a1]$

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We take the reduction rules RR for xij's from Yamane page 509:

(I) $i = m < j < n$, $\underline{i\quad\ j}$, $\underline{m\quad\ n}$ (II) $i < m < n < j$, $\underline{i\quad\ \ j}$, $\underline{m\quad n}$ (III) $i < m < j = n$, $\underline{i\quad\ \ j}$, $\underline{m\quad\ n}$

(IV) $i < m < j < n$, $\underline{i\quad\ j}$ (V) $i < j = m < n$, $\underline{i\quad j}$ (VI) $i < j < m < n$. $\underline{i\ \ j}$

$$q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 \quad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{III})}.$$
$$[e_{ij}, e_{mn}] = 0 \quad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{II})} \bigcup C_{(\mathrm{VI})}.$$
$$[e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} \quad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{IV})}.$$
$$q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = q e_{in} \quad \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})}.$$

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In[•]:=
```
RR_k_[xmn_, xij_] /; τ[xij] == τ[xmn] == "x" := Module[{i, j, m, n},
  {i, j} = ι@xij; {m, n} = ι@xmn;
  Which[
   (i == m ∧ j < n) ∨ (i < m < j ∧ n == j), q^-2 U_k[xij, xmn], (* I∨III *)
   (i < m < j ∧ n < j) ∨ (j < m), U_k[xij, xmn], (* II∨VI *)
   i < m < j ∧ n > j, U_k[xij, xmn] + (q^-2 - q^2) U_k[x[i, n], x[m, j]], (* IV *)
   j == m, q^2 U_k[xij, xmn] - q U_k[x[i, n]] (* V *)
  ]
 ]
```

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2
```

Time: 0.03 seconds Out[•]=

$$-q\,U_2[x24]\,U_3[x12]+q^2\,U_2[x23,x34]\,U_3[x12]$$

```
In[•]:= U1[x34] U2[x23] U3[x12] // m1,2→2 // m2,3→0
```

Time: 0.06 seconds Out[•]=

$$q^2\,U_0[x14]-q^3\,U_0[x12,x24]-q^3\,U_0[x13,x34]+q^4\,U_0[x12,x23,x34]$$

```
In[•]:= Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
   ]]
]
```

Time: 40.53 seconds Out[•]=

```
{}
```

```
In[•]:= Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[X] U2[Y] U3[Z];
    {X, Y, Z} → (P // m1,2→2 // m2,3→0) == (P // m2,3→2 // m1,2→0)
   ]]
]
```

» {x12, x13, x23, a1, a2, A1[0], A2[-2], A1[2], A2[0], A1[-1], A1[0], A2[2], A1[1], A1[1], A1[-2]}

Out[•]=

```
{}
```

```
Module[{Bas = $PBWGens, P},
 DeleteCases[_ → True][Flatten@Table[As[W, Bas], As[X, Bas], As[Y, Bas], As[Z, Bas],
    P = U1[W] U2[X] U3[Y] U4[Z];
    {W, X, Y, Z} → (P // m2,3→2 // m2,4→2 // m1,2→1) ==
      (P // m3,4→3 // m2,3→2 // m1,2→1) == (P // m1,2→1 // m3,4→3 // m1,3→1) ==
      (P // m1,2→1 // m1,3→1 // m1,4→1) == (P // m2,3→2 // m1,2→1 // m1,4→1)
   ]]
]
```

» {x12, x13, x23, a1, a2}

Out[•]=

```
{}
```

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The Co-Product, Co-Associativity, and the Hopf Axiom

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In[•]:=

```
Do[As[i, 1, $n - 1],
  Δj_,k_[a[i]] = Uj[a[i]] Uk[] + Uj[] Uk[a[i]];
  Δj_,k_[A[i][n_]] = Uj[A[i][n]] Uk[A[i][n]];
  Δj_,k_[x[i, i + 1]] = Uj[x[i, i + 1]] Uk[A[i][1]] + Uj[A[i][-1]] Uk[x[i, i + 1]]
 ];
Do[As[i, 1, $n - 2], As[n, i + 2, $n],
 Δj_,k_[x[i, n]] = mk,k1→k[mj,j1→j[
    q Δj,k[x[i, n - 1]] Δj1,k1[x[n - 1, n]] - q^-1 Δj,k[x[n - 1, n]] Δj1,k1[x[i, n - 1]]
   ]]
]
```

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
          Table[As[P, Uᵢ /@ Bas], P → Δᵢ→ⱼ,ₖ[P]]
         ]
```

Time: 0.12 seconds Out[•]=

$$U_i[x12] \to U_j[x12]\,U_k[\,] - 2\in\hbar\,U_j[x12]\,U_k[a1] + U_j[\,]\,U_k[x12] + 2\in\hbar\,U_j[a1]\,U_k[x12] + 2\in^2\hbar^2\,U_j[a1,a1]\,U_k[x12] + 2\in^2\hbar^2\,U_j[x12]\,U_k[a1,a1]$$

$$U_i[x13] \to U_j[x13]\,U_k[\,] - 2\in\hbar\,U_j[x13]\,U_k[a1] - 2\in\hbar\,U_j[x13]\,U_k[a2] + U_j[\,]\,U_k[x13] + 2\in\hbar\,U_j[a1]\,U_k[x13] + 2\in\hbar\,U_j[a2]\,U_k[x13] + 2\in^2\hbar^2\,U_j[a1,a1]\,U_k[x13] + 4\in^2\hbar^2\,U_j[a1,a2]\,U_k[x13] + 2\in^2\hbar^2\,U_j[a2,a2]\,U_k[x13] + 4\gamma\in\hbar\,U_j[x12]\,U_k[x23] + 8\gamma\in^2\hbar^2\,U_j[x12,a2]\,U_k[x23] + 2\in^2\hbar^2\,U_j[x13]\,U_k[a1,a1] + 4\in^2\hbar^2\,U_j[x13]\,U_k[a1,a2] + 2\in^2\hbar^2\,U_j[x13]\,U_k[a2,a2] - 8\gamma\in^2\hbar^2\,U_j[x12]\,U_k[x23,a1]$$

$$U_i[x14] \to U_j[x14]\,U_k[\,] - 2\in\hbar\,U_j[x14]\,U_k[a1] - 2\in\hbar\,U_j[x14]\,U_k[a2] - 2\in\hbar\,U_j[x14]\,U_k[a3] + U_j[\,]\,U_k[x14] + 2\in\hbar\,U_j[a1]\,U_k[x14] + 2\in\hbar\,U_j[a2]\,U_k[x14] + 2\in\hbar\,U_j[a3]\,U_k[x14] + 2\in^2\hbar^2\,U_j[a1,a1]\,U_k[x14] + 4\in^2\hbar^2\,U_j[a1,a2]\,U_k[x14] + 4\in^2\hbar^2\,U_j[a1,a3]\,U_k[x14] + 2\in^2\hbar^2\,U_j[a2,a2]\,U_k[x14] + 4\in^2\hbar^2\,U_j[a2,a3]\,U_k[x14] + 2\in^2\hbar^2\,U_j[a3,a3]\,U_k[x14] + 4\gamma\in\hbar\,U_j[x12]\,U_k[x24] + 8\gamma\in^2\hbar^2\,U_j[x12,a2]\,U_k[x24] + 8\gamma\in^2\hbar^2\,U_j[x12,a3]\,U_k[x24] + 4\gamma\in\hbar\,U_j[x13]\,U_k[x34] + 8\gamma\in^2\hbar^2\,U_j[x13,a3]\,U_k[x34] + 2\in^2\hbar^2\,U_j[x14]\,U_k[a1,a1] + 4\in^2\hbar^2\,U_j[x14]\,U_k[a1,a2] + 4\in^2\hbar^2\,U_j[x14]\,U_k[a1,a3] + 2\in^2\hbar^2\,U_j[x14]\,U_k[a2,a2] + 4\in^2\hbar^2\,U_j[x14]\,U_k[a2,a3] + 2\in^2\hbar^2\,U_j[x14]\,U_k[a3,a3] - 8\gamma\in^2\hbar^2\,U_j[x12]\,U_k[x24,a1] - 8\gamma\in^2\hbar^2\,U_j[x13]\,U_k[x34,a1] - 8\gamma\in^2\hbar^2\,U_j[x13]\,U_k[x34,a2]$$

$$U_i[x23] \to U_j[x23]\,U_k[\,] - 2\in\hbar\,U_j[x23]\,U_k[a2] + U_j[\,]\,U_k[x23] + 2\in\hbar\,U_j[a2]\,U_k[x23] + 2\in^2\hbar^2\,U_j[a2,a2]\,U_k[x23] + 2\in^2\hbar^2\,U_j[x23]\,U_k[a2,a2]$$

$$U_i[x24] \to U_j[x24]\,U_k[\,] - 2\in\hbar\,U_j[x24]\,U_k[a2] - 2\in\hbar\,U_j[x24]\,U_k[a3] + U_j[\,]\,U_k[x24] + 2\in\hbar\,U_j[a2]\,U_k[x24] + 2\in\hbar\,U_j[a3]\,U_k[x24] + 2\in^2\hbar^2\,U_j[a2,a2]\,U_k[x24] + 4\in^2\hbar^2\,U_j[a2,a3]\,U_k[x24] + 2\in^2\hbar^2\,U_j[a3,a3]\,U_k[x24] + 4\gamma\in\hbar\,U_j[x23]\,U_k[x34] + 8\gamma\in^2\hbar^2\,U_j[x23,a3]\,U_k[x34] + 2\in^2\hbar^2\,U_j[x24]\,U_k[a2,a2] + 4\in^2\hbar^2\,U_j[x24]\,U_k[a2,a3] + 2\in^2\hbar^2\,U_j[x24]\,U_k[a3,a3] - 8\gamma\in^2\hbar^2\,U_j[x23]\,U_k[x34,a2]$$

$$U_i[x34] \to U_j[x34]\,U_k[\,] - 2\in\hbar\,U_j[x34]\,U_k[a3] + U_j[\,]\,U_k[x34] + 2\in\hbar\,U_j[a3]\,U_k[x34] + 2\in^2\hbar^2\,U_j[a3,a3]\,U_k[x34] + 2\in^2\hbar^2\,U_j[x34]\,U_k[a3,a3]$$

$$U_i[a1] \to U_j[a1]\,U_k[\,] + U_j[\,]\,U_k[a1]$$

$$U_i[a2] \to U_j[a2]\,U_k[\,] + U_j[\,]\,U_k[a2]$$

$$U_i[a3] \to U_j[a3]\,U_k[\,] + U_j[\,]\,U_k[a3]$$

```
In[•]:= Short[{Δ4,5[a2], Δ7,9[A2[77]], Δ1,2[x12], Δ1,2[x15]}, 8]
```

Time: 0.02 seconds Out[•]=

$$\Big\{U_4[a2]\,U_5[\,] + U_4[\,]\,U_5[a2],\ U_7[A2[77]]\,U_9[A2[77]],$$

$$U_1[A1[-1]]\,U_2[x12] + U_1[x12]\,U_2[A1[1]],\ U_1[A1[-1],A2[-1],A3[-1],A4[-1]]\,U_2[x15] + \left(-\frac{1}{q^2}+q^2\right)U_1[x12,A2[-1],A3[-1],A4[-1]]\,U_2[x25,A1[1]] + \left(-\frac{1}{q^2}+q^2\right)U_1[x13,A3[-1],A4[-1]]\,U_2[x35,A1[1],A2[1]] + \left(-\frac{1}{q^2}+q^2\right)U_1[x14,A4[-1]]\,U_2[x45,A1[1],A2[1],A3[1]] +$$

$$U_1[x15]\,U_2[A1[1],A2[1],A3[1],A4[1]]\Big\}$$

```
In[•]:= Δ1→1,2[U1[x13, a2] U0[x34]]
```

Time: 0.03 seconds Out[•]=

$$U_0[x34]\,U_1[A1[-1], A2[-1], a2]\,U_2[x13] + U_0[x34]\,U_1[A1[-1], A2[-1]]\,U_2[x13, a2] + \left(-\frac{1}{q^2} + q^2\right) U_0[x34]\,U_1[x12, A2[-1], a2]\,U_2[x23, A1[1]] + U_0[x34]\,U_1[x13, a2]\,U_2[A1[1], A2[1]] + \left(-\frac{1}{q^2} + q^2\right) U_0[x34]\,U_1[x12, A2[-1]]\,U_2[x23, A1[1], a2] + U_0[x34]\,U_1[x13]\,U_2[A1[1], A2[1], a2]$$

```
In[•]:= $k = 1
```

Time: 0.01 seconds Out[•]=

```
1
```

```
In[•]:= Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U1[X];
    X → (P // Δ1→1,2 // Δ2→2,3) == (P // Δ1→1,3 // Δ1→1,2)
   ]]
]
» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A2[1], A2[0],
 A1[-2], A3[-2], A1[-2], A1[-2], A3[2], A3[-2], A2[-2], A1[2]}
```

Out[•]=

```
{}
```

```
In[•]:= $k = 1; $n = 4
```

Time: 0.02 seconds Out[•]=

```
4
```

```
In[•]:= Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → 0][Flatten@Table[As[X, Bas], As[Y, Bas],
    P = U1[X] U2[Y];
    {X, Y} → UCF[(P // m1,2→1 // Δ1→1,2) - (P // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)]
   ]]
]
» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A3[0],
 A2[-1], A2[0], A2[-2], A1[1], A1[-2], A2[2], A3[1], A1[0], A3[-2]}
```

Out[•]=

```
{}
```

Is Δ a quantization of the co-bracket?

```
In[•]:= $k = 1
```

Time: 0.01 seconds Out[•]=

```
1
```

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, U₁ /@ Bas],
   P → UCF[(Δ₁→₁,₂[P] - Δ₁→₂,₁[P]) //.
       {q^n_. :> 1 + n ħ / 2, Uᵢ_[xs___, A_[n_], as___] :> Uᵢ[xs] + n ħ Uᵢ[xs, a @@ ι[A], as] / 2}
     ] /. ħ^n_ /; n > 1 :> 0
   ]
  ]
```

Time: 0.09 seconds Out[•]=

$$U_1[x12] \to -4 \in \hbar\, U_1[x12]\, U_2[a1] + 4 \in \hbar\, U_1[a1]\, U_2[x12]$$

$$U_1[x13] \to -4 \in \hbar\, U_1[x13]\, U_2[a1] - 4 \in \hbar\, U_1[x13]\, U_2[a2] - 4\,\gamma \in \hbar\, U_1[x23]\, U_2[x12] + 4 \in \hbar\, U_1[a1]\, U_2[x13] + 4 \in \hbar\, U_1[a2]\, U_2[x13] + 4\,\gamma \in \hbar\, U_1[x12]\, U_2[x23]$$

$$U_1[x14] \to -4 \in \hbar\, U_1[x14]\, U_2[a1] - 4 \in \hbar\, U_1[x14]\, U_2[a2] - 4 \in \hbar\, U_1[x14]\, U_2[a3] - 4\,\gamma \in \hbar\, U_1[x24]\, U_2[x12] - 4\,\gamma \in \hbar\, U_1[x34]\, U_2[x13] + 4 \in \hbar\, U_1[a1]\, U_2[x14] + 4 \in \hbar\, U_1[a2]\, U_2[x14] + 4 \in \hbar\, U_1[a3]\, U_2[x14] + 4\,\gamma \in \hbar\, U_1[x12]\, U_2[x24] + 4\,\gamma \in \hbar\, U_1[x13]\, U_2[x34]$$

$$U_1[x23] \to -4 \in \hbar\, U_1[x23]\, U_2[a2] + 4 \in \hbar\, U_1[a2]\, U_2[x23]$$

$$U_1[x24] \to -4 \in \hbar\, U_1[x24]\, U_2[a2] - 4 \in \hbar\, U_1[x24]\, U_2[a3] - 4\,\gamma \in \hbar\, U_1[x34]\, U_2[x23] + 4 \in \hbar\, U_1[a2]\, U_2[x24] + 4 \in \hbar\, U_1[a3]\, U_2[x24] + 4\,\gamma \in \hbar\, U_1[x23]\, U_2[x34]$$

$$U_1[x34] \to -4 \in \hbar\, U_1[x34]\, U_2[a3] + 4 \in \hbar\, U_1[a3]\, U_2[x34]$$

$$U_1[a1] \to 0$$

$$U_1[a2] \to 0$$

$$U_1[a3] \to 0$$

pdf

The Antipode

pdf

```
In[•]:= Do[As[i, 1, $n - 1],
   S_j_[a[i]] = -U_j[a[i]];
   S_j_[A[i][n_]] = U_j[A[i][-n]];
   S_j_[x[i, i + 1]] = -q^-2 U_j[x[i, i + 1]] (*?*)
  ];
 Do[As[i, 1, $n - 2], As[n, i + 2, $n],
  S_j_[x[i, n]] = m_j1,j→j[
    q S_j[x[i, n - 1]] S_j1[x[n - 1, n]] - q^-1 S_j[x[n - 1, n]] S_j1[x[i, n - 1]]
   ]
 ]
```

```
In[•]:= $k = 2
```

Time: 0.00 seconds Out[•]=

2

```
In[•]:= Module[{Bas = $PBWGens, P},
 Bas = Echo@Join[Bas, Table[A[RandomInteger[{1, $n - 1}]][RandomInteger[{-2, 2}]], {10}]];
 DeleteCases[_ → True][Flatten@Table[As[X, Bas],
    P = U₁[X];
    X → (P // Δ₁→₂,₃ // S₂ // m₂,₃→₁)
   ]]
]
```

```
» {x12, x13, x14, x23, x24, x34, a1, a2, a3, A2[0],
 A1[0], A2[0], A2[1], A2[0], A3[0], A3[-1], A1[1], A1[-2], A3[0]}
```

```
Out[•]=
{x12 → 0, x13 → 0, x14 → 0, x23 → 0, x24 → 0, x34 → 0, a1 → 0, a2 → 0,
 a3 → 0, A2[0] → U₁[], A1[0] → U₁[], A2[0] → U₁[], A2[1] → U₁[], A2[0] → U₁[],
 A3[0] → U₁[], A3[-1] → U₁[], A1[1] → U₁[], A1[-2] → U₁[], A3[0] → U₁[]}
```

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The Generating Functions for the 𝔸 operations

```
In[•]:= BeginProfile[]
```

Time: 0.30 seconds Out[•]=

ProfileRoot

```
In[•]:= $k = 1;
```

pdf

NonCommutativeMultiply, B, adPower, Ad, TT, Seepage

pdf

```
In[•]:= Supp[ℰ_] := Union@Cases[{ℰ}, Uᵢ_[___] ↦ i, ∞];
```

```
In[•]:= Supp[2 ℏ U₁[x45] U₂[x34]]
```

Time: 0.01 seconds Out[•]=

{1, 2}

pdf

```
In[•]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ⊠ y_ := Module[{i, is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = mᵢ→σ@i[z], {i, is}];
   z = Expand[y z]; Do[z = mσ@i,i→i[z], {i, is}]; z];
UB[x_, y_] := PPUB@UCF[x ⊠ y - y ⊠ x];
```

```
In[•]:= (2 ℏ₁ U₁[x45] U₂[x34]) ⊠ (2 ℏ₂ U₁[x45] U₂[x34])
```

Time: 0.03 seconds Out[•]=

4 ℏ₁ ℏ₂ U₁[x45, x45] U₂[x34, x34]

pdf

```
In[•]:= (*adPower[x_,k_,Env_][c_*y_QU]:=UCF[c adPower[x,k,Env][y]];
adPower[x_,k_,Env_][y_Plus] := adPower[x,k,Env]/@ y;*)
adPower[_, 0, Env_][y_] := y;
(*adPower[x_,k_,Env_][0]=0;*)
adPower[x_, k_, Env_][y_] :=
  adPower[x, k, Env][y] = UCF@UB[adPower[x, k - 1, Env][y], x];
adPower[x_, k_][y_] := adPower[x, k, $E][y];
```

```
In[•]:= Table[As[k, 0, 5], adPower[U0[x12], k][U0[x23]]]
```

Time: 0.06 seconds Out[•]=

$$\{U_0[x23],\ (-1-\gamma\in\hbar)\,U_0[x13]+2\,\gamma\in\hbar\,U_0[x12,x23],\ 0,\ 0,\ 0,\ 0\}$$

```
In[•]:= Column@Module[{Bas = $PBWGens, P},
  Table[As[P, Ui /@ Bas], P → Δi→j,k[P]]
 ]
```

Time: 0.08 seconds Out[•]=

$$U_i[x12]\to U_j[x12]\,U_k[]+\in\hbar\,U_j[x12]\,U_k[a1]+U_j[]\,U_k[x12]+\in\hbar\,U_j[a1]\,U_k[x12]$$
$$U_i[x13]\to U_j[x13]\,U_k[]+\in\hbar\,U_j[x13]\,U_k[a1]+\in\hbar\,U_j[x13]\,U_k[a2]+$$
$$U_j[]\,U_k[x13]+\in\hbar\,U_j[a1]\,U_k[x13]+\in\hbar\,U_j[a2]\,U_k[x13]+4\,\gamma\in\hbar\,U_j[x12]\,U_k[x23]$$
$$U_i[x23]\to U_j[x23]\,U_k[]+\in\hbar\,U_j[x23]\,U_k[a2]+U_j[]\,U_k[x23]+\in\hbar\,U_j[a2]\,U_k[x23]$$
$$U_i[a1]\to U_j[a1]\,U_k[]+U_j[]\,U_k[a1]$$
$$U_i[a2]\to U_j[a2]\,U_k[]+U_j[]\,U_k[a2]$$

pdf

```
In[•]:= Ad[0] = Identity;
```

pdf

```
In[•]:= Ad[c_. Ui_[xjk_] (EUs : U_[] ...)][y_] /; τ[xjk] == "x" :=  PPAd@Module[{k = 0, s = 0, t},
   While[(t = adPower[Ui[xjk], k][y]) =!= 0,
    s += c^k t / k!;
    ++k
   ];
   UCF[s ⊠ EUs]
  ]
```

```
In[•]:= Ad[∈ ℏ Uj[x12] Uk[]][Uj[x23] Uk[a4]]
```

Time: 0.09 seconds Out[•]=

$$-\in\hbar\,U_j[x13]\,U_k[a4]+U_j[x23]\,U_k[a4]$$

```
In[•]:= Ad[7 ℏ Uj[x12] Uk[]][Uj[x23] Uk[a4]]
```

Time: 0.08 seconds Out[•]=

$$\left(-7\,\hbar-7\,\gamma\in\hbar^2\right)U_j[x13]\,U_k[a4]+U_j[x23]\,U_k[a4]+14\,\gamma\in\hbar^2\,U_j[x12,x23]\,U_k[a4]$$

pdf

```
In[•]:= Wtaj_[x_] := Wtaj[x] = UB[U0[aj], U0[x]] / U0[x]
```

```
In[•]:= Table[Wt_a[j][x23], {j, 1, $n}]
```

Time: 0.08 seconds Out[•]=

$$\left\{-\frac{\gamma}{2}, \gamma, -\frac{\gamma}{2}, 0, 0\right\}$$

pdf

```
In[•]:= Ad[c_. U_i_[aj_] (EUs : U_[] ...)][y_] /; τ[aj] == "a" :=
          (y /. U_i[gs___] :> U_i[gs] ℯ^(c Total[Wt_aj /@ {gs}]))
```

```
In[•]:= Ad[π 𝕚 U_j[a1]][U_j[x23] U_k[a4]]
```

Time: 0.03 seconds Out[•]=

$$e^{-\frac{1}{2}\,\mathbb{i}\,\pi\,\gamma}\,U_j[x23]\,U_k[a4]$$

pdf

```
In[•]:= TT_λ_[] = 0; (* Translated Tangent *)
        TT_λ_[xs___, x_] := PP_TT@UCF[Ad[x][TT_λ[xs]] + ∂_λ x];
        TT_λ_[xs_List] := TT_λ @@ xs
```

```
In[•]:= TT_λ[λ ξ12 U_0[x12] U_1[], λ ξ23 U_0[x23] U_1[]]
```

Time: 0.08 seconds Out[•]=

$$\xi 12\,U_0[x12]\,U_1[] + \lambda\,\xi 12\,\xi 23\,U_0[x13]\,U_1[] + \xi 23\,U_0[x23]\,U_1[]$$

```
In[•]:= $PBWGens
```

Time: 0.00 seconds Out[•]=

```
{x12, x13, x23, a1, a2}
```

```
In[•]:= Join[
         Cases[$PBWGens, g_ :> Which[
             τ[g] === "x", λ (g^*)_i U_k[g],
             τ[g] === "a", (g^*)_i U_k[g]
           ]],
         Cases[$PBWGens, g_ :> Which[
             τ[g] === "x", λ (g^*)_j U_k[g],
             τ[g] === "a", (g^*)_j U_k[g]
           ]]
        ]
```

Time: 0.03 seconds Out[•]=

$$\{\lambda\,\xi 12_i\,U_k[x12],\ \lambda\,\xi 13_i\,U_k[x13],\ \lambda\,\xi 23_i\,U_k[x23],\ \alpha 1_i\,U_k[a1],\ \alpha 2_i\,U_k[a2],$$
$$\lambda\,\xi 12_j\,U_k[x12],\ \lambda\,\xi 13_j\,U_k[x13],\ \lambda\,\xi 23_j\,U_k[x23],\ \alpha 1_j\,U_k[a1],\ \alpha 2_j\,U_k[a2]\}$$

```
In[•]:= 𝔒_i_[poly_] := Total[
          CoefficientRules[poly, $PBWGens] /.
            (p_ → c_) :> c U_i @@ Flatten@MapThread[ConstantArray, {$PBWGens, p}]
         ]
```

```
In[•]:= 𝒪ᵢ[x12² x13 + 7 x23¹⁷]
```

Time: 0.03 seconds Out[•]=

```
Uᵢ[x12, x12, x13] +
 7 Uᵢ[x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23, x23]
```

pdf

```
In[•]:= Seepageᵢ_[{}, c_] = c;
        Seepageᵢ_[ℱs_List, P_] := Module[{lx},
          lx = $PBWGens[[Length@ℱs]];
          CF@Collect[P, lx, (Ad[Last[ℱs]] Uᵢ[lx]][𝒪ᵢ[Seepageᵢ[Most@ℱs, #]]] /. Uᵢ → Times) &]
         ]
```

pdf

```
In[•]:= Invertᵢ_[ℰ_] := Module[{A, B, β, j}, (* ℰ=A U[]+ϵ B *)
          A = Coefficient[ℰ, Uᵢ[]] /. ϵ → 0;
          B = UCF[(ℰ - A Uᵢ[])];
          UCF[
           A⁻¹ (Uᵢ[] + Sum[As[j, 1, $k], (-1)ʲ NonCommutativeMultiply @@ ConstantArray[B / A, j]])]
         ]

        {$k = 1, $n = 3}
```

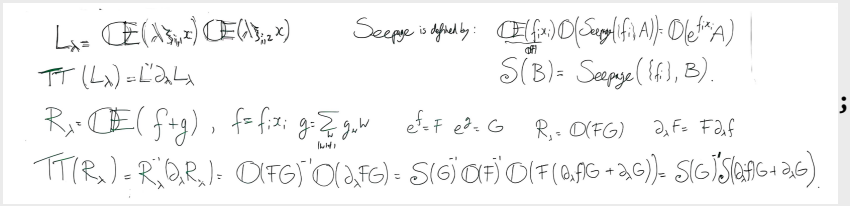
Time: 0.01 seconds Out[•]=

```
{2, 3}
```

pdf

```
In[•]:= sol[kk_, nn_] := PP"sol"[kk,nn]@Block[{$k = kk, $n = nn}, Module[{(*TTL, *)},
            ES[s_][ℰ_] := (EchoLabel[s][Short[ℰ, 5]]; ℰ);
            TTL = ES["TTL"]@TTλ@
               Join[
                Cases[$PBWGens, g_ ↦ Which[
                   τ[g] === "x", λ (g*)ᵢ Uₖ[g],
                   τ[g] === "a", (g*)ᵢ Uₖ[g]
                  ]],
                Cases[$PBWGens, g_ ↦ Which[
                   τ[g] === "x", λ (g*)ⱼ Uₖ[g],
                   τ[g] === "a", (g*)ⱼ Uₖ[g]
                  ]]
               ];
            mons[-1] = {};
            mons[p_] := mons[p - 1] ⋃ Union@Cases[Coefficient[TTL, ϵ, p], Uₖ[___], ∞];
            ff = Table[
               Sum[ϵᵖ f_{p,u}[λ], {p, 0, $k}],
               {u, $PBWGens}
              ] /. f_{p_,u_}[λ] /; τ[u] == "a" ↦ If[p == 0, (u*)ᵢ + (u*)ⱼ, 0];
            fs = Flatten[Table[
```

```
    f_{k,Times@@m},
    {k, 0, $k}, {m, mons[k]}
   ]];
 gg = Sum[
   ϵ^p f_{p,Times@@u}[λ] Times @@ u,
   {p, 0, $k}, {u, Complement[mons[p], U_k /@ $PBWGens]}
  ];
 G = Series[e^{gg}, {ϵ, 0, $k}];
```



```
 S_i_[B_] := UCF[SS@O_i[Seepage_i[ff, SS@UCF@B]]];
 SG = ES["SG"]@S_k[G];
 TTR = ES["TTR"]@UCF[Invert_k[SG] ⊠ S_k[(∂_λ (ff.$PBWGens) G + ∂_λ G)]];
 diff = ES["diff"]@UCF[TTL - TTR];
 eqns = ES["eqns"]@Flatten[Table[
     Coefficient[Coefficient[diff, m], ϵ, k] == 0,
     {k, 0, $k}, {m, mons[k]}
    ]];
 inits = (#[0] == 0) & /@ fs;
 ES["DSolve"][
  DSolve[eqns ⋃ inits, fs, λ] /. F_Function :> ReplacePart[F, 2 → CF[F[[2]]]]
 ]]
```

In[•]:=
```
sol[2, 2]
```

» TTL $\left(e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \xi 12_i + e^{\gamma\,\alpha 1_j}\ \xi 12_j\right)\ U_k[x12]$

» SG $U_k[]$

» TTR $U_k[x12]\ \left(e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ f_{0,x12}'[\lambda] + e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \epsilon\ f_{1,x12}'[\lambda] + e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \epsilon^2\ f_{2,x12}'[\lambda]\right)$

» diff

$U_k[x12]\ \left(e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \xi 12_i + e^{\gamma\,\alpha 1_j}\ \xi 12_j - e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ f_{0,x12}'[\lambda] - e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \epsilon\ f_{1,x12}'[\lambda] - e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \epsilon^2\ f_{2,x12}'[\lambda]\right)$

» eqns $\left\{e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ \xi 12_i + e^{\gamma\,\alpha 1_j}\ \xi 12_j - e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ f_{0,x12}'[\lambda] == 0,\right.$
$\left. -e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ f_{1,x12}'[\lambda] == 0,\ -e^{\gamma\,(\alpha 1_i+\alpha 1_j)}\ f_{2,x12}'[\lambda] == 0\right\}$

» DSolve

$\left\{\left\{f_{0,x12} \to \text{Function}\left[\{\lambda\},\ \lambda\ \xi 12_i + e^{-\gamma\,\alpha 1_i}\ \lambda\ \xi 12_j\right],\ f_{1,x12} \to \text{Function}[\{\lambda\}, 0],\ f_{2,x12} \to \text{Function}[\{\lambda\}, 0]\right\}\right\}$

Out[•]=
$$\left\{\left\{f_{0,x12} \to \text{Function}\left[\{\lambda\},\ \lambda\ \xi 12_i + e^{-\gamma\,\alpha 1_i}\ \lambda\ \xi 12_j\right],\right.\right.$$
$$\left.\left. f_{1,x12} \to \text{Function}[\{\lambda\}, 0],\ f_{2,x12} \to \text{Function}[\{\lambda\}, 0]\right\}\right\}$$

In[•]:=
```
sol[1, 3]
```

» TTL $\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right)U_k[x12]+$

$\ll5\gg+\left(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\in\lambda\hbar\,\xi13_i\,\xi23_i-2\,e^{-\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i-3\alpha2_j)}\,\gamma\in\lambda\hbar\,\xi13_j\,\xi23_i+\right.$

$\left.2\,e^{\frac{1}{2}\gamma(\alpha1_i+\alpha2_i+3\alpha2_j)}\,\gamma\in\lambda\hbar\,\xi13_i\,\xi23_j+2\,e^{\frac{3\gamma\alpha2_j}{2}}\,\gamma\in\lambda\hbar\,\xi13_j\,\xi23_j\right)U_k[x13,\,x23]$

» SG $U_k[\,]+e^{\frac{3}{2}\gamma(\alpha1_i+\alpha1_j)}\in U_k[x12,\,x13]\,f_{1,x12\,x13}[\lambda]+$

$U_k[x13,\,x13]\left(e^{\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\in f_{0,x23}[\lambda]\,f_{1,x12\,x13}[\lambda]+e^{\gamma(\alpha1_i+\alpha1_j+\ll1\gg+\alpha2_j)}\in f_{1,\ll3\gg^2}[\lambda]\right)+$

$e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\in U_k[x12,\,x23]\,f_{1,x12\,x23}[\lambda]+$

$U_k[x13,\,x23]\left(e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\in f_{0,x23}[\lambda]\,f_{1,x12\,x23}[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\in f_{1,x13\,x23}[\lambda]\right)$

» TTR $U_k[x12]\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]+e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\in f_{1,x12}'[\lambda]\right)+\ll5\gg+U_k[x13,\,x23]$

$\left(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\in\hbar\,f_{0,x23}[\lambda]\,f_{0,x13}'[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\in f_{0,x23}[\lambda]\,f_{1,x12\,x23}'[\lambda]+e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\in f_{1,x13\,x23}'[\lambda]\right)$

» diff $U_k[x12]\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-\right.$

$\left.e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\in f_{1,x12}'[\lambda]\right)+$

$U_k[x13]\left(e^{\ll1\gg}\,\xi13_i+e^{\ll1\gg}\,\ll1\gg+\ll22\gg\right)+\ll3\gg+U_k[x12,\,x23]\,(\ll1\gg+\ll8\gg)+$

$U_k[x13,\,x23]\left(2\,e^{\frac{3}{2}\gamma(\alpha2_i+\alpha2_j)}\,\gamma\in\lambda\hbar\,\xi13_i\,\xi23_i-2\,e^{-\frac{1}{2}\gamma(\alpha1_i-2\,\ll2\gg_i-3\alpha2_j)}\,\gamma\in\lambda\hbar\,\xi13_j\,\xi23_i+\ll8\gg\right)$

» eqns

$\left\{e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]==0,\;e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\xi13_i+\right.$

$e^{\frac{1}{2}\gamma(\alpha1_j+\alpha2_j)}\,\xi13_j+e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,\lambda\,\xi12_i\,\xi23_i-\ll1\gg+\ll1\gg+e^{\frac{1}{2}\gamma(\ll1\gg)}\,\lambda\,\xi12_j\,\xi23_j-$

$\left.e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,f_{0,x23}[\lambda]\,f_{0,x12}'[\lambda]-e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j)}\,f_{0,x13}'[\lambda]==0,\;\ll6\gg,\;\ll1\gg,\;\ll1\gg\right\}$

» DSolve $\left\{\left\{f_{0,x12}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\right],\right.\right.$

$f_{0,x23}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i)}\,\lambda\,\xi23_j\right],$

$f_{0,x13}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi13_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\lambda\,\xi13_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi23_i\right],$

$\ll5\gg,\;f_{1,x13^2}\to\text{Function}[\{\lambda},\,0],\;f_{1,x12}\to\text{Function}[\{\lambda},\,0]\left.\right\}\right\}$

Out[•]=

$\left\{\left\{f_{0,x12}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\right],\right.\right.$

$f_{0,x23}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i)}\,\lambda\,\xi23_j\right],$

$f_{0,x13}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi13_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\lambda\,\xi13_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi23_i\right],$

$f_{1,x23}\to\text{Function}[\{\lambda},\,0],\;f_{1,x13}\to\text{Function}\left[\{\lambda},\,-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi23_i\right],$

$f_{1,x12\,x23}\to\text{Function}\left[\{\lambda},\,2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi23_i\right],$

$f_{1,x12\,x13}\to\text{Function}\left[\{\lambda},\,-2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi12_j\,\xi13_i\right],$

$f_{1,x13\,x23}\to\text{Function}\left[\{\lambda},\,-2\,e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi13_j\,\xi23_i\right],$

$f_{1,x13^2}\to\text{Function}[\{\lambda},\,0],\;f_{1,x12}\to\text{Function}[\{\lambda},\,0]\left.\right\}\right\}$

In[•]:= **sol[1, 4]**

» TTL $\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j\right)U_k[x12]+\ll21\gg+$

$\left(2\,e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\,\gamma\in\lambda\hbar\,\xi24_i\,\xi34_i-2\,e^{-\frac{1}{2}\gamma(\alpha1_j+\alpha2_i-2\alpha3_i-3\alpha3_j)}\,\gamma\in\lambda\hbar\,\xi24_j\,\xi34_i+\right.$

$\left.2\,e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-\alpha2_i-\alpha3_i-3\alpha3_j)}\,\gamma\in\lambda\hbar\,\xi24_i\,\xi34_j+2\,e^{-\frac{1}{2}\gamma(\alpha1_j-3\alpha3_j)}\,\gamma\in\lambda\hbar\,\xi24_j\,\xi34_j\right)U_k[x24,\,x34]$

» SG $U_k[\,]+e^{\frac{1}{2}\gamma(3\alpha1_i+3\alpha1_j-\alpha3_i-\alpha3_j)}\in U_k[x12,\,x13]\,f_{1,x12\,x13}[\lambda]+$

$\ll14\gg+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-\alpha2_i-\alpha2_j-\alpha3_i-\alpha3_j)}\in U_k[x23,\,x34]\,f_{1,x23\,x34}[\lambda]+$

$U_k[x24,\,x34]\left(e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\in f_{0,x34}[\lambda]\,f_{1,x23\,x34}[\lambda]+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\in f_{1,x24\,x34}[\lambda]\right)$

» TTR $U_k[x12]\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]+e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\in f_{1,x12}'[\lambda]\right)+$

$\ll21\gg+U_k[x24,\,x34]\left(2\,e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\,\gamma\in\hbar\,f_{0,x34}[\lambda]\,f_{0,x24}'[\lambda]+\right.$

$\left.e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\in f_{0,x34}[\lambda]\,f_{1,x23\,x34}'[\lambda]+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha1_j-3\alpha3_i-3\alpha3_j)}\in f_{1,x24\,x34}'[\lambda]\right)$

» diff $U_k[x12]\left(e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-\right.$

$\left.e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\in f_{1,x12}'[\lambda]\right)+$

$U_k[x13]\left(e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j-\alpha3_i-\alpha3_j)}\,\xi13_i+e^{\ll1\gg}\,\xi13_j+\ll22\gg\right)+U_k[x12,\,x13]\,(\ll1\gg)+\ll17\gg+$

$\ll1\gg\ \ll1\gg+U_k[x23,\,x34]\left(-2\,e^{\frac{1}{2}\gamma(\ll6\gg+\alpha3_j)}\,\gamma\in\lambda\hbar\,\xi23_i\,\xi34_i+\ll8\gg\right)+U_k[x24,\,x34]\,(\ll1\gg)$

» eqns $\left\{e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,\xi12_i+e^{\frac{1}{2}\gamma(2\alpha1_j-\alpha2_j)}\,\xi12_j-e^{\frac{1}{2}\gamma(2\alpha1_i+2\alpha1_j-\alpha2_i-\alpha2_j)}\,f_{0,x12}'[\lambda]==0,\right.$

$e^{\frac{1}{2}\gamma(\alpha1_i+\alpha1_j+\alpha2_i+\alpha2_j-\alpha3_i-\alpha3_j)}\,\xi13_i+e^{\frac{1}{2}\gamma(\ll1\gg)}\,\xi13_j+\ll9\gg==0,$

$e^{\frac{1}{2}\gamma(\ll1\gg)}\,\xi14_i+\ll27\gg==0,\ \ll1\gg==0,\ \ll21\gg,\ \ll1\gg==0,\ \ll1\gg==0,\ \ll1\gg==0,$

$\left.2\,e^{\frac{1}{2}\gamma(-\alpha1_i-\alpha1_j+3\alpha3_i+3\alpha3_j)}\,\gamma\lambda\hbar\,\xi24_i\,\xi34_i-2\,e^{\frac{1}{2}\gamma(-\alpha1_j-\alpha2_i+2\alpha3_i+3\alpha3_j)}\,\gamma\lambda\hbar\,\xi24_j\,\xi34_i+\ll8\gg==0\right\}$

» DSolve $\left\{\left\{f_{0,x12}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\right],\right.\right.$

$\left.\left.f_{0,x23}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i+\alpha3_i)}\,\lambda\,\xi23_j\right],\ \ll26\gg,\ f_{1,x12}\to\text{Function}[\{\lambda},\,0]\right\}\right\}$

Out[•]=

$\left\{\left\{f_{0,x12}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi12_i+e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda\,\xi12_j\right],\right.\right.$

$f_{0,x23}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi23_i+e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i+\alpha3_i)}\,\lambda\,\xi23_j\right],$

$f_{0,x13}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi13_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i-\alpha3_i)}\,\lambda\,\xi13_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi23_i\right],$

$f_{0,x34}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi34_i+e^{\frac{1}{2}\gamma(\alpha2_i-2\alpha3_i)}\,\lambda\,\xi34_j\right],$

$f_{0,x24}\to\text{Function}\left[\{\lambda},\,\lambda\,\xi24_i+e^{\frac{1}{2}\gamma(\alpha1_i-\alpha2_i-\alpha3_i)}\,\lambda\,\xi24_j-e^{\frac{1}{2}\gamma(\alpha1_i-2\alpha2_i+\alpha3_i)}\,\lambda^2\,\xi23_j\,\xi34_i\right],$

$f_{0,x14}\to\text{Function}\left[\{\lambda},\right.$

$\left.\lambda\,\xi14_i+e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha3_i)}\,\lambda\,\xi14_j-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\lambda^2\,\xi12_j\,\xi24_i-e^{-\frac{1}{2}\gamma(\alpha1_i+\alpha2_i-\alpha3_i)}\,\lambda^2\,\xi13_j\,\xi34_i\right],$

$f_{1,x23}\to\text{Function}[\{\lambda},\,0],\ f_{1,x13}\to\text{Function}\left[\{\lambda},\,-e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\lambda^2\hbar\,\xi12_j\,\xi23_i\right],$

$f_{1,x12\,x23}\to\text{Function}\left[\{\lambda},\,2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\lambda^2\hbar\,\xi12_j\,\xi23_i\right],$

$f_{1,x12\,x13}\to\text{Function}\left[\{\lambda},\,-2\,e^{-\frac{1}{2}\gamma(2\alpha1_i-\alpha2_i)}\,\gamma\lambda^2\hbar\,\xi12_j\,\xi13_i\right],$

$$\mathtt{f_{1,x13\,x23}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 13_j\,\xi 23_i\Big],$$

$$\mathtt{f_{1,x13^2}} \to \mathtt{Function}[\{\lambda\},\ 0],\ \mathtt{f_{1,x12\,x24}} \to \mathtt{Function}\Big[\{\lambda\},\ 2\,e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 12_j\,\xi 24_i\Big],$$

$$\mathtt{f_{1,x12\,x14}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 12_j\,\xi 14_i\Big],$$

$$\mathtt{f_{1,x34}} \to \mathtt{Function}[\{\lambda\},\ 0],\ \mathtt{f_{1,x24}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 23_j\,\xi 34_i\Big],$$

$$\mathtt{f_{1,x14}} \to \mathtt{Function}\Big[\{\lambda\},\ -e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 12_j\,\xi 24_i - e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 13_j\,\xi 34_i\Big],$$

$$\mathtt{f_{1,x23\,x34}} \to \mathtt{Function}\Big[\{\lambda\},\ 2\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 23_j\,\xi 34_i\Big],$$

$$\mathtt{f_{1,x23\,x24}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 23_j\,\xi 24_i\Big],$$

$$\mathtt{f_{1,x14\,x23}} \to \mathtt{Function}\Big[\{\lambda\},\ -4\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 13_j\,\xi 24_i\Big],$$

$$\mathtt{f_{1,x13\,x34}} \to \mathtt{Function}\Big[\{\lambda\},\ 2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 13_j\,\xi 34_i\Big],\ \mathtt{f_{1,x13\,x24}} \to$$
$$\mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 12_j\,\xi 23_i\,\xi 24_i - \frac{2}{3}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+2\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_i +$$
$$\frac{2}{3}\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_i - 2\,e^{-\frac{1}{2}\gamma(3\alpha 2_i-2\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 13_j\,\xi 23_j\,\xi 34_i\Big],$$

$$\mathtt{f_{1,x13\,x14}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 13_j\,\xi 14_i +$$
$$2\,e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 12_j\,\xi 14_i\,\xi 23_i + 2\,e^{-\frac{1}{2}\gamma(3\alpha 1_i-\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 12_j\,\xi 13_j\,\xi 24_i +$$
$$\frac{1}{6}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+2\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_i - \frac{1}{6}\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_i -$$
$$e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_i + e^{\frac{1}{2}\gamma(-2\alpha 1_i+\alpha 2_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_i +$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 2_i-2\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_i\,\xi 34_j - \frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 2_i+2\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_i\,\xi 34_j +$$
$$e^{\gamma(-\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_j - e^{-\gamma(\alpha 1_i-\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_j\Big],$$

$$\mathtt{f_{1,x24\,x34}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 24_j\,\xi 34_i\Big],\ \mathtt{f_{1,x24^2}} \to \mathtt{Function}[\{\lambda\},\ 0],$$

$$\mathtt{f_{1,x14\,x34}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 14_j\,\xi 34_i\Big],$$

$$\mathtt{f_{1,x14\,x24}} \to \mathtt{Function}\Big[\{\lambda\},\ -2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 3_i)}\,\gamma\,\lambda^2\,\hbar\,\xi 14_j\,\xi 24_i -$$
$$\frac{2}{3}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+2\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 14_i\,\xi 23_j\,\xi 34_i + \frac{2}{3}\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 14_i\,\xi 23_j\,\xi 34_i +$$
$$2\,e^{-\gamma\alpha 2_i}\,\gamma\,\lambda^3\,\hbar\,\xi 14_j\,\xi 23_j\,\xi 34_i + 2\,e^{-\frac{1}{2}\gamma(\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 13_j\,\xi 24_i\,\xi 34_i +$$
$$\frac{4}{3}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 13_i\,\xi 24_j\,\xi 34_i - \frac{4}{3}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^3\,\hbar\,\xi 13_i\,\xi 24_j\,\xi 34_i +$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_i\,\xi 24_j\,\xi 34_i - \frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_i\,\xi 24_j\,\xi 34_i +$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(2\alpha 1_i-3\alpha 2_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_j\,\xi 34_i - \frac{1}{2}\,e^{-\frac{1}{2}\gamma(-2\alpha 1_i+3\alpha 2_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_j\,\xi 34_i -$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_i\,\xi 34_j + \frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\,\lambda^4\,\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_i\,\xi 34_j +$$

$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i-3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_j\,\xi 23_i\,\xi 24_j\,\xi 34_j-\frac{1}{2}\,e^{-\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i+3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_j\,\xi 23_i\,\xi 24_j\,\xi 34_j+$$
$$\frac{1}{2}\,e^{\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_j\,\xi 34_j-\frac{1}{2}\,e^{-\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 23_j\,\xi 24_j\,\xi 34_j+$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_i\,\xi 23_j\,\xi 34_i\,\xi 34_j-\frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_i\,\xi 23_j\,\xi 34_i\,\xi 34_j+$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i-3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_j\,\xi 23_i\,\xi 34_j^2-\frac{1}{2}\,e^{-\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i+3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_j\,\xi 23_i\,\xi 34_j^2-$$
$$\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 1_i-3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_i\,\xi 23_j\,\xi 34_j^2+\frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+3\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 13_i\,\xi 23_j\,\xi 34_j^2\Big],$$

$$f_{1,x14^2}\to \text{Function}\Big[\{\lambda\},\ \frac{1}{6}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+2\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 14_i\,\xi 23_j\,\xi 34_i-$$
$$\frac{1}{6}\,e^{\frac{1}{2}\gamma(\alpha 1_i-2\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 14_i\,\xi 23_j\,\xi 34_i-\frac{1}{3}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 13_i\,\xi 24_j\,\xi 34_i+$$
$$\frac{1}{3}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 13_i\,\xi 24_j\,\xi 34_i-\frac{1}{10}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_i\,\xi 24_j\,\xi 34_i+$$
$$\frac{1}{10}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_i\,\xi 24_j\,\xi 34_i-\frac{1}{10}\,e^{\frac{1}{2}\gamma(2\alpha 1_i-3\alpha 2_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_j\,\xi 34_i+$$
$$\frac{1}{10}\,e^{-\frac{1}{2}\gamma(-2\alpha 1_i+3\alpha 2_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_j\,\xi 34_i+\frac{2}{5}\,e^{-\frac{1}{2}\gamma(2\alpha 1_i-\alpha 2_i)}\,\gamma\lambda^5\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_i^2-$$
$$\frac{2}{5}\,e^{\frac{1}{2}\gamma(-2\alpha 1_i+\alpha 2_i)}\,\gamma\lambda^5\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_i^2-\frac{2}{3}\,e^{\frac{1}{2}\gamma(\alpha 2_i-2\alpha 3_i)}\,\gamma\lambda^3\hbar\,\xi 13_i\,\xi 14_i\,\xi 34_j+$$
$$\frac{2}{3}\,e^{-\frac{1}{2}\gamma(-\alpha 2_i+2\alpha 3_i)}\,\gamma\lambda^3\hbar\,\xi 13_i\,\xi 14_i\,\xi 34_j-\frac{1}{2}\,e^{\frac{1}{2}\gamma(\alpha 2_i-2\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 14_i\,\xi 23_i\,\xi 34_j+$$
$$\frac{1}{2}\,e^{-\frac{1}{2}\gamma(-\alpha 2_i+2\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_i\,\xi 14_i\,\xi 23_i\,\xi 34_j-e^{\gamma(-\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_j\,\xi 13_i\,\xi 24_i\,\xi 34_j+$$
$$e^{-\gamma(\alpha 1_i-\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^4\hbar\,\xi 12_j\,\xi 13_i\,\xi 24_i\,\xi 34_j-\frac{2}{5}\,e^{\gamma(-\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 12_j\,\xi 23_i\,\xi 24_i\,\xi 34_j+$$
$$\frac{2}{5}\,e^{-\gamma(\alpha 1_i-\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 12_j\,\xi 23_i\,\xi 24_i\,\xi 34_j+\frac{1}{10}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_i\,\xi 34_j-$$
$$\frac{1}{10}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_i\,\xi 34_j-\frac{1}{10}\,e^{\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i-3\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i$$
$$\xi 12_j\,\xi 23_i\,\xi 24_j\,\xi 34_j+\frac{1}{10}\,e^{-\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i+3\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 12_j\,\xi 23_i\,\xi 24_j\,\xi 34_j-$$
$$\frac{1}{10}\,e^{\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_j\,\xi 34_j+\frac{1}{10}\,e^{-\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i^2\,\xi 23_j\,\xi 24_j\,\xi 34_j-$$
$$\frac{4}{5}\,e^{\frac{1}{2}\gamma(\alpha 2_i-2\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_i\,\xi 34_i\,\xi 34_j+\frac{4}{5}\,e^{-\frac{1}{2}\gamma(-\alpha 2_i+2\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_i\,\xi 34_i\,\xi 34_j-$$
$$\frac{8}{5}\,e^{\gamma(-\alpha 1_i+\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_i\,\xi 34_j+\frac{8}{5}\,e^{-\gamma(\alpha 1_i-\alpha 2_i+\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i$$
$$\xi 34_i\,\xi 34_j-\frac{1}{10}\,e^{\frac{1}{2}\gamma(\alpha 1_i-\alpha 2_i-\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_j\,\xi 34_i\,\xi 34_j+\frac{1}{10}\,e^{-\frac{1}{2}\gamma(-\alpha 1_i+\alpha 2_i+\alpha 3_i)}$$
$$\gamma\lambda^5\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_j\,\xi 34_i\,\xi 34_j-\frac{2}{5}\,e^{\frac{1}{2}\gamma(-2\alpha 1_i+3\alpha 2_i-4\alpha 3_i)}\,\gamma\lambda^5\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_j^2+$$

$$\frac{2}{5}\,e^{-\frac{1}{2}\gamma\,(2\,\alpha 1_i-3\,\alpha 2_i+4\,\alpha 3_i)}\,\gamma\,\lambda^5\,\hbar\,\xi 12_j\,\xi 13_i\,\xi 23_i\,\xi 34_j^2-\frac{1}{10}\,e^{\frac{1}{2}\gamma\,(-\alpha 1_i+\alpha 2_i-3\,\alpha 3_i)}\,\gamma\,\lambda^5\,\hbar\,\xi 12_i\,\xi 13_j\,\xi 23_i\,\xi 34_j^2+$$
$$\frac{1}{10}\,e^{-\frac{1}{2}\gamma\,(\alpha 1_i-\alpha 2_i+3\,\alpha 3_i)}\,\gamma\,\lambda^5\,\hbar\,\xi 12_i\,\xi 13_j\,\xi 23_i\,\xi 34_j^2+\frac{1}{10}\,e^{\frac{1}{2}\gamma\,(\alpha 1_i-3\,\alpha 3_i)}\,\gamma\,\lambda^5\,\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_j\,\xi 34_j^2-$$
$$\frac{1}{10}\,e^{-\frac{1}{2}\gamma\,(-\alpha 1_i+3\,\alpha 3_i)}\,\gamma\,\lambda^5\,\hbar\,\xi 12_i\,\xi 13_i\,\xi 23_j\,\xi 34_j^2\Big],\ f_{1,x12}\to \text{Function}[\{\lambda\},\ 0]\Big\}\Big\}$$